Approximation Methods for Pricing Problems under the Nested Logit Model with Price Bounds

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Nested Logit Model

Store

Competitor Space

• n products, $N = \{1, \ldots, n\}$
Nested Logit Model

- Store
  - Preference weight of product j: $w_j$

- Competitor Space
  - n products, $N = \{1, \ldots, n\}$
  - Preference weight of product j: $w_j$
Nested Logit Model

- **Store**
  - $w_1$
  - $w_j$
  - $w_n$

- **Competitor Space**
  - $w_0 = 1$

- $n$ products, $N = \{1, \ldots, n\}$
- Preference weight of product $j$: $w_j$
- Total preference weight of competitor space normalized to 1
Decision: Prices

• price of product j:

\[ w_0 = 1 \]
Decision: Prices

- price of product j: $p_j$

\[ w_0 = 1 \]
Decision: Prices

• Price of product j: $p_j$

\[ w_j = e^{\alpha_j - \beta_j p_j} \]
Decision: Prices

• price of product j: \( p_j \)

\[
w_j = e^{\alpha_j - \beta_j p_j}
\]

• prices are constrained: \( l_j \leq p_j \leq u_j \)
• price of product j: $p_j$

$$p_j = \frac{\alpha_j}{\beta_j} - \frac{1}{\beta_j} \log w_j$$

• prices are constrained: $l_j \leq p_j \leq u_j$
Decision: Prices

- **price of product j**: $p_j$
  
  $$p_j = \kappa_j - \eta_j \log w_j$$

- **prices are constrained**: $l_j \leq p_j \leq u_j$
**Decision: Prices**

- **Price of product j:** $p_j$

  $p_j = \kappa_j - \eta_j \log w_j$

- **Prices are constrained:** $L_j \leq w_j \leq U_j$
Nested Logit Model

• If the customer buys from our firm, product j is purchased with probability

$$\frac{w_j}{\sum_{k \in N} w_k}$$
Nested Logit Model

\[ w_0 = 1 \]

\[ \frac{w_j}{\sum_{k \in N} w_k} \]
Nested Logit Model

Store

Competitor Space

\[ w_0 = 1 \]

\[
\frac{w_j}{\sum_{k \in N} w_k}
\]
Nested Logit Model

- Expected revenue obtained given that the customer buys from our firm

\[
R(w) = \sum_{j \in N} (\kappa_j - \eta_j \log w_j) \frac{w_j}{\sum_{k \in N} w_k}
\]

\[
w_0 = 1
\]
Nested Logit Model

Store

\[ \gamma \in [0, 1] \]

\[ w_1, w_j, w_n \]

Competitor Space

\[ w_0 = 1 \]
Nested Logit Model

Customer buys from our firm with probability

\[
Q(w) = \frac{(\sum_{j \in N} w_j)^\gamma}{1 + (\sum_{j \in N} w_j)^\gamma}
\]

\[
\gamma \in [0, 1]
\]

Store

\[
\begin{align*}
\gamma & \in [0, 1] \\
\sum_{j \in N} w_j & = 1 \\
\end{align*}
\]

Competitor Space

\[
\sum_{j \in N} w_j = 1
\]

\[
\begin{align*}
w_0 & = 1 \\
\end{align*}
\]
Problem Statement

• Expected revenue: $\Pi(w) = Q(w)R(w)$

Store

\[ w_1, \gamma \in [0, 1], w_j, w_n \]

Competitor Space

\[ w_0 = 1 \]
Problem Statement

- **Store**

  \( w_1 \) \( \gamma \in [0, 1] \) \( w_j \) \( w_n \)

- **Competitor Space**

  \( w_0 = 1 \)

- **Expected revenue:**

  \[
  \Pi(w) = Q(w)R(w)
  \]

- **Maximization:**

  \[
  Z^* = \max_{L \leq w \leq U} \Pi(w)
  \]
Contributions

• Prior work: Li and Huh (2011), Gallego and Wang (2011) - unconstrained pricing under NL model

Our Approach:
Contributions

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Our Approach:

• Handles arbitrary price sensitivity values
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• Allows for the inclusion of price bounds
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Our Approach:

• Handles arbitrary price sensitivity values

• Allows for the inclusion of price bounds

• Provides a performance guarantee not found in existing literature
Transformation of Objective

\[ Z^* \geq \Pi(w) = Q(w)R(w) \]

\[ (1 + \left( \sum_{j \in N} w_j \right)^\gamma)Z^* \geq \left( \sum_{j \in N} w_j \right)^\gamma \frac{\sum_{j \in N} (\kappa_j - \eta_j \log w_j)w_j}{\sum_{j \in N} w_j} \]

\[ \forall L \leq w \leq U \]
Transformation of Objective

\[ Z^* \geq \left( \sum_{j \in N} w_j \right)^\gamma \frac{\sum_{j \in N}(\kappa_j - \eta_j \log w_j)w_j}{1 + \left( \sum_{j \in N} w_j \right)^\gamma \sum_{j \in N} w_j} \quad \forall L \leq w \leq U \]

\[ (1 + \left( \sum_{j \in N} w_j \right)^\gamma) Z^* \geq \left( \sum_{j \in N} w_j \right)^\gamma \frac{\sum_{j \in N}(\kappa_j - \eta_j \log w_j)w_j}{1 + \left( \sum_{j \in N} w_j \right)^\gamma \sum_{j \in N} w_j} \quad \forall L \leq w \leq U \]
Transformation of Objective

\[ Z^* \geq \frac{\left( \sum_{j \in N} w_j \right)^\gamma}{1 + \left( \sum_{j \in N} w_j \right)^\gamma} \cdot \frac{\sum_{j \in N} (\kappa_j - \eta_j \log w_j) w_j}{\sum_{j \in N} w_j} \quad \forall L \leq w \leq U \]
Transformation of Objective

\[
Z^* \geq \frac{\left( \sum_{j \in N} w_j \right)^\gamma}{1 + \left( \sum_{j \in N} w_j \right)^\gamma} \frac{\sum_{j \in N} (\kappa_j - \eta_j \log w_j) w_j}{\sum_{j \in N} w_j} \sum_{j \in N} w_j \quad \forall L \leq w \leq U
\]

\[
\left( \sum_{j \in N} w_j \right)^\gamma \left( \frac{\sum_{j \in N} (\kappa_j - \eta_j \log w_j) w_j}{\sum_{j \in N} w_j} - Z^* \right) \quad \forall L \leq w \leq U
\]
Transformation of Objective

\[ Z^* \geq \frac{(\sum_{j \in N} w_j)^{\gamma}}{1 + (\sum_{j \in N} w_j)^{\gamma}} \frac{\sum_{j \in N} (\kappa_j - \eta_j \log w_j)w_j}{\sum_{j \in N} w_j} \forall L \leq w \leq U \]

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\[ Z^* = \]
Transformation of Objective

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\[ Z^* = \max_{L \leq w \leq U} \left\{ \left( \sum_{j \in N} w_j \right)^{\gamma} \left( \frac{\sum_{j \in N} (\kappa_j - \eta_j \log w_j)w_j}{\sum_{j \in N} w_j} - Z^* \right) \right\} \]
Transformation of Objective

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• Use single decision variable to capture \( \sum_{j \in N} w_j \)
Transformation of Objective

\[ Z^* = \max_{L \leq w \leq U} \left\{ \left( \sum_{j \in N} w_j \right)^\gamma \left( \frac{\sum_{j \in N} (k_j - \eta_j \log w_j) w_j}{\sum_{j \in N} w_j} - Z^* \right) \right\} \]

- Use single decision variable to capture \( \sum_{j \in N} w_j \)
Transformation of Objective

\[
\max_{y \in ?} \left\{ y^\gamma \left( \frac{?}{y} - Z^* \right) \right\}
\]

\[
Z^* = \max_{L \leq w \leq U} \left\{ \left( \sum_{j \in N} w_j \right)^\gamma \left( \frac{\sum_{j \in N} (\kappa_j - \eta_j \log w_j) w_j}{\sum_{j \in N} w_j} - Z^* \right) \right\}
\]

• Use single decision variable to capture \( \sum_{j \in N} w_j \)
Transformation of Objective

\[
\max_{y \in \mathbb{?}} \left\{ y^\gamma \left( \frac{?}{y} - Z^* \right) \right\}
\]

- Capture numerator with

\[
g(y) = \max \left\{ \sum_{j \in N} (\kappa_j - \eta_j \log w_j)w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\}
\]

\[
Z^* = \max_{L \leq w \leq U} \left\{ \left( \sum_{j \in N} w_j \right)^\gamma \left( \frac{\sum_{j \in N} (\kappa_j - \eta_j \log w_j)w_j}{\sum_{j \in N} w_j} - Z^* \right) \right\}
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- Capture numerator with

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\]

- Restrict attention to \( y \in [\bar{L}, \bar{U}] \),

\[
\bar{L} = \sum_{j \in N} L_j, \bar{U} = \sum_{j \in N} \min\{\max\{\exp(\kappa_j/\eta_j - 1), L_j\}, U_j\}
\]

- Use single decision variable to capture \( \sum_{j \in N} w_j \)
Fixed Point Problem

\( Z^* \) is the solution to

\[
z = \max_{y \in [\bar{L}, \bar{U}]} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma z \right\}
\]
Recovering Optimal Prices

\[ y \]

• If we know \( \text{Z}^* \), solving

\[
\max_{y \in [\bar{L}, \bar{U}]} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma \text{Z}^* \right\}
\]

recovers the optimal solution

\[
g(y) = \max \left\{ \sum_{j \in N} (\kappa_j - \eta_j \log w_j) w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\}
\]
Recovering Optimal Prices

\[ g(\cdot) \]

\[ y \]

\[ \text{If we know } Z^*, \text{ solving} \]

\[
\max_{y \in [\bar{L}, \bar{U}]} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma Z^* \right\}
\]

recovers the optimal solution

\[ g(y) = \max \left\{ \sum_{j \in N} (\kappa_j - \eta_j \log w_j) w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\} \]
Recovering Optimal Prices

• If we know $Z^*$, solving

$$g(y) = \max \left\{ \sum_{j \in N} (\kappa_j - \eta_j \log w_j) w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\}$$

recovers the optimal solution

$$\max_{y \in [\bar{L}, \bar{U}]} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma Z^* \right\}$$
Recovering Optimal Prices

• If we know $Z^*$, solving

$$\max_{y \in [\bar{L}, \bar{U}]} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma Z^* \right\}$$

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Recovering Optimal Prices

If we know \( Z^* \), solving

\[
\max_{y \in [\bar{L}, \bar{U}]} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma Z^* \right\}
\]

recovers the optimal solution

\[
g(y) = \max \left\{ \sum_{j \in N} (\kappa_j - \eta_j \log w_j)w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\}
\]
Recovering Optimal Prices

If we know $\mathbf{Z}^*$, solving

$$
\max_{y \in [\bar{L}, \bar{U}]} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma \mathbf{Z}^* \right\}
$$

recovers the optimal solution

$$
g(y) = \max \left\{ \sum_{j \in N} (\kappa_j - \eta_j \log w_j)w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\}
$$
Recovering Optimal Prices

If we know $Z^*$, solving

$$
\max_{y \in [\bar{L}, \bar{U}]} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma Z^* \right\}
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recovers the optimal solution

$$
g(y) = \max \left\{ \sum_{j \in N} (\kappa_j - \eta_j \log w_j)w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\}
$$
Recovering Optimal Prices

\[ g(\cdot) \]

\[ y \rightarrow w_1(y) \rightarrow p_1(y) \]

\[ \vdots \]

\[ w_n(y) \rightarrow \]

- If we know \( Z^* \), solving

\[
\max_{y \in [\bar{L}, \bar{U}]} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma Z^* \right\}
\]

recovers the optimal solution

\[ g(y) = \max \left\{ \sum_{j \in N} (\kappa_j - \eta_j \log w_j)w_j : \sum_{j \in N} w_j \leq y, \ w_j \in [L_j, U_j] \ \forall j \in N \right\} \]
Recovering Optimal Prices

If we know $Z^*$, solving

$$\max_{y \in [\bar{L}, \bar{U}]} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma Z^* \right\}$$

recovers the optimal solution

$$g(y) = \max \left\{ \sum_{j \in N} (\kappa_j - \eta_j \log w_j) w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\}$$
Recovering Optimal Prices

If we know $Z^*$, solving

$$\max_{y \in [\bar{L}, \bar{U}]} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma Z^* \right\}$$

recovers the optimal solution

• Problem: structural challenges
Recovering Optimal Prices

If we know $Z^*$, solving

$$\max_{y \in [\bar{L}, \bar{U}]} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma Z^* \right\}$$

recovers the optimal solution

Problem: structural challenges
Problematic Example

\[ y^\gamma \frac{g(y)}{y} - y^\gamma Z^* \]
Approximation Framework

- Construct a set of grid points $\{\tilde{y}^t : t = 1, \ldots, T\}$, solve

$$
z = \max_{y \in \{\tilde{y}^t : t = 1, \ldots, T\}} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma z \right\}
$$
Approximation Framework

- Construct a set of grid points \( \{ \tilde{y}^t : t = 1, \ldots, T \} \), solve

\[
\hat{z} = \max_{y \in \{ \tilde{y}^t : t = 1, \ldots, T \}} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma z \right\}
\]

- Objective: given \( \rho > 0 \), obtain an approximate solution with expected revenue

\[
\hat{z} \geq \frac{1}{1 + \rho} Z^*
\]

- Keep grid size small
Approximation Framework

• Construct a set of grid points \( \{\tilde{y}^t : t = 1, \ldots, T\} \), solve

\[
\hat{z} = \max_{y \in \{\tilde{y}^t : t = 1, \ldots, T\}} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma z \right\}
\]

• Objective: given \( \rho > 0 \), obtain an approximate solution with expected revenue \( \hat{z} \geq \frac{1}{1 + \rho} Z^* \)

• Keep grid size small

Thm: A grid with \( g(\tilde{y}^t) \leq (1 + \rho) g(\tilde{y}^{t+1}) \) \( \forall t = 1, \ldots, T - 1 \) yields desired approx. guarantee
Partitioning Intervals

- Exist $O(n)$ intervals $\{I^k : k = 1, \ldots, K\}$ partitioning $[\bar{L}, \bar{U}]$

Free Products $F^K$
- Snapped to $L_j$
- Snapped to $U_j$
Partitioning Intervals

• Associated product sets $F^k, L^k, U^k$
  partitioning $\mathbb{N}$

$\bar{L}$ $\mathcal{I}^k$ $\bar{U}$

$\mathcal{F}^k$ Free Products

$(w_j)_{j \in \mathbb{N}}$

$\mathcal{L}^k$ Snapped to $L_j$

$\mathcal{U}^k$ Snapped to $U_j$
Partitioning Intervals

\[ g(y) \text{ uniquely determined by free products on each interval} \]

\[ \tilde{L} \rightarrow \mathcal{I}_k^j \rightarrow \tilde{U} \]

\[ (\omega_j)_{j \in \mathbb{N}} \]

- \( \mathcal{F}^k \): Free Products
- \( \mathcal{L}^k \): Snapped to \( L_j \)
- \( \mathcal{U}^k \): Snapped to \( U_j \)
Partitioning Intervals

\[
g(y) = \max \left\{ \sum_{j \in \mathcal{F}^k} w_j (\kappa_j - \eta_j \log w_j) : \sum_{j \in \mathcal{F}^k} w_j \leq y - (\sum_{j \in \mathcal{L}^k} L_j + \sum_{j \in \mathcal{U}^k} U_j) \right\} \\
+ \sum_{j \in \mathcal{L}^k} u_j L_j + \sum_{j \in \mathcal{U}^k} l_j U_j
\]

\[
g(\cdot) \rightarrow (w_j)_{j \in \mathcal{N}}
\]

- \(\mathcal{F}^k\): Free Products
- \(\mathcal{L}^k\): Snapped to \(L_j\)
- \(\mathcal{U}^k\): Snapped to \(U_j\)
Grid Construction

• Intermediate points of the form

\[ \tilde{Y}^{kq} = \sum_{j \in \mathcal{L}^k} L_j + \sum_{j \in \mathcal{U}^k} U_j + (1 + \rho)^q, \quad q = \ldots, -1, 0, 1, \ldots \]

together with the interval endpoints, compose our grid.
Grid Construction

- Intermediate points of the form

\[ \tilde{Y}_{kq} = \sum_{j \in L^k} L_j + \sum_{j \in U^k} U_j + (1 + \rho)^q, q = \ldots, -1, 0, 1, \ldots \]

together with the interval endpoints, compose our grid.
Grid Properties

\[ g(y) \leq (1 + \rho) g(\tilde{Y}^{kq}) \ \forall y \in I^k \cap [\tilde{Y}^{kq}, \tilde{Y}^{k,q+1}] \]

\[ \forall q, \forall k \]
Grid Properties

\[ g(y) \leq (1 + \rho)g(\tilde{Y}^{kq}) \quad \forall y \in \mathcal{I}^k \cap [\tilde{Y}^{kq}, \tilde{Y}^{k,q+1}] \quad \forall q, \forall k \]

• Our grid satisfies the properties needed for the desired approximation guarantee
Grid Properties

\[ g(y) \leq (1 + \rho)g(\tilde{Y}^{k,q}) \forall y \in I^k \cap [\tilde{Y}^{k,q}, \tilde{Y}^{k,q+1}] \]

• Our grid satisfies the properties needed for the desired approximation guarantee

# of grid points:

\[ O(n + n \log(n \max_j U_j / \min_j L_j) / \log(1 + \rho)) \]
Multiple Product Categories

**Product Categories**

- $\gamma_i \in [0, 1]
- w_{ij}
- w_{in}
- $w_{11}$

**Competitor Space**

- $w_0 = 1$

- $m$ categories, $M = \{1, \ldots, m\}$
Multiple Product Categories

- Probability that customer buys product j, given customer chooses category i:

$$ \frac{w_{ij}}{\sum_{k \in N} w_{ik}} $$
Multiple Product Categories

Product Categories

$\gamma_i \in [0, 1]$  

$w_{i1}$  

$w_{ij}$  

$w_{in}$

Competitor Space

$w_0 = 1$

$$\frac{w_{ij}}{\sum_{k \in N} w_{ik}}$$
• Expected revenue given customer buys from category $i$:

$$R_i(w_i) = \sum_{j \in N} (\kappa_{ij} - \eta_{ij} \log w_{ij}) \frac{w_{ij}}{\sum_{k \in N} w_{ik}}$$
Customer buys from category $i$ with probability

$$Q_i(w_1, \ldots, w_m) = \frac{(\sum_{j \in N} w_{ij})^{\gamma_i}}{1 + \sum_{l \in M} (\sum_{j \in N} w_{lj})^{\gamma_l}}$$
Multiple Product Categories

Product Categories

\[ \gamma_i \in [0, 1] \]

\[ w_{ij} \]

\[ w_{in} \]

Competitor Space

\[ w_0 = 1 \]

• Expected revenue:

\[ \Pi(w_1, \ldots, w_m) = \sum_{i \in M} Q_i(w_1, \ldots, w_m) R_i(w_i) \]
Multiple Product Categories

Product Categories

\[ \gamma_i \in [0, 1] \]

\[ w_i \]

\[ w_{ij} \]

\[ w_{in} \]

Competitor Space

\[ w_0 = 1 \]

- Expected revenue:

\[ \Pi(w_1, \ldots, w_m) = \sum_{i \in M} Q_i(w_1, \ldots, w_m) R_i(w_i) \]

\[ Z^* = \max_{(w_1, \ldots, w_m): L_i \leq w_i \leq U_i, i \in M} \Pi(w_1, \ldots, w_m) \]
General Fixed Point Problem

$Z^*$ is the solution to

\[
z = \sum_{i \in M} \max_{y_i \in [\bar{L}_i, \bar{U}_i]} \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\}
\]
General Approximate Problem

\[ z = \sum_{i \in M} y_i \in \{ \tilde{y}_i^t : t = 1, \ldots, T_i \} \max \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\} \]

\[ \min \quad z \]

\[ \text{s.t.} \quad z \geq \sum_{i \in M} y_i \in \{ \tilde{y}_i^t : t = 1, \ldots, T_i \} \max \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\} \]
General Approximate Problem

\[ z = \sum_{i \in M} \max_{y_i \in \{\tilde{y}_i^t : t = 1, \ldots, T_i\}} \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\} \]

\[
\min \ z \\
\text{s.t. } z \geq \sum_{i \in M} \max_{y_i \in \{\tilde{y}_i^t : t = 1, \ldots, T_i\}} \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\}
\]

\[ x_i \geq (\tilde{y}_i^t)^{\gamma_i} \frac{g_i(\tilde{y}_i^t)}{\tilde{y}_i^t} - (\tilde{y}_i^t)^{\gamma_i} z, \quad \forall t \in \{1, \ldots, T_i\}, \forall i \in M \]
General Approximate Problem

\[ z = \sum_{i \in M} \max_{y_i \in \{\tilde{y}_i^t : t = 1, \ldots, T_i \}} \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\} \]

\[ \min z \]

s.t. \[ z \geq \sum_{i \in M} \max_{y_i \in \{\tilde{y}_i^t : t = 1, \ldots, T_i \}} \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\} \]

\[ x_i \geq (\tilde{y}_i^t)^{\gamma_i} \frac{g_i(\tilde{y}_i^t)}{\tilde{y}_i^t} - (\tilde{y}_i^t)^{\gamma_i} z, \quad \forall t \in \{1, \ldots, T_i \}, \forall i \in M \]
General Approximate Problem

\[ z = \sum_{i \in M} \max_{y_i \in \{ \bar{y}_i^t : t = 1, \ldots, T_i \}} \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\} \]
General Approximate Problem

$$z = \sum_{i \in M} \max_{y_i \in \{\tilde{y}_i^t : t = 1, \ldots, T_i\}} \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\}$$

- The approximate problem is a linear program

$$\min_{(x,z)} z$$

s.t. 

$$z \geq \sum_{i \in M} x_i$$

$$x_i \geq (\tilde{y}_i^t)^{\gamma_i} \frac{g_i(\tilde{y}_i^t)}{\tilde{y}_i^t} - (\tilde{y}_i^t)^{\gamma_i} z, \quad \forall t \in \{1, \ldots, T_i\}, \forall i \in M.$$
General Approximate Problem

\[ z = \sum_{i \in M} y_i \in \{ \tilde{y}_i^t : t = 1, \ldots, T_i \} \max \left\{ y_i^\gamma_i g_i(y_i) \frac{g_i(y_i)}{y_i} - y_i^\gamma_i z \right\} \]

- The approximate problem is a linear program

\[
\begin{align*}
\min_{(x, z)} & \quad z \\
\text{s.t.} & \quad z \geq \sum_{i \in M} x_i \\
& \quad x_i \geq (\tilde{y}_i^t)^\gamma_i g_i(\tilde{y}_i^t) - (\tilde{y}_i^t)^\gamma_i z, \quad \forall t \in \{1, \ldots, T_i\}, \forall i \in M.
\end{align*}
\]

- We apply the same method of grid construction
General Approximate Problem

\[
z = \sum_{i \in M} y_i \in \{ \tilde{y}_i^t : t = 1, \ldots, T_i \} \max \left\{ y_i \frac{g_i(y_i)}{y_i} - y_i \gamma_i z \right\}
\]

• The approximate problem is a linear program

\[
\min_{(x, z)} z
\]

s.t. \[
z \geq \sum_{i \in M} x_i
\]

\[
x_i \geq (\tilde{y}_i^t) \gamma_i \frac{g_i(\tilde{y}_i^t)}{\tilde{y}_i^t} - (\tilde{y}_i^t) \gamma_i z, \quad \forall t \in \{1, \ldots, T_i\}, \forall i \in M.
\]

• We apply the same method of grid construction

\[
\sigma = \max_{i \in M} \left\{ \frac{\max_j U_{i,j}}{\min_j L_{i,j}} \right\}
\]
General Approximate Problem

\[
    z = \sum_{i \in M} \max_{y_i \in \{\tilde{y}_i^t : t = 1, \ldots, T_i\}} \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\}
\]

• The approximate problem is a linear program

\[
    \min_{(x,z)} \quad z \\
    \text{s.t.} \quad z \geq \sum_{i \in M} x_i \\
    x_i \geq (\tilde{y}_i^t)^{\gamma_i} \frac{g_i(\tilde{y}_i^t)}{\tilde{y}_i^t} - (\tilde{y}_i^t)^{\gamma_i} z, \quad \forall t \in \{1, \ldots, T_i\}, \forall i \in M.
\]

• We apply the same method of grid construction
General Approximate Problem

\[ z = \sum_{i \in M} \max_{y_i \in \{\tilde{y}_i^t : t = 1, \ldots, T_i\}} \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\} \]

• The approximate problem is a linear program

\[
\begin{align*}
\min_{(x, z)} & \quad z \\
\text{s.t.} & \quad z \geq \sum_{i \in M} x_i \\
& \quad x_i \geq (\tilde{y}_i^t)^{\gamma_i} \frac{g_i(\tilde{y}_i^t)}{\tilde{y}_i^t} - (\tilde{y}_i^t)^{\gamma_i} z, \quad \forall t \in \{1, \ldots, T_i\}, \forall i \in M.
\end{align*}
\]

• We apply the same method of grid construction

LP has \( O(mn + mn \log(n\sigma)/\log(1 + \rho)) \) constraints, \( m + 1 \) variables
Upper Bound LP

• Given any set of grid points \( \{\hat{y}_i^d : d = 1, \ldots, D_i\} \) for each \( i \in M \), solve

\[
\min_{(x,z)} z \\
\text{s.t.} \quad z \geq \sum_{i \in M} x_i \\
x_i \geq (y_i^d)^{g_i} g_i \frac{(\hat{y}_i^d+1)}{\hat{y}_i^d} - (y_i^d)^{\gamma_i} z, \quad \forall d \in \{1, \ldots, D_i - 1\}, \forall i \in M.
\]

• Allows comparison of the performance of our algorithm, other heuristics, etc.
Algorithm Performance

Avg. % optimality gap vs. max. U/L ratio

Max. % optimality gap vs. max. U/L ratio
Thank you