Approximation Methods for Pricing Problems under the Nested Logit Model with Price Bounds

W. Zachary Rayfield  Paat Rusmevichientong  Huseyin Topaloglu
School of ORIE, Marshall School of Business, School of ORIE,
Cornell University  University of Southern California  Cornell University
Pricing under the Nested Logit Model

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Pricing under the Nested Logit Model
Pricing under the Nested Logit Model
Problem Formulation

Store

Competitor Space
Problem Formulation

- $n$ products, $N = \{1, \ldots, n\}$
Problem Formulation

- n products, \( N = \{1, \ldots, n\} \)
- Preference weight of product \( j \): \( w_j \)
Problem Formulation

- $n$ products, $N = \{1, \ldots, n\}$
- Preference weight of product $j$: $w_j$
- Total preference weight of competitor space normalized to 1
Problem Formulation

- Price of product $j$:
Problem Formulation

Store

• Price of product $j$: $p_j$

Competitor Space

$w_0 = 1$
Problem Formulation

Store

Competitor Space

\( w_0 = 1 \)

• Price of product \( j \): \( p_j \)

\[
w_j = e^{\alpha_j - \beta_j p_j}
\]
Problem Formulation

- Price of product $j$: $p_j$

\[ w_j = e^{\alpha_j p_j - \beta_j p_j} \]

- Prices are constrained:
Problem Formulation

• Price of product \( j \): \( p_j \)

\[
w_j = e^{\alpha_j - \beta_j p_j}
\]

• Prices are constrained: \( l_j \leq p_j \leq u_j \)
Problem Formulation

Store

Competitor Space

\( w_0 = 1 \)

- Price of product \( j \): \( p_j \)
  \[
  w_j = e^{\alpha_j - \beta_j p_j} \quad \Rightarrow \quad p_j(w_j) = \frac{\alpha_j}{\beta_j} - \frac{1}{\beta_j} \log w_j
  \]

- Prices are constrained: \( l_j \leq p_j \leq u_j \)
Problem Formulation

Store

- **Price of product j**: $p_j$
  
  \[ w_j = e^{\alpha_j - \beta_j p_j} \]

  \[ p_j(w_j) = \frac{\alpha_j}{\beta_j} - \frac{1}{\beta_j} \log w_j \]

  \[ L_j \leq w_j \leq U_j \]

Competitor Space

\[ w_0 = 1 \]
Problem Formulation

Store

- Price of product $j$: $p_j$
  
  $$w_j = e^{\alpha_j - \beta_j p_j} \quad \Rightarrow \quad p_j(w_j) = \frac{\alpha_j}{\beta_j} - \frac{1}{\beta_j} \log w_j$$

- Prices are constrained: $L_j \leq w_j \leq U_j$

- If customer buys from our firm, product $j$ purchased with probability
  
  $$\frac{w_j}{\sum_{k \in N} w_k}$$

Competitor Space

- $w_0 = 1$
Problem Formulation

• Price of product $j$: $p_j$
  
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• Prices are constrained: $L_j \leq w_j \leq U_j$

• Expected revenue | customer buys from our firm:

$$R(w) = \sum_{j \in N} p_j(w_j) \frac{w_j}{\sum_{k \in N} w_k}$$

$w_0 = 1$
Problem Formulation

• Price of product $j$: $p_j$

$$w_j = e^{\alpha_j - \beta_j p_j} \quad \Rightarrow \quad p_j(w_j) = \frac{\alpha_j}{\beta_j} - \frac{1}{\beta_j} \log w_j$$

• Prices are constrained: $L_j \leq w_j \leq U_j$

• Expected revenue | customer buys from our firm:

$$R(w) = \sum_{j \in N} p_j(w_j) \frac{w_j}{V(w)}$$
Problem Formulation

Store

\[ w_1 \]

\[ w_j \]

\[ w_n \]

Competitor Space

\[ w_0 = 1 \]
Problem Formulation

Store

\[ V(w) = \sum_{j \in N} w_j \]

Competitor Space

\[ w_0 = 1 \]
Problem Formulation

Store

\[ V(w) = \sum_{j \in N} w_j \]

• Customer buys from our firm with probability

\[ \frac{V(w)^\gamma}{1 + V(w)^\gamma} \]

Competitor Space

\[ w_0 = 1 \]

\[ \gamma \in [0, 1] \]
Problem Formulation

\[ V(\mathbf{w}) = \sum_{j \in N} w_j \]

\[ \gamma \in [0, 1] \]

\[ w_0 = 1 \]

\[ \text{Customer buys from our firm with probability } \frac{V(\mathbf{w})^\gamma}{1 + V(\mathbf{w})^\gamma} \]

Expected Revenue: \[ \Pi(\mathbf{w}) = \frac{V(\mathbf{w})^\gamma}{1 + V(\mathbf{w})^\gamma} R(\mathbf{w}) \]
Problem Formulation

\[ V(w) = \sum_{j \in N} w_j \]

- Customer buys from our firm with probability \( \frac{V(w)^\gamma}{1 + V(w)^\gamma} \)

Expected Revenue: \( \Pi(w) = \frac{V(w)^\gamma}{1 + V(w)^\gamma} R(w) \)

\[ Z^* = \max_{L \leq w \leq U} \Pi(w) \]
\[
Z^* \geq \Pi(w) = \frac{V(w)^\gamma}{1 + V(w)^\gamma} R(w) \quad \forall L \leq w \leq U \quad (\text{tight at opt.})
\]
Transformation of Objective

\[
Z^* \geq \Pi(w) = \frac{V(w)^\gamma}{1 + V(w)^\gamma} R(w) \quad \forall L \leq w \leq U \quad \text{(tight at opt.)}
\]

\[
Z^*(1 + V(w)^\gamma) \geq V(w)^\gamma R(w)
\]
Transformation of Objective

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\[ Z^*(1 + V(w)^\gamma) \geq V(w)^\gamma R(w) \]

\[ Z^* \geq V(w)^\gamma (R(w) - Z^*) \]
Transformation of Objective

\[ Z^* \geq \Pi(w) = \frac{V(w)\gamma}{1 + V(w)\gamma} R(w) \quad \forall L \leq w \leq U \quad \text{(tight at opt.)} \]

\[ Z^*(1 + V(w)\gamma) \geq V(w)\gamma R(w) \]

\[ Z^* \geq V(w)\gamma (R(w) - Z^*) \]

\[ Z^* = \max_{L \leq w \leq U} V(w)\gamma (R(w) - Z^*) \]
$Z^* \geq \Pi(w) = \frac{V(w)^\gamma}{1 + V(w)^\gamma} R(w) \quad \forall L \leq w \leq U$  (tight at opt.)

$Z^*(1 + V(w)^\gamma) \geq V(w)^\gamma R(w)$

$Z^* \geq V(w)^\gamma (R(w) - Z^*)$

$Z^* = \max_{L \leq w \leq U} V(w)^\gamma (R(w) - Z^*)$

**Use single decision variable to capture** $V(w)$
Transformation of Objective

\[ Z^* \geq \Pi(w) = \frac{V(w)^\gamma}{1 + V(w)^\gamma} R(w) \quad \forall L \leq w \leq U \quad \text{(tight at opt.)} \]

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\[ Z^* \geq V(w)^\gamma (R(w) - Z^*) \]

\[ Z^* = \max_{L \leq w \leq U} V(w)^\gamma (R(w) - Z^*) \]

**Use single decision variable to capture** \( V(w) \)

\[ R(w) = \frac{\sum_{j \in N} p_j(w_j)w_j}{V(w)} \]
Transformation of Objective

\[ Z^* \geq \Pi(w) = \frac{V(w)^\gamma}{1 + V(w)^\gamma} R(w) \quad \forall L \leq w \leq U \quad \text{(tight at opt.)} \]

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\[ Z^*(1 + V(w)\gamma) \geq V(w)\gamma R(w) \]

\[ Z^* \geq V(w)\gamma(R(w) - Z^*) \]

\[ Z^* = \max_{L \leq w \leq U} \left( V(w)\gamma(R(w) - Z^*) \right) \]

Use single decision variable to capture \( V(w) \)

\[ R(w) = \frac{\sum_{j \in N} p_j(w_j)w_j}{V(w)} \]

\[ Z^* = \max_{y \in \mathbb{R}} \left\{ y^\gamma \left( \frac{y - Z^*}{y} \right) \right\} \]
Transformation of Objective

\[ Z^* = \max_{L \leq w \leq U} (V(w)^\gamma (R(w) - Z^*)) \]

Use single decision variable to capture \( V(w) \)

\[ R(w) = \frac{\sum_{j \in N} p_j(w_j)w_j}{V(w)} \]

\[ Z^* = \max_{y \in ?} \left\{ y^\gamma \left( \frac{?}{y} - Z^* \right) \right\} \]
Capture numerator with

\[ g(y) = \max \left\{ \sum_{j \in N} p_j(w_j)w_j : \sum_{j \in N} w_j \leq y, \; w_j \in [L_j, U_j] \forall j \in N \right\} \]

\[ Z^* = \max_{L \leq w \leq U} (V(w)^\gamma (R(w) - Z^*)) \]

Use single decision variable to capture \( V(w) \)

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(Concave, non-linear knapsack problem)

\[ Z^* = \max_{L \leq w \leq U} \left( V(w) \gamma (R(w) - Z^*) \right) \]

Use single decision variable to capture \( V(w) \)

\[ R(w) = \frac{\sum_{j \in N} p_j(w_j)w_j}{V(w)} \]

\[ Z^* = \max_{y \in ?} \left\{ y^\gamma \left( \frac{?}{y} - Z^* \right) \right\} \]
Transformation of Objective

Capture numerator with

$$g(y) = \max \left\{ \sum_{j \in N} p_j(w_j)w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\}$$

(Concave, non-linear knapsack problem)

Restrict attention to $[\bar{L}, \bar{U}]$

$$Z^* = \max_{y \in \gamma} \left\{ y \gamma \left( \frac{?}{y} - Z^* \right) \right\}$$
Transformation of Objective

Capture numerator with

\[ g(y) = \max \left\{ \sum_{j \in N} p_j(w_j)w_j : \sum_{j \in N} w_j \leq y, \ w_j \in [L_j, U_j] \ \forall j \in N \right\} \]

(Concave, non-linear knapsack problem)

Restrict attention to \([\bar{L}, \bar{U}]\)

Fixed-point relationship:

\[ Z^* = \max_{y \in [\bar{L}, \bar{U}]} \left\{ y^\gamma \left( \frac{g(y)}{y} - Z^* \right) \right\} \]
Recovering Optimal Prices

If $Z^*$ known:

Recover the optimal solution by solving

$$g(y) = \max \left\{ \sum_{j \in N} p_j(w_j)w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\}$$

$$y^\gamma \frac{g(y)}{y} - y^\gamma Z^*$$
Recovering Optimal Prices

If \( Z^* \) known:

Recover the optimal solution by solving

\[
\max_{y \in [\bar{L}, \bar{U}]} \left\{ y^{\gamma} \frac{g(y)}{y} - y^{\gamma} Z^* \right\}
\]

\[
g(y) = \max \left\{ \sum_{j \in N} p_j(w_j) w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\}
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Recovering Optimal Prices

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$$

$$
g(y) = \max \left\{ \sum_{j \in N} p_j(w_j)w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\}
$$
Recovering Optimal Prices

$$y \xrightarrow{g(\cdot)} w_1(y) \xrightarrow{\vdots} w_n(y)$$

If $Z^*$ known:

Recover the optimal solution by solving

$$\max_{y \in [\bar{L}, \bar{U}]} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma Z^* \right\}$$

$$g(y) = \max \left\{ \sum_{j \in N} p_j(w_j)w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\}$$
Recovering Optimal Prices

If \( Z^* \) known:

Recover the optimal solution by solving

\[
\max_{y \in [\bar{L}, \bar{U}]} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma Z^* \right\}
\]

where

\[
g(y) = \max \left\{ \sum_{j \in N} p_j(w_j)w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\}
\]
To find $Z^*$, solve

$$z = \max_{y \in [L, U]} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma z \right\}$$
To find $Z^*$, solve

$$
z = \max_{y \in [\bar{L}, \bar{U}]} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma z \right\}
$$

Problem: RHS function not necessarily concave in $y$
To find $Z^*$, solve

$$z = \max_{y \in [\bar{L}, \bar{U}]} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma z \right\}$$

Problem: RHS function not necessarily concave in $y$

Ex:
Approximation Framework

• Construct a set of grid points \( \{\tilde{y}^t : t = 1, \ldots, T \} \), solve

\[
z = \max_{y \in \{\tilde{y}^t : t = 1, \ldots, T \}} \left\{ y\gamma \frac{g(y)}{y} - y\gamma z \right\}
\]
Approximation Framework

• Construct a set of grid points $\{\tilde{y}^t : t = 1, \ldots, T\}$, solve

$$z = \max_{y \in \{\tilde{y}^t : t = 1, \ldots, T\}} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma z \right\}$$

• Objective: given $\rho > 0$, obtain an approximate solution $\hat{z}$ (w/corresponding prices):
Approximation Framework

• Construct a set of grid points \( \{ \tilde{y}^t : t = 1, \ldots, T \} \), solve

\[
z = \max_{y \in \{ \tilde{y}^t : t = 1, \ldots, T \}} \left\{ \gamma \frac{g(y)}{y} - y'z \right\}
\]

• Objective: given \( \rho > 0 \), obtain an approximate solution \( \hat{z} \) (w/corresponding prices):

Want \( \hat{z} \geq \frac{1}{1 + \rho} Z^* \)
Approximation Framework

• Construct a set of grid points \( \{\tilde{y}^t : t = 1, \ldots, T\} \), solve

\[
z = \max_{y \in \{\tilde{y}^t : t = 1, \ldots, T\}} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma z \right\}
\]

• Objective: given \( \rho > 0 \), obtain an approximate solution \( \hat{z} \) (w/corresponding prices):

Want \( \hat{z} \geq \frac{1}{1 + \rho} Z^* \)

• Keep grid size small
Approximation Framework

• Construct a set of grid points \( \{\tilde{y}_t : t = 1, \ldots, T\} \), solve

\[
 z = \max_{y \in \{\tilde{y}_t : t = 1, \ldots, T\}} \left\{ y^\gamma \frac{g(y)}{y} - y^\gamma z \right\}
\]

• Objective: given \( \rho > 0 \), obtain an approximate solution \( \hat{z} \) (w/corresponding prices):

Want \( \hat{z} \geq \frac{1}{1 + \rho} Z^* \)

• Keep grid size small

**Thm:** A grid \( \{\tilde{y}_t : t = 1, \ldots, T\} \) with

\[
g(\tilde{y}_t + 1) \leq (1 + \rho) g(\tilde{y}_t) \ \forall t = 1, \ldots, T - 1
\]

guarantees a worst-case expected revenue of \( \frac{1}{1 + \rho} Z^* \)
Knapsack Properties

\[ g(y) = \max \left\{ \sum_{j \in N} p_j(w_j)w_j : \sum_{j \in N} w_j \leq y, \ w_j \in [L_j, U_j] \ \forall j \in N \right\} \]

- \( g(\cdot) \) concave, increasing on \([\bar{L}, \bar{U}]\)
- Optimal decision variable values monotonically increasing with \( y \)
Knapsack Properties

\[ g(y) = \max \left\{ \sum_{j \in N} p_j(w_j)w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\} \]

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- \( g(\cdot) \) concave, increasing on \([\bar{L}, \bar{U}]\)
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![Diagram showing knapsack properties with \(L_j\), \(U_j\), \(w_1\) to \(w_n\), and \(\bar{L}\)]
Knapsack Properties

\[ g(y) = \max \left\{ \sum_{j \in N} p_j(w_j) w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\} \]

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- Optimal decision variable values monotonically increasing with \( y \)

\( \bar{L} \)

\( \bar{U} \)

\( \bar{L} \)

\( \bar{U} \)
Knapsack Properties

\[ g(y) = \max \left\{ \sum_{j \in N} p_j(w_j)w_j : \sum_{j \in N} w_j \leq y, \ w_j \in [L_j, U_j] \ \forall j \in N \right\} \]

- \( g(\cdot) \) concave, increasing on \([\bar{L}, \bar{U}]\)
- Optimal decision variable values monotonically increasing with \(y\)

\[ U_j \]
\[ FREE \]
\[ L_j \]
\[ \bar{L} \]
Knapsack Properties

\[ g(y) = \max \left\{ \sum_{j \in N} p_j(w_j)w_j : \sum_{j \in N} w_j \leq y, \ w_j \in [L_j, U_j] \ \forall j \in N \right\} \]

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We can compute these points explicitly...
Knapsack Properties

\[ g(y) = \max \left\{ \sum_{j \in N} p_j(w_j)w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\} \]

- \( g(\cdot) \) concave, increasing on \([\bar{L}, \bar{U}]\)
- Optimal decision variable values monotonically increasing with \( y \)

We can compute these points explicitly . . . . . . . . .
\# of points = \( O(n) \)
\[ g(y) = \max \left\{ \sum_{j \in N} p_j(w_j)w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\} \]
\[
g(y) = \max \left\{ \sum_{j \in N} p_j(w_j)w_j : \sum_{j \in N} w_j \leq y, w_j \in [L_j, U_j] \forall j \in N \right\}
\]
Grid Points

Intermediate points \( \tilde{Y}^{kq} = \sum_{j \in \mathcal{L}^k} L_j + \sum_{j \in \mathcal{U}^k} U_j + (1 + \rho)^q, \, q \in \mathbb{Z} \)

\[
g(y) = \max \left\{ \sum_{j \in N} p_j(w_j)w_j : \sum_{j \in N} w_j \leq y, \, w_j \in [L_j, U_j] \forall j \in N \right\}
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Intermediate points \( \tilde{Y}^{kq} = \sum_{j \in \mathcal{L}^k} L_j + \sum_{j \in \mathcal{U}^k} U_j + (1 + \rho)^q, \ q \in \mathbb{Z} \)

Free product capacity increments exponentially
Intermediate points
\[ \tilde{Y}^{kq} = \sum_{j \in \mathcal{L}_k} L_j + \sum_{j \in \mathcal{U}_k} U_j + (1 + \rho)^q, \quad q \in \mathbb{Z} \]

Free product capacity increments exponentially
\[ g(\tilde{Y}^{k,q+1}) \leq (1 + \rho)g(\tilde{Y}^{kq}) \forall q \in \mathbb{Z} \]

Grid Points

\[ g(y) = \max \left\{ \sum_{j \in N} p_j(w_j)w_j : \sum_{j \in N} w_j \leq y, \ w_j \in [L_j, U_j] \forall j \in N \right\} \]
Intermediate points \( \tilde{Y}^{kq} = \sum_{j \in L^k} L_j + \sum_{j \in U^k} U_j + (1 + \rho)^q, \quad q \in \mathbb{Z} \)

Free product capacity increments exponentially

\[ g(\tilde{Y}^{k,q+1}) \leq (1 + \rho) g(\tilde{Y}^{kq}) \quad \forall q \in \mathbb{Z} \]

(Grid satisfies theorem conditions)
Multiple Product Categories

Product Categories

\[ \gamma_i \in [0, 1] \]

\[ w_{ij} \]

\[ w_{in} \]

Competitor Space

\[ w_0 = 1 \]
• m categories, $M = \{1, \ldots, m\}$
• m categories, $M = \{1, \ldots, m\}$
• Probability that customer buys product $j$ | customer chooses category $i$:

$$\frac{w_{ij}}{V_i(w_i)}$$
• $m$ categories, $M = \{1, \ldots, m\}$
• Probability that customer buys product $j$ | customer chooses category $i$: $w_{ij} V_i(w_i)$
• Expected revenue | customer buys from category $i$:

$$R_i(w_i) = \sum_{j \in N} p_{ij}(w_{ij}) \frac{w_{ij}}{V_i(w_i)}$$
Multiple Product Categories

- Customer buys from category $i$ with probability
  \[ \frac{V_i(w_i)^{\gamma_i}}{1 + \sum_{l \in M} V_l(w_l)^{\gamma_l}} \]

Expected Revenue:
\[ \Pi(w_1, \ldots, w_m) = \sum_{i \in M} \frac{V_i(w_i)^{\gamma_i}}{1 + \sum_{l \in M} V_l(w_l)^{\gamma_l}} R_i(w_i) \]

\[ Z^* = \max_{(w_1, \ldots, w_m): L_i \leq w_i \leq U_i, i \in M} \Pi(w_1, \ldots, w_m) \]
General Approximate Problem

\[ z = \sum_{i \in M} \max_{y_i \in \{ \tilde{y}_i^t : t = 1, \ldots, T_i \}} \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\} \]

\text{min } \quad z

\text{s.t. } \quad z \geq \sum_{i \in M} \max_{y_i \in \{ \tilde{y}_i^t : t = 1, \ldots, T_i \}} \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\} \]
General Approximate Problem

\[
z = \sum_{i \in M} y_i \in \{ \tilde{y}_{i}^{t} : t = 1, \ldots, T_{i} \} \max \left\{ y_{i}^{\gamma_{i}} \frac{g_i(y_i)}{y_i} - y_{i}^{\gamma_{i}} z \right\}
\]

\[
\begin{align*}
\min & \quad z \\
\text{s.t.} & \quad z \geq \sum_{i \in M} y_i \in \{ \tilde{y}_{i}^{t} : t = 1, \ldots, T_{i} \} \max \left\{ y_{i}^{\gamma_{i}} \frac{g_i(y_i)}{y_i} - y_{i}^{\gamma_{i}} z \right\} \\
& \quad x_i \geq (\tilde{y}_{i}^{t})^{\gamma_{i}} \frac{g_i(\tilde{y}_{i}^{t})}{\tilde{y}_{i}^{t}} - (\tilde{y}_{i}^{t})^{\gamma_{i}} z, \quad \forall t \in \{1, \ldots, T_{i}\}, \forall i \in M
\end{align*}
\]
General Approximate Problem

\[
z = \sum_{i \in M} \max_{y_i \in \{\tilde{y}_i^t : t = 1, \ldots, T_i \}} \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\}
\]

\[
\min \quad z
\]

s.t. \[
z \geq \sum_{i \in M} \max_{y_i \in \{\tilde{y}_i^t : t = 1, \ldots, T_i \}} \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\}
\]

\[
x_i \geq (\tilde{y}_i^t)^{\gamma_i} \frac{g_i(\tilde{y}_i^t)}{\tilde{y}_i^t} - (\tilde{y}_i^t)^{\gamma_i} z, \quad \forall t \in \{1, \ldots, T_i \}, \forall i \in M
\]
General Approximate Problem

\[ z = \sum_{i \in M} \max_{y_i \in \{\tilde{y}_i^t : t = 1, \ldots, T_i\}} \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\} \]
General Approximate Problem

\[ z = \sum_{i \in M} \max_{y_i \in \{\tilde{y}_i^t : t = 1, \ldots, T_i\}} \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\} \]

• The approximate problem is a linear program

\[
\begin{align*}
\min_{(x,z)} & \quad z \\
\text{s.t.} & \quad z \geq \sum_{i \in M} x_i \\
& \quad x_i \geq (\tilde{y}_i^t)^{\gamma_i} \frac{g_i(\tilde{y}_i^t)}{\tilde{y}_i^t} - (\tilde{y}_i^t)^{\gamma_i} z, \quad \forall t \in \{1, \ldots, T_i\}, \forall i \in M.
\end{align*}
\]
General Approximate Problem

\[ z = \sum_{i \in M} \max_{y_i \in \{ \tilde{y}_i^t : t = 1, \ldots, T_i \}} \left\{ y_i^{\gamma_i} \frac{g_i(y_i)}{y_i} - y_i^{\gamma_i} z \right\} \]

• The approximate problem is a \textit{linear program}

\[
\begin{align*}
\min_{(x,z)} & \quad z \\
\text{s.t.} & \quad z \geq \sum_{i \in M} x_i \\
 & \quad x_i \geq (\tilde{y}_i^t)^{\gamma_i} \frac{g_i(\tilde{y}_i^t)}{\tilde{y}_i^t} - (\tilde{y}_i^t)^{\gamma_i} z, \quad \forall t \in \{1, \ldots, T_i\}, \forall i \in M.
\end{align*}
\]

• We apply the same method of grid construction

• LP has \(m+1\) variables, \# constraints \(\approx m \times \) (grid size)
• Given any set of grid points \( \{\overline{y}^t_i : t = 1, \ldots, \tau_i\} \) for each \( i \in M \), solve

\[
\begin{align*}
\min_{(x, z)} & \quad z \\
\text{s.t.} & \quad z \geq \sum_{i \in M} x_i \\
& \quad x_i \geq (\overline{y}^t_i)^{\gamma_i} \frac{g_i(\overline{y}^{t+1}_i)}{\overline{y}^t_i} - (\overline{y}^t_i)^{\gamma_i} z, \quad \forall t \in \{1, \ldots, \tau_i - 1\}, \forall i \in M.
\end{align*}
\]

• Allows comparison of the performance of our algorithm, other heuristics, etc.
Algorithm Performance

\[ \rho = 0.005 \]

- APP - Avg. Gap with Upper Bound
- Rounding Heuristic - Avg. Gap with Upper Bound

- 25 total products
- 100 total products
- 225 total products

S, M, L - small/medium/large degree of constraint violation
Sample Computation Times

- 25 total products / Small violation
- 225 total products / Large violation
Thank you!