**Problem**

We are selling products grouped into $m$ nests, with $n$ products in each nest. The customer assigns a preference weight $w_{ij}$ to product $j$ in nest $i$ that depends on the product’s price. The customer makes a purchasing decision via a two-stage process: she first selects a nest (or opts to buy nothing), then a product $j$ within nest $i$. The probability of selecting nest $i$ is $Q_i(w_1, \ldots, w_m)$ and the expected revenue we obtain given that nest $i$ is selected is $R_i(w_i) = \frac{1}{1 + \sum_{j \in L_i} \nu_{ij}}$, where $M = \{1, \ldots, m\}$, $N = \{1, \ldots, n\}$ and $\gamma_i \in [0, 1]$ measures product dissimilarity within nests. Due to a one-to-one relationship, control of the preference weights equates to control of the prices $p_i - \nu_{ij} \log w_{ij}$, where $1/\nu_{ij}$ represents product price sensitivity and $p_i/\nu_{ij}$ represents product quality. We wish to maximize the total expected revenue subject to lower and upper bound constraints, i.e.

$$Z^* = \max \{ \sum_{i \in M} Q_i(w_1, \ldots, w_m)R_i(w_i) : L_i \leq w_i \leq U, \forall i \in M \},$$

where preference weight lower bounds (upper bounds) correspond to price upper bounds (lower bounds).

**Grid Construction**

We can find $O(n)$ intervals $[\nu^k_i, \mu^k_i] : k = 1, \ldots, K_i$ partitioning $[L_i, U_i]$ such that the set of active constraints corresponding to $g_i(y_i)$ is fixed for all $y_i \in [\nu^k_i, \mu^k_i]$ (and we can identify which constraints are active). Referring to the approximate problem, given $\rho > 0$, we construct the grid $[\bar{y}_i^k : t = 1, \ldots, T_i]$ by laying out the points $[\nu^k_i, \mu^k_i]$ in order and filling in with intermediary points of the form $\bar{y}_i^k = \sum_{j \in L_i} 1_{ij} + \sum_{j \in U_i} 1_{ij} + (1 + \rho)\bar{y}_i$, $g \in Z$ where $L_i^k$ and $U_i^k$ are the sets of products whose preference weights are snapped to their lower and upper bounds, respectively, on interval $k$. For each $g_i$, we have the property

$$g_i(y_i) \leq (1 + \rho)g_i(\bar{y}_i^k) \forall y_i \in [\nu^k_i, \mu^k_i],$$

so we only need enough intermediary points to “fill in” each of these intervals $k$ in the appropriate manner. Letting $\sigma_i = \max_{y_i \in U_i} \min_{y_i \in L_i}$, we need $O(\log(n)\sigma_i)/(1 + \rho)$ intermediary points for each $g_i$ to satisfy the conditions of the theorem to the right. We can obtain the solution $\bar{z}$ by solving a linear program with $m + 1$ variables and $O(m + m \log(\log \max_i \sigma_i)/(1 + \rho))$ constraints, giving us an approximation guarantee of $1 + \rho$ for the original problem. We can do this for any $\rho > 0$, giving us a fully-polynomial-time approximation scheme.

**Fixed Point Representation**

Let $g_i(y_i)$ be the optimal objective value of the nonlinear knapsack problem

$$\max \{ \sum_{j \in M} (k_{ij} - \eta_{ij} \log w_{ij})w_{ij} \}
\quad \text{s.t.} \sum_{j \in N} w_{ij} \leq y_i
w_{ij} \in [L_{ij}, U_{ij}], \forall j \in N.$$

**Proposition.** $Z^*$ is the solution to the fixed point problem

$$z = \sum_{i \in M} \max_{y_i \in [L_i, U_i]} \left\{ y_i^g g_i(y_i) - y_i^g z \right\}.$$  

where $L_i = \sum_{j \in N} L_{ij}$ and $U_i = \sum_{j \in N} U_{ij}$.

**Approximate Problem**

We evaluate the max in (1) over a discrete collection of points.

$$z = \sum_{i \in M} \max_{y_i \in \bar{y}_i^k : t = 1, \ldots, T_i} \left\{ y_i^g g_i(y_i) - y_i^g z \right\}.$$  

We can express this problem as a linear program with $m + 1$ variables and $\sum_{i \in M} T_i$ constraints,

$$\min_{(x,z)} \quad z \quad \text{s.t.} \quad \sum_{i = 1}^m x_i \geq \gamma_i (y_{ij}^{k+1})^{1 - \gamma_i} - (y_{ij}^k)^{1 - \gamma_i}, \forall t, \forall i.$$  

**Theorem (Grid Requirements).** For some $\rho \geq 0$, assume that the collection of grid points $\{\bar{y}_i^k : t = 1, \ldots, T_i\}$ satisfy

$$g_i(y_{ij}^{k+1}) \leq (1 + \rho)g_i(y_{ij}^k)$$

for all $t = 1, \ldots, T_i - 1$. If $\bar{z}$ denotes the value of $z$ that satisfies (2), then we have

$$(1 + \rho)\bar{z} \geq Z^*.$$  

**Structural Challenges**

The functions that we maximize over for each $z$ in the fixed point representation are not necessarily unimodal when the problem is constrained and/or conditions on the price sensitivities and dissimilarity parameters are not met. We turn our attention to developing approximate methods for the constrained problem.

**Contributions**

Expanding upon the work done on unconstrained versions of the problem in [1] and [2]:

1. We show that the constrained problem reduces to a single-dimensional fixed point problem, but involves a function that may be computationally expensive to evaluate.
2. We develop a fully-polynomial-time approximation scheme (FPTAS) for the constrained revenue maximization problem.
3. Our approach can be extended to allow control of the offer assortments in addition to prices, in which case the running time increases by a factor of $n$.

**References**
