The Effectiveness of Guardrails in Online Fair Allocation

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The problem of online fair allocation has received significant attention in recent years, owing to its applications in food bank operations and vaccine allocation. The canonical setting is as follows: a principal has a budget $B$ of resources to allocate over $T$ rounds. Each round sees a random number of arrivals, and the principal must commit to an allocation for these individuals before moving on to the next round. The goal is to decide per-person allocations that are ‘approximately fair’ in hindsight, i.e. have low waste and envy. Existing works propose a variety of techniques with strong guarantees in simple models, where individuals have additive preferences, and the only source of resource depletion is the algorithm’s allocations. In this work, we extend the scope of one of these techniques — allocating with algorithmic guardrails — in two orthogonal and practically motivated directions: considering (i) a wide range of utility functions, and (ii) settings with perishable resources. Our results demonstrate the effectiveness of allocating with guardrails, and moreover give insight into the impact of different modeling choices for this problem.

1 INTRODUCTION

This work is motivated by a class of problems faced many non-profit organizations: the fair online allocation of goods under uncertainty. One concrete and well-studied motivating application of this class of problems is mobile food pantry operations run by food banks. These programs are set up as follows. Each day, a volunteer drives a truck filled with supplies from the food bank to various towns, and sets up temporary distribution sites. At a given distribution site, the operator must decide how much food to distribute, without knowing the demand in future locations. A key challenge in doing so is determining allocations that ensure equity across individuals despite uncertainty in the future demand, all the while maximizing the utilization of available resources, i.e., efficiency.

This classical resource allocation problem is seen in broader stockpile allocation settings and is well-studied in economics, computer science, and operations research [25, 26]. In particular, Sinclair et al. [32] studied this problem for the specific setting in which individuals have additive utilities across a set of goods that is fixed across time, and characterized a fundamental trade-off between envy — the maximum amount by which any individual prefers another’s allocation to their own — and efficiency. They then designed a simple allocation mechanism based on the use of algorithmic guardrails — high-probability upper and lower bounds on the unique fair solution in hindsight — that achieves any desired point along the identified envy-efficiency Pareto frontier.

While these results provide a promising first step toward understanding the trade-offs between various operational considerations, they fail to capture two important realities of food bank operations. First, the assumption that agents’ preferences over resources are additive is limiting, in that it does not account for settings in which resources exhibit complementarities (modeled via e.g., Leontief, or filling, utilities), in addition to omitting popular utility functions in the extant literature (e.g., Cobb-Douglas utilities). Second, though the assumption that the set of resources is fixed throughout the time horizon may be suitable in many settings, it is often the case — especially within the context of food pantries — that resources experience exogenous depletion over time, e.g., due to perishing. This paper is motivated by the goal of designing fair online algorithms for these more realistic settings, and in particular understanding the robustness of the seemingly powerful guardrail allocation framework within such settings. We detail our two-fold contribution in this regard below.
1.1 Our Contributions

We consider a model in which, over $T$ discrete time periods, a decision-maker must decide how many of $B$ goods to allocate to a stream of online arrivals, drawn from a known distribution. Individuals are characterized by a given type, each associated with a utility function. The goal is to find an allocation that trades off between two metrics: envy (which serves as a measure of (un)fairness), and waste (which serves as a measure of (in)efficiency).

In our first contribution, we extend the results of Sinclair et al. [32] by considering the class of utility functions that are concave, one-positively homogeneous, and non-decreasing. This class of functions subsumes popular models of utilities such as additive, Leontief, and Cobb-Douglas utility functions. For such functions, we show that the guardrail algorithm developed by Sinclair et al. [32], coined GUARDED-HOPE, is asymptotically optimal, in its ability to achieve every point on the envy-efficiency Pareto frontier.

Informal Theorem 1 (General Utility Functions, see Theorem 4.3). For any choice of $L_T$ and under a wide range of utility function models, with high probability GUARDED-HOPE with parameter $L_T$ achieves:

$$\text{Envy} \leq L_T, \quad \Delta\text{efficiency} \lesssim \min\{\sqrt{T}, 1/L_T\},$$

where $\text{Envy}$ and $\Delta\text{efficiency}$ respectively correspond to the envy and waste incurred by the algorithm.

The key difficulty in establishing this result is in deriving sharp bounds on the sensitivity of the Eisenberg-Gale (EG) program [16] upon which the guardrail construction relies. In particular, showing that solutions to this program are robust to mild perturbations in the arrival process relies on a property known as competitive monotonicity, previously known only for the setting of weak gross substitutes [23].

In our second main contribution, we consider a setting in which the budget of resources, rather than being fixed over time, is exogenously depleted according to a known perishing distribution for each unit of good. In this setting, there is the additional challenge that the algorithm not only needs to decide how much to give out to individuals in a round $t$, but also which of the resources. This timing aspect of the allocation is the key point of departure from the previous model: since items can no longer be allocated after they perish, the algorithm must judiciously choose between the remaining units in each period and allocate those that are likely to perish in subsequent periods in order to minimize waste. Our main result in this regard is to show that, for “offset-expiring” processes (an assumption we make formal in Section 5), a cautious algorithm that allocates the lower guardrail in each period, and allocates items in increasing order of expected perishing times, achieves optimal envy and efficiency. For this algorithm, we have the following main result.

Informal Theorem 2 (Perishing Resources, see Theorem 5.2). For a wide range of perishing processes, with high probability GUARDED-HOPE achieves:

$$\text{Envy} = 0, \quad \Delta\text{efficiency} \lesssim \sqrt{T}.$$

This result says that perishing resources do not fundamentally affect the envy-efficiency trade-off. The intuition behind this is that, when resources perish earlier than expected, they adversely affect both envy and waste; in contrast, uncertainty in the number of arrivals affects the two in opposite directions, hence leading to a trade-off. For clarity of exposition, we formally establish this for one point of the envy-efficiency trade-off curve. However, we believe that, with appropriate assumptions on the perishing process, this insight extends to the entire trade-off curve.

A more significant contribution from a modeling perspective is that the class of perishing processes we consider extends far past current state-of-the-art models of online resource allocation.
of perishable resources, which are limited to known lifetimes for goods, or i.i.d. perishing times. While the former is naturally handled by Sinclair et al. [32], we argue that the latter setting is unreasonably pessimistic, in that achieving low waste is hopeless. Moreover, in most practical settings, the principal has some practical idea of which items are more at risk of perishing early on — for example, food banks can use the expiry date of canned goods, or the ripeness of fruits and vegetables to determine which are at most risk of spoiling. Our offset-expiring model provides a formalism for this notion.

Overall, our results highlight the use of algorithmic guardrails as an extremely powerful tool for fair resource allocation in online settings. Note that both our algorithms are the natural extension of the original GUARDED-HOPE algorithm of Sinclair et al. [32] to these settings. The technical novelty lies in showing these extensions preserve the trade-off curve under mild conditions. However, the effectiveness of this algorithmic framework in providing guarantees in new settings, as well as its strong performance in experiments, give hope that its scope is far-reaching.

1.2 Paper Organization

We next survey related work in Section 2. In Section 3, we present the general online resource allocation problem, as well as introducing the fairness notions of interest. We then consider the setting of general utilities without perishing in Section 4, establishing that the guardrail algorithm developed in Sinclair et al. [32] achieves the optimal envy-efficiency trade-off for this wider range of utility functions. In Section 5, we design and analyze a separate guardrail-based algorithm for the setting with perishable goods. We conclude by comparing the numerical performance of our algorithms against state-of-the-art benchmarks in Section 6, using data from a collaborating food bank.

2 RELATED WORK

Fairness in resource allocation has a long history in the economics and computation literature, beginning with Varian’s seminal work [33, 34]. We highlight the most closely related works below, especially as they relate to online fair allocation; see [3] for a comprehensive survey.

Fair allocation without perishable resources. We first detail various models and objectives considered in settings without perishable goods. There exists a long line of work in which the resource becomes available to the decision-maker online, whereas agents are fixed [1, 2, 7–9, 11, 13, 22, 27, 28, 36]. These models lie in contrast to the one we consider, wherein resources are fixed and individuals arrive online. Papers that consider this latter setting include Kalinowski et al. [24], who consider maximizing utilitarian welfare with indivisible goods, rather than focusing on fairness guarantees with divisible goods. Gerding et al. [21] consider a scheduling setting wherein agents have fixed and known arrival and departure times, as well as demand for the resource. A series of papers also consider the problem of fair division with minimal disruptions relative to previous allocations, as measured by a fairness ratio, a competitive ratio analog of counterfactual envy in our setting [14, 19, 20]. A number of papers also seek to design algorithms with attractive competitive ratios with respect to the Nash Social Welfare objective [6, 8], or the max-min objective [25, 26].

The above papers situate themselves within the adversarial, or worst-case, tradition. A separate line of work considers fair resource allocation in stochastic settings [15, 17], as we do. The algorithms developed in these papers, however, are non-adaptive: they decide on the entire allocation upfront, before observing any of the realized demand. In contrast, we consider a model where the principal makes the allocation decision in each round after observing the number of arrivals. Freeman et al. [18] consider a problem in which agents’ utilities are realized from an unknown distribution, and the budget resets in each round. They present algorithms for Nash social welfare maximization.
and discuss some of their properties. Our work is most closely related (and indeed, builds upon) Sinclair et al. [32], who first introduced the envy-freeness and efficiency tradeoff we are interested in. We improve upon their results in showing the applicability of the GUARDED-HOPE framework to a broader class of utilities, which subsumes additive utilities that they considered in their original paper. Moreover, we consider a model in which goods also perish over time, which none of the aforementioned works consider.

**Perishable resources.** Though online resource allocation of perishable goods has a long history in the operations research literature (see, e.g., Nahmias [29] for a comprehensive survey of earlier literature), the best of our knowledge, the question of fairly allocating perishable goods has attracted relatively little attention. Motivated by the problem of electric vehicle charging, Gerding et al. [21] consider an online scheduling problem where agents arrive and compete for a perishable resource which spoils at the end of every period, and as a result must be allocated at every time step. They consider a range of objectives, including: maximum total resource allocated, maximum number of satisfied agents, as well as envy-freeness. Bateni et al. [10] similarly consider a setting wherein goods perish immediately. Our paper, in contrast, considers stochastic perishing over the course of multiple rounds. Alijani et al. [4] similarly consider a setting with stochastic perishing; in the problem they consider, a principal seeks to sell perishable items to a stream of buyers in order to maximize social welfare. They show that, when items have independent perishing times satisfying the monotone hazard rate condition, the competitive ratio of any policy is lower bounded by a constant greater than one. This negative result is in line with our discussion regarding i.i.d. perishing times representing the worst-case for the decision-maker. Also related is recent work on stochastic matching with unknown arrivals and abandonments. For instance, Aouad and Saritaç [5] design constant-factor approximations for a setting where agents of different types arrive according to a Poisson process and abandon the system once their exponentially distributed sojourn time elapses, and the principal seeks to maximize cumulative rewards, or minimize cumulative costs.

3 PRELIMINARIES

In this section, we state the most general form of the model, relegating specifics for the applications we consider to subsequent sections.

**Notation.** We use $\mathbb{R}_+$ to denote the set of non-negative reals, $\|X\|_{\infty} = \max_{i,j} |X_{i,j}|$ to denote the matrix maximum norm, and $cX$ to denote entry-wise multiplication for a constant $c$. When comparing vectors, we use $X \leq Y$ to denote that each component $X_i \leq Y_i$. Finally, given $n \in \mathbb{N}^+$, we let $[n] = \{1, \ldots, n\}$.

3.1 General Model

Our model closely follows that of Sinclair et al. [32], which considers a principal who, over $T$ distinct rounds, must divide $K$ divisible resources among a population of individuals. The principal has initial fixed budget $B_k \in \mathbb{R}$ for each resource $k \in [K]$. Let $B = (B_k)_{k \in [K]}$. In each round $t \in [T]$, a random set of individuals arrives, requesting a share of the resources. Each individual is characterized by their type $\theta \in \Theta$, with $|\Theta| < \infty$. Specifically, each type $\theta \in \Theta$ is associated with a known utility function $u(x, \theta) : \mathbb{R}^K \times \Theta \mapsto \mathbb{R}$, for a given allocation $x \in \mathbb{R}^K$ of resources. We assume that, for all $\theta \in \Theta$, $u(\cdot, \theta)$ is concave, one-positively homogeneous, and non-decreasing. We provide examples of common utility functions that satisfy these conditions in Section 4.

We let $N_{t,\theta}$ denote the number of type $\theta$ arrivals in round $t$, with $N_{t,\theta}$ drawn independently from a known distribution $\mathcal{F}_{t,\theta}$. For a fixed vector of arrivals $(N_{t,\theta})_{t \in [T], \theta \in \Theta}$, we will often use $N_{t,\theta}$ to denote the total number of individuals who arrived in round $t$ and afterwards (i.e., $N_{t,\theta} = \sum_{t'=t}^T N_{t',\theta}$). We similarly let $N_{t,\theta} = \sum_{t'=1}^{t'} N_{t',\theta}$, and use $N_{[t,t'),\theta}$ to denote arrivals of type $\theta$ between...
rounds $t$ and $t'$. We use $N_{t,\theta}$ to denote the total number of type $\theta$ individuals across all rounds (i.e., $N_{t,\theta} = \sum_{t \in [T]} N_{t,\theta}$), and let $N = \sum_{t \in [T]} \sum_{\theta \in \Theta} N_{t,\theta}$. Whenever the $\theta$ subscript is omitted, it is assumed that we are considering the aggregate quantity, summed over all $\theta$.

Finally, for any online algorithm which observes the number of arrivals of each type $N_{t,\theta}$ at the beginning of round $\theta$, we let $X_{t,\theta}^{alg} \in \mathbb{R}^{T \times [\Theta] \times [K]}$ be the sequence of allocations determined by the algorithm. Importantly, we assume that, for $t \in [T]$, $\theta \in \Theta$, the algorithm allocates $X_{t,\theta}$ uniformly across all $N_{t,\theta}$ individuals.

3.1.1 Additional assumptions and notation. As in Sinclair et al. [32], we assume that, for all $t \in [T]$, $\theta \in \Theta$, that $N_{t,\theta} \geq 1$ almost surely. We moreover let $\sigma_{t,\theta}^2 = \text{Var} (N_{t,\theta}) < \infty$, and assume $\rho_{t,\theta} = \left| N_{t,\theta} - \mathbb{E} [N_{t,\theta}] \right| < \infty$ almost surely. We denote $\mu_{\max} = \max_{t,\theta} \mathbb{E} [N_{t,\theta}]$, $\sigma_{\min}^2 = \min_{t,\theta} \sigma_{t,\theta}^2$, $\sigma_{\max}^2 = \max_{t,\theta} \sigma_{t,\theta}^2$, and $\rho_{\max} = \max_{t,\theta} \rho_{t,\theta}$.$^1$ Finally, we define $\beta_{\text{avg}} = \frac{B}{\mathbb{E}[N]}$ to be the average resource per individual, and assume $\beta_{\text{avg}} \in \Theta (1)$.

3.2 Notions of Fairness and Efficiency

Before defining our notions of fairness and efficiency for the online setting, in which the decision-maker observes the arrivals $(N_{t,\theta})_{t \in [T], \theta \in \Theta}$ at the beginning of each round $t \in [T]$, we introduce standard notions of fairness and efficiency in the offline setting, in which the decision-maker knows the entire vector of arrivals $(N_{t,\theta})_{t \in [T], \theta \in \Theta}$ at the beginning of time.

3.2.1 Fairness and Efficiency in Offline Allocations. The notion of offline fairness we consider is that of Varian Fairness [33], widely used in the operations research and economics literature.

Definition 3.1 (Fair Allocation). Given types $\Theta$, number of individuals of each type $(N_{t,\theta})_{t \in [T], \theta \in \Theta}$, and utility functions $(u(\cdot, \theta))_{t \in \Theta}$, an allocation $X = \{X_{t,\theta} \in \mathbb{R}^K \mid \sum_{t=1}^T \sum_{\theta \in \Theta} N_{t,\theta} X_{t,\theta} \leq B \}$ is fair if it simultaneously satisfies the following:

1. Envy-Freeness (EF): $u(X_{t,\theta}, \theta) \geq u(X_{t',\theta'}, \theta)$ for all $t, t' \in [T]$, $\theta, \theta' \in \Theta$.
2. Pareto Efficiency (PE): Consider feasible allocations $Y \in \mathbb{R}^{T \times [\Theta] \times [K]}$, $X \in \mathbb{R}^{T \times [\Theta] \times [K]}$ such that $Y \neq X$, and $u(Y_{t,\theta}, \theta) > u(X_{t,\theta}, \theta)$ for $t \in [T]$, $\theta \in \Theta$. Then, there exists $t' \in [T]$, $\theta' \in \Theta$ such that $u(Y_{t',\theta'}, \theta') > u(X_{t',\theta'}, \theta')$.
3. Proportionality (PROP): For all $t \in [T]$, $\theta \in \Theta$, $u(X_{t,\theta}, \theta) \geq u(B/N, \theta)$.

Though computing offline fair allocations is in general an intractable task, one of the most celebrated results of classical economics shows that, for a special class of utility functions, there exists a convex program, known as the Eisenberg-Gale (EG) program [16], for which solutions are indeed fair. Specifically, the EG program maximizes a quantity known as Nash Social Welfare, defined as

$$\text{NSW}(X) = \left( \prod_{t \in [T]} \prod_{\theta \in \Theta} u(X_{t,\theta}, \theta)^{N_{t,\theta}} \right)^{1/\sum_{t \in [T]} N_{t,\theta}} \quad (1)$$

for a given allocation $X \in \mathbb{R}^{T \times [\Theta] \times [K]}$. Taking the log of this quantity and omitting a constant factor of $1/\sum_{t \in [T], \theta \in \Theta} N_{t,\theta}$, the EG program is given by:

$$\max_{X \in \mathbb{R}^{T \times [\Theta] \times [K]}} \sum_{t \in [T]} \sum_{\theta \in \Theta} N_{t,\theta} \log u(X_{t,\theta}, \theta) \quad (2)$$

$^1$See [32] for further discussion on relaxing the required confidence to more general arrival distributions governed by a Markovian process.
We emphasize that, in contrast to the online problem we are interested in, which only has access to \((N_t,\theta)_{t \in \Theta}\) at the beginning of round \(t\), the EG program is offline, i.e., it has access to \(\text{all} arrival\)s at the beginning of time. Moreover, when the utility functions \((u(\cdot, \theta))_{\theta \in \Theta}\) are concave, solutions to (2) are efficiently computable.

Unfortunately, though we have just argued that computing fair allocations is possible in the offline setting, Sinclair et al. [32] showed that the impossibility of any online algorithm simultaneously achieving the three desired properties of envy-freeness, Pareto efficiency and proportionality. This then motivates the goal of finding approximately fair allocations in the online setting.

### 3.2.2 Approximate Fairness and Efficiency in Online Allocations

We formally define the three notions of online fairness we are interested in, first introduced in [32]. Let \(X_{t,\theta}^{alg} \in \mathbb{R}^{T \times |\Theta| \times K}\) be the allocations determined an arbitrary online algorithm.

#### Definition 3.2 (Counterfactual Envy, Hindsight Envy, and Efficiency)

Given individuals with types \(\Theta\), sizes \((N_t,\theta)_{t \in [T], \theta \in \Theta}\), and resource budgets \((B_k)_{k \in [K]}\), for any online allocation \((X_{t,\theta}^{alg})_{t \in [T], \theta \in \Theta} \in \mathbb{R}^{K}\), we define:

- **Counterfactual Envy**: The counterfactual distance of \(X_{t,\theta}^{alg}\) to envy-freeness as
  \[
  \Delta_{EF} \triangleq \max_{t \in [T], \theta \in \Theta} \| u(X_{t,\theta}^{alg}, \theta) - u(X_{t,\theta}^{opt}, \theta) \|_\infty
  \]  
  (3)

  where \(X^{opt}\) is a solution to the offline EG program, given values \((N_t,\theta)_{t \in [T], \theta \in \Theta}\).

- **Hindsight Envy**: The hindsight distance of \(X_{t,\theta}^{alg}\) to envy-freeness as
  \[
  \text{ENVY} \triangleq \max_{t, t' \in [T], \theta, \theta' \in \Theta} u(X_{t',\theta'}^{alg}, \theta) - u(X_{t,\theta}^{alg}, \theta).
  \]  
  (4)

- **Efficiency**: The distance to efficiency as
  \[
  \Delta_{\text{efficiency}} \triangleq \sum_{k \in [K]} \left( B_k - \sum_{t \in [T]} \sum_{\theta \in \Theta} N_{t,\theta} X_{t,\theta,k}^{alg} \right)
  \]  
  (5)

Note that counterfactual envy is well-defined as, even though the solutions to the EG program need not be unique under arbitrary utilities, the *utilities* induced by the optimal offline allocation are (cf. Eisenberg [16], Theorem 1), for the class of utility functions we consider in this paper.

At a high level, counterfactual envy \(\Delta_{EF}\) can be viewed as the online algorithm’s distance to fairness in hindsight, since, under mild assumptions (see Section 4), solutions to the EG program satisfy ex-post envy-freeness, Pareto efficiency and proportionality. Though this latter envy-freeness metric is with respect to an offline solution, hindsight envy measures how differently the online algorithm treated any two individuals, across types and time. Finally, the efficiency of the online algorithm \(\Delta_{\text{efficiency}}\) measures how wasteful the algorithm was in hindsight.\(^2\)

These metrics are, in a sense, at odds with each other, which marks the inherent difficulty of this problem. To see this, consider the following two extreme scenarios. On the one hand, an algorithm can trivially achieve a hindsight envy of zero by allocating nothing to individuals in any round; this, however, would result in both high counterfactual envy, in addition to maximal inefficiency. On the

\(^2\)The reader may notice that this definition of distance to efficiency does not exactly mirror the offline notion of Pareto efficiency defined in the previous section. However, it is easy to see that under strict monotonicity of utility functions, a distance to efficiency of zero is a necessary precondition to Pareto efficiency.
other hand, a distance to efficiency of zero can trivially be satisfied by exhausting the budget in the initial round, at a cost of maximal envy as individuals arriving at later rounds would be envious of the allocation given in the first. Sinclair et al. [32] formalized this tension via the following lower bounds, which we re-state here for completeness.

**Theorem 3.3 (Theorems 1 and 2 in [32]).** Under any arrival distribution satisfying the assumptions outlined above, there exists a problem instance with \( K = 1 \), additive utilities, and no perishing, such that any algorithm must incur \( \Delta_{\text{EF}} \gtrsim \frac{1}{\sqrt{T}} \). Moreover, any algorithm that achieves \( \Delta_{\text{EF}} \leq L_T = o(1) \) or \( \text{Envy} \leq L_T = o(1) \) must also incur waste \( \Delta_{\text{efficiency}} \gtrsim \min\{\sqrt{T}, 1/L_T\} \).\(^3\)

We next discuss a general algorithmic framework, first introduced in Sinclair et al. [32] for the setting of additive utilities with no perishing, which we will show achieves these lower bounds for the two different settings considered in this work.

### 3.3 Achieving Optimal Envy and Efficiency via Guardrails

Having established that solutions to the EG program are fair for the class of utility functions we consider, a natural first attempt in dealing with the uncertainty inherent to the online nature of the program would be to solve the EG program once at the beginning of the time horizon under the assumption that arrivals are given by \( N_{t, \theta} = \mathbb{E}[N_{t, \theta}] \) for all \( t \in [T], \theta \in \Theta \). Using the fact that, under mild conditions on the arrival distribution, the number of arrivals in each round concentrates around its mean, one would hope to obtain concentration around the fair EG solution.

Unfortunately, such an approach is not careful enough to account for how uncertainty propagates through the metrics of interest. In particular, under mild conditions, the random number of arrivals deviates from its mean by an additive factor of \( \sqrt{T} \) with constant probability, by the Berry-Esseen theorem [12]. As a result, allocating according to the mean would imply that, with constant probability the algorithm exhausts its budget too soon, thus leading to high hindsight, as well as counterfactual, envy.

Sinclair et al. [32] tackled this issue for the setting with additive utilities by solving the EG program twice: once, under an upper bound \( (\bar{n}_\theta)_{\theta \in \Theta} \), and once under a lower bound \( (n_\theta)_{\theta \in \Theta} \), on the random number of arrivals. Importantly, the lower bound \( n_\theta \leq N_\theta \) holds with high probability. The optimal envy-efficiency trade-off is then achieved by carefully constructing \( n_\theta \) and \( \bar{n}_\theta \) such that: (i) the induced guardrails are not too far apart, thus ensuring low envy, and (ii) the upper guardrail is high enough, and used often enough, that waste is minimized.

### 4 GENERAL UTILITY FUNCTIONS

We begin by showing that under careful analysis GUARDED-HOPE can be leveraged for a broad class of utility functions to achieve the envy-efficiency trade-offs presented in Theorem 3.3, extending past the limiting additive utility assumption from Sinclair et al. [32]. The main technical contribution involves showing that under appropriately chosen upper and lower threshold allocations that the utility of the optimal fair allocation in hindsight \( X_{\text{opt}}^t \) (i.e. the solution to (2) with known arrivals \( N_{t, \theta} \)) is sandwiched between the utilities of these two guardrail allocations. We show this by deriving a property of the EG program under a broad range of utilities known as competitive monotonicity, which was previously known only for the setting of weak gross substitutes [23].

\(^3\)We note that these bounds also hold for the more general setting we consider, as even with arbitrary utility functions the optimal fair allocation in the case of a single resource is \( X_{t, \theta}^{\text{opt}} = \frac{\bar{B}_t}{N_t} \), for \( t \in [T], \theta \in \Theta \), where \( N = \sum_{t, \theta} N_{t, \theta} \). This follows from the fact that solutions to the EG program are envy-free, and the unique envy-free allocation and Pareto-efficient solution is \( B/N \) in the case of a single resource.
4.1 Model

In this section, we consider the class of utility functions that satisfy the following assumption.

**Assumption 1.** For all $\theta \in \Theta$, $(u(\cdot, \theta))_{\theta \in \Theta}$ are such that any solution to the Eisenberg-Gale program under vector of arrivals $(N_{t, \theta})_{t \in [T], \theta \in \Theta}$ satisfies envy-freeness, Pareto efficiency, and proportionality.

Before providing examples of utility functions for which Assumption 1 holds, we establish a useful property of solutions to the EG program, that will simplify much of the subsequent analyses.

**Proposition 4.1.** Suppose $u(x, \theta)$ is concave in $x$ for all $\theta \in \Theta$. Then there exists a time-invariant solution $X^{opt}$ to the EG program. That is, for all $\theta \in \Theta$, $X_{t, \theta}^{opt} = X_{t', \theta}^{opt}$ for all $t, t' \in [T]$.

We note that time-invariance of an optimal solution to the EG program does not in general hold in the setting with perishable resources, considered in the following section.

In the remainder of this section, we consider the following equivalent formulation of the EG program:

$$\max_{X \in \mathbb{R}^{K \times K}} \sum_{\theta \in \Theta} N_{\theta} \log u(X_{\theta}, \theta) \tag{EG}$$

s.t. $$\sum_{\theta \in \Theta} N_{\theta} X_{\theta, k} \leq B_k \ \forall k \in [K]$$

$$X_{\theta, k} \geq 0 \ \forall \theta \in \Theta, k \in [K].$$

With this simplification in hand, we show that, under a slight additional assumption on agents’ utility functions (in particular, that they be increasing rather than simply non-decreasing), solutions to the EG program satisfy the three fairness properties of interest.

**Proposition 4.2.** Suppose $u(\cdot, \theta)$ is concave, one-positively homothetic, and strictly increasing, for all $\theta \in \Theta$. Then, any solution to the EG program is Pareto-efficient, envy-free, and proportional.

Examples of such utility functions include:

- **Linear:** $u(x, \theta) = \langle w_\theta, x \rangle$ for some vector of per-resource preferences $w_\theta \in \mathbb{R}^K$.
- **Cobb-Douglas:** $u(x, \theta) = \prod_{k=1}^{K} w_{\theta, k} x_{\theta, k}^{\alpha_{\theta, k}}$, where $w_\theta \in \mathbb{R}_{>0}^K$, $\alpha_\theta \in \mathbb{R}_{>0}^K$, and $\sum_{k \in [K]} \alpha_{\theta, k} = 1$.
- **Leontief + $\epsilon$-linear:** $u(x, \theta) = \min\{x_k / w_{\theta, k}\} + \epsilon \langle w'_\theta, x_\theta \rangle$, where $w_\theta, w'_\theta \in \mathbb{R}_{>0}^K$, and $\epsilon > 0$.

Note that the popular class of utility functions, Leontief utilities, fails to satisfy the requirements of Proposition 4.2. In particular, due to the fact that these functions are not strictly increasing, solutions to the EG program are not unique and may in general result in high waste. In Appendix C we however show that our algorithmic framework can nonetheless be leveraged to obtain fairness bounds in this latter setting by instead considering Leontief utilities with an $\epsilon$-linear perturbation, for an appropriately chosen value of $\epsilon$. Such an $\epsilon$-perturbation trick is more widely useful for utilities which are homothetic and concave, but not strictly increasing.

The proof of Proposition 4.2 relies on the fact that, for the class of utilities we consider, solutions to the EG program correspond precisely to allocations in a competitive equilibrium of the corresponding Fisher market with equal incomes. We then leverage the seminal result of Varian [33] which states that, under mild conditions, competitive equilibria are envy-free and Pareto efficient (see Fig. 1). With some additional work, proportionality follows.

4.2 The Algorithm

Before formally describing the algorithm, we define some useful notation. For an arbitrary vector of arrivals $(\tilde{N}_\theta)_{\theta \in \Theta}$, let $x((\tilde{N}_\theta)_{\theta \in \Theta})$ be an optimal solution to the EG program, given $(\tilde{N}_\theta)_{\theta \in \Theta}$.
Moreover, let $X^{opt} = x((N_\theta)_{\theta \in \Theta})$, where $(N_\theta)_{\theta \in \Theta}$ is the realized vector of arrivals. GUARDED-HOPE is presented in Algorithm 1.

**Algorithm 1**: Histogram of Preference Estimates with Guardrails (GUARDED-HOPE)

**Input**: Budget $B = B_1^{alg}$, confidence terms $(Conf_{t, t', \theta})_{t, t' \in [T], \theta \in \Theta}$, desired bound on envy $L_T$, arrival confidence bounds $(\overline{\pi}_\theta)_{\theta \in \Theta}$, $(\underline{\pi}_\theta)_{\theta \in \Theta}$

**Output**: An allocation $X^{alg} \in \mathbb{R}^{T \times |\Theta| \times K}$

Solve for $X = x((N_\theta)_{\theta \in \Theta})$ as the solution to (EG) with arrival vector $(\overline{n}_\theta)_{\theta \in \Theta}$

Solve for $X = x((\overline{\pi}_\theta)_{\theta \in \Theta})$ as the solution to (EG) with arrival vector $(\overline{\pi}_\theta)_{\theta \in \Theta}$

// solve for guardrails

for $t = 1, \ldots, T$ do

for each resource $k \in [K]$ do

if $B_{t,k}^{alg} < \sum_{\theta \in \Theta} N_{t,\theta} \overline{x}_{\theta,k}$ then // insufficient budget to allocate lower guardrail

Set $x^{alg}_{t,\theta,k} = \frac{B_{t,k}^{alg}}{\sum_{\theta \in \Theta} N_{t,\theta}}$ for each $\theta \in \Theta$

else if $B_{t,k}^{alg} - \sum_{\theta \in \Theta} N_{t,\theta} \overline{x}_{\theta,k} \geq \sum_{\theta \in \Theta} \underline{x}_{\theta,k} (E[N_{t,\theta}] + Conf_{t, T, \theta})$ then // use upper guardrail

Set $x^{alg}_{t,\theta,k} = \overline{x}_{\theta,k}$ for each $\theta \in \Theta$

else // use lower guardrail

Set $x^{alg}_{t,\theta,k} = \underline{x}_{\theta,k}$ for each $\theta \in \Theta$

end

Update $B_{t+1,k}^{alg} = B_{t,k}^{alg} - \sum_{\theta \in \Theta} N_{t,\theta} x^{alg}_{t,\theta,k}$

end

return $X^{alg}$

In each round, the algorithm first checks if it has enough budget remaining in the current round to allocate the lower allocation of the two EG solutions, denoted by $\overline{X}$ and referred to as the lower guardrail. If not, it simply splits the remaining budget equally across all arrivals. Otherwise, the algorithm checks if it can afford to allocate the higher allocation of the two EG solutions, denoted by $\overline{X}$ and referred to as the upper guardrail, assuming a high-probability upper bound on the number of future arrivals guaranteeing them the lower guardrail allocation. If so, it chooses this higher allocation $\overline{X}$. Otherwise, it allocates $X$.

Note that, in any given round, the allocation chosen by GUARDED-HOPE is fair, by construction (i.e., it is envy-free when considering individuals in the system in the same period). The key difficulty is choosing $(\overline{\pi}_\theta)_{\theta \in \Theta}$ and $(\underline{\pi}_\theta)_{\theta \in \Theta}$ (and hence $X = x((\overline{\pi}_\theta)_{\theta \in \Theta})$ and $\overline{X} = x((\overline{\pi}_\theta)_{\theta \in \Theta})$) such that, despite switching between upper and lower guardrail allocations to manage the budget consumption, the online algorithm remains on the envy-freeness and efficiency frontier across periods, defined by Theorem 3.3.
4.3 Performance Guarantee

Before presenting the envy-freeness and efficiency bounds achieved by GUARDED-HOPE, we define some useful notation. Let $\bar{X} = x((\bar{n}_\theta)_{\theta \in \Theta})$ and $X = x((n_\theta)_{\theta \in \Theta})$, for some $\bar{n}_\theta$ and $n_\theta$ to be defined in the theorem statement.

We have the following main result.

**Theorem 4.3.** Fix $L_T = o(1)$, and let GUARDED-HOPE be initialized with
\[
    \text{CONF}_{t,t',\theta} = \sqrt{2(t'-t)\rho_{\max}^2 \log(2T^2|\Theta|/\delta)} \quad \forall \ t' > t, \ \theta \in \Theta
\]
\[
    \bar{n}_\theta = \mathbb{E}[N_\theta] \left(1 + \max_\theta \frac{\text{CONF}_{0,T,\theta}}{\mathbb{E}[N_\theta]} \right) \quad \forall \ \theta \in \Theta
\]
\[
    n_\theta = \mathbb{E}[N_\theta] (1 - c) \quad \forall \ \theta \in \Theta, \ c = \frac{L_T}{C_1} \left(1 + \max_\theta \frac{\text{CONF}_{0,T,\theta}}{\mathbb{E}[N_\theta]} \right) - \max_\theta \frac{\text{CONF}_{0,T,\theta}}{\mathbb{E}[N_\theta]}.
\]

Then, with probability at least $1 - \delta$, GUARDED-HOPE achieves:
\[
    \text{ENVI} \leq L_T \quad \Delta_{\text{EF}} \leq \max\{1/\sqrt{T}, L_T\} \quad \Delta_{\text{efficiency}} \leq \min\{\sqrt{T}, 1/L_T\} \quad \Delta_{\text{prop}} \leq \max\{1/\sqrt{T}, L_T\},
\]
where $\leq$ drops poly-logarithmic factors of $T$, $o(1)$ terms, and absolute constants.

In the remainder of the section, for ease of notation we let $\text{CONF}_\theta = \text{CONF}_{0,T,\theta}$. We first give the high-level ideas behind the proof of these bounds.

The theorem relies on a careful analysis of the allocations decided by GUARDED-HOPE under event $\mathcal{E} = \{|N_{(t,t'),\theta} - \mathbb{E}[N_{(t,t'),\theta}]| \leq \text{CONF}_{t,t',\theta} \forall t' > t, \forall \theta \in \Theta\}$, which can easily be shown to hold with probability at least $1 - \delta$ via an application of Hoeffding’s inequality. Then, given $\mathcal{E}$, we first establish that in every round, the algorithm has enough budget to allocate at least the lower guardrail $X_{\theta,k}$ to all future arrivals. This then implies that, by construction, either $\bar{X}_\theta$ or $X_\theta$ is allocated in every round. Now, recall the tension faced by GUARDED-HOPE. In theory, allocating $X$ in every round would achieve low envy, at the cost of high inefficiency. We show, however, that the algorithm is adaptively cautious: under event $\mathcal{E}$, there exists a round past which it always allocates the upper guardrail $\bar{X}$, thus guaranteeing low waste.

Once these two facts about the algorithm are established, the remainder of the work lies in (1) analyzing agents’ utilities under $\bar{X}$ and $X$, static solutions to the offline EG program, for which we can leverage the fact that these are fair, by assumption, and (2) relating these back to $X^{opt}$, the offline optimal solution, all as a function of $L_T$.

We provide the formal statements of these facts below, deferring their proofs to the appendix. We first define constants $C_1 = \max_{\theta \in \Theta} u(B, \theta)$, $C_2 = B_{\min}/\mathbb{E}[N]$, $C_3 = \|B\|_{\infty}$, where $B_{\min} = \min_{k \in [K]} B_k$. Consider event $\mathcal{E}$, as defined above. Lemma 4.4 implies that it suffices to restrict our attention to $\mathcal{E}$ for the high-probability statement.

**Lemma 4.4.** With probability at least $1 - \delta$, for every $\theta \in \Theta$ and $t' > t$ we have that $|N_{(t,t'),\theta} - \mathbb{E}[N_{(t,t'),\theta}]| \leq \text{CONF}_{t,t',\theta}$.

Next, Lemmas 4.5 and 4.6 formalize both that GUARDED-HOPE only ever allocates the guardrails under $\mathcal{E}$, and that it is adaptively cautious.

**Lemma 4.5.** Given event $\mathcal{E}$, for every resource $k$ and time $t \in [T]$,
\[
    B_{t,k}^{alg} \geq \sum_{\theta \in \Theta} N_{t,\theta,k} \bar{X}_{\theta,k}.
\]
As a result, given $\mathcal{E}$, $X_{t,\theta,k}^{alg} \in \{\bar{X}_{\theta,k}, X_{\theta,k}\}$ for all $t \in [T], \theta \in \Theta, k \in [K]$.
Lemma 4.6. Consider event $\mathcal{E}$, and let $t_{0,k}$ be the last round for which $X_{t,\theta,k}^{alg} = X_{\theta,k}$, (else, define $t_{0,k} = 0$ if $X_{t,\theta,k}^{alg} = X_{\theta,k}$ for all $t \in [T]$) for each resource $k \in [K]$. Then, $t_{0,k} \geq \max\{0, T - cL_T^{-2}\}$ for all $k \in [K]$, where $c = \tilde{O}(1)$.

Finally, Lemma 4.7 relates the guardrail allocations to each other, in addition to the optimal offline solution. In particular, it states that agents’ utilities under two different allocations differ by at most $L_T$, and that the allocations themselves are within a factor of $L_T$ of each other. Moreover, under a lower bound on $L_T$, not only is the true vector of arrivals sandwiched by the arrival confidence bounds, but so are agents’ utilities.

Lemma 4.7. The following holds for our guardrail allocations $X_{\theta}$ and $\overline{X}_{\theta}$:

1. $u(x((\bar{n}_\theta)_{\theta \in \Theta})_\theta, \theta) - u(x((\bar{n}_\theta)_{\theta \in \Theta})_\theta, \theta) \leq L_T$
2. $\|x((\bar{n}_\theta)_{\theta \in \Theta})_\theta - x((\bar{n}_\theta)_{\theta \in \Theta})_\theta\|_\infty \geq \frac{C_4}{C_1} L_T$
3. $\|x((\bar{n}_\theta)_{\theta \in \Theta})_\theta - x((\bar{n}_\theta)_{\theta \in \Theta})_\theta\|_\infty \leq \frac{C_4}{C_1} L_T$.

If in addition $L_T \geq 2C_1 \max_{\theta} \frac{\text{Conf}_{\nu_{opt}}}{\mathbb{E}[N_{\theta}]}$, under event $\mathcal{E}$ we have:

4. $\bar{n}_\theta \leq N_\theta \leq \bar{n}_\theta$ for all $\theta \in \Theta$
5. $u(x((\bar{n}_\theta)_{\theta \in \Theta})_\theta, \theta) \leq u(x_{\theta}^{opt}, \theta) \leq u(x((\bar{n}_\theta)_{\theta \in \Theta})_\theta, \theta)$.

A key technical contribution of this work is in deriving Property 5 in the above theorem. Though Sinclair et al. [32] leveraged the fact that the utility functions were additive, a more careful analysis shows that this fact still holds for concave, non-decreasing and one-positively homothetic utilities. Moreover, in proving this fact we derive a property of the EG program under these utilities known as competitive monotonicity, which to the best of our knowledge was simply known for the setting of weak gross substitutes [23].

Given these building blocks of our main result, we now prove the theorem.

Proof of Theorem 4.3. Efficiency: Consider first $\Delta_{\text{efficiency}}$. We have:

$$
\Delta_{\text{efficiency}} = \sum_{k} \left( B_k - \sum_{t \in [T]} \sum_{\theta} X_{t,\theta,k}^{alg} N_{t,\theta} \right) = \sum_{k} \left( B_{t_{0,k}}^{alg} - \sum_{t \geq t_{0,k}} \sum_{\theta} N_{t,\theta} X_{t,\theta,k}^{alg} \right)
$$

(a) $\leq \sum_{k} B_{t_{0,k}}^{alg} - \sum_{\theta} \left( X_{\theta,k} N_{t_{0,k},\theta} + \sum_{t > t_{0,k}} \overline{X}_{\theta,k} N_{t,\theta} \right)$

(b) $\leq \sum_{k} \sum_{\theta} N_{t_{0,k},\theta} \overline{X}_{\theta,k} + \sum_{\theta} X_{\theta,k} \left( \mathbb{E}[N_{t_{0,k},\theta}] + \text{Conf}_{t_{0,k},T,\theta} \right) - \sum_{\theta} \left( X_{\theta,k} N_{t_{0,k},\theta} + \sum_{t > t_{0,k}} \overline{X}_{\theta,k} N_{t,\theta} \right)$

$\leq \sum_{k} \sum_{\theta} X_{\theta,k} \left( \mathbb{E}[N_{t_{0,k},\theta}] + \text{Conf}_{t_{0,k},T,\theta} - N_{t_{0,k},\theta} \right) - \left( \overline{X}_{\theta,k} - X_{\theta,k} \right) \left( N_{t_{0,k},\theta} - N_{t_{0,k},\theta} \right)$

$\leq 2 \sum_{k} \sum_{\theta} X_{\theta,k} \text{Conf}_{t_{0,k},T,\theta} + N_{t_{0,k},\theta} \left( \overline{X}_{\theta,k} - X_{\theta,k} \right)$

where (a) follows from the fact that, by the definition of $t_{0,k}$, $X_{t_{0,k},\theta,k}^{alg} = X_{\theta,k}$, and $X_{t,\theta,k}^{alg} = \overline{X}_{\theta,k}$ for all $t > t_{0,k}$; (b) follows from the condition in the algorithm for allocating the lower allocation at time
\( t_{0,k} \), which upper bounds \( B_{t_{0,k}}^{\text{alg}} \), and (c) follows from the fact that, under \( E \), \( E[N_{t_{0,k},\theta}] - N_{t_{0,k},\theta} \leq \text{CONF}_{t_{0,k},T,\theta}^\text{conf} \).

Plugging in the definition of \( \text{CONF}_{t_{0,k},T,\theta} \) and the fact that \( (\bar{X}_{\theta,k} - X_{\theta,k}) \leq \frac{C_3}{C_1} L_T \) by Lemma 4.7, we have:

\[
\Delta_{\text{efficiency}} \leq 2C_3 \sum_k \sum_{\theta} \sqrt{2 \rho_{\text{max}}^2 (T - t_{0,k}) \log(2T^2|\Theta|/\delta)} + (\mu_{\text{max}} + \rho_{\text{max}}) \frac{C_3}{C_1} L_T.
\]

Taking this and plugging in the lower bound on \( t_{0,k} \) from Lemma 4.6, we get that:

\[
\Delta_{\text{efficiency}} \leq 2C_3 K|\Theta| \sqrt{2 \rho_{\text{max}}^2 \log(2T^2|\Theta|/\delta)} \min\{\sqrt{T}, \sqrt{2c/L_T}\} + 2(\mu_{\text{max}} + \rho_{\text{max}}) \frac{C_3}{C_1} L_T.
\]

Note that, for \( L_T = o(1) \), the second term is dominated by the first, which gives us the desired bound on \( \Delta_{\text{efficiency}} \).

**Hindsight Envy:** Fix \( t, t' \in [T] \), and \( \theta, \theta' \in \Theta \). By Lemma 4.5, \( X_{t,\theta,k}^{\text{alg}} \in \{\bar{X}_{\theta,k}, X_{\theta,k}\} \) for all \( t \in [T], \theta \in \Theta, k \in [K] \). Using the fact that \( u \) is non-decreasing, we have:

\[
u(X_{t',\theta}, \theta) - u(X_{t,\theta}, \theta) \leq u(\bar{X}_{\theta}, \theta) - u(\bar{X}_{\theta}, \theta)
\]

\[
\leq \sum_{k} \sum_{\theta} \sqrt{2 \rho_{\text{max}}^2 (T - t_{0,k}) \log(2T^2|\Theta|/\delta)} + (\mu_{\text{max}} + \rho_{\text{max}}) \frac{C_3}{C_1} L_T.
\]

where (a) follows from the fact that \( \bar{X} \) is envy-free by construction, and (b) follows from the bound in Lemma 4.7. Taking the max over \( t, t', \theta, \theta' \) gives the result.

**Counterfactual Envy:** We next obtain the bound on \( \Delta_{\text{EF}} \). Consider first the setting where \( L_T \geq 2C_1 \sqrt{\frac{2 \rho_{\text{max}} \log(2T^2|\Theta|/\delta)}{T}} \), such that Properties 4 and 5 of Lemma 4.7 are satisfied. Again, using Lemma 4.5 along with the fact that \( u(X_{\theta,\theta}, \theta) \leq u(X_{\theta,\theta}^{\text{opt}}, \theta) \leq u(\bar{X}_{\theta}, \theta) \), by Lemma 4.7, we have that, given \( E \):

\[ |u(X_{t,\theta}^{\text{alg}}, \theta) - u(X_{\theta}^{\text{opt}}, \theta)| \leq |u(\bar{X}_{\theta}, \theta) - u(X_{\theta}^{\text{opt}}, \theta)| \leq L_T. \]

Consider now the case where \( L_T < 2C_1 \sqrt{\frac{2 \rho_{\text{max}} \log(2T^2|\Theta|/\delta)}{T}} \), then, one of the following two chain of inequalities holds for all \( \theta \in \Theta \): (1) \( u(X_{\theta,\theta}, \theta) \leq u(X_{\theta,\theta}^{\text{opt}}, \theta) \leq u(\bar{X}_{\theta}, \theta) \), or (2) \( u(X_{\theta,\theta}^{\text{opt}}, \theta) \leq u(\bar{X}_{\theta}, \theta) \), since \( u \) is non-decreasing. In the first case, we have \( |u(X_{t,\theta}^{\text{alg}}, \theta) - u(X_{\theta}^{\text{opt}}, \theta)| \leq L_T \) as above.

Else, let \( \bar{X}_{\theta} \) be the upper guardrail solution corresponding to \( L_T = 2C_1 \sqrt{\frac{2 \rho_{\text{max}} \log(2T^2|\Theta|/\delta)}{T}} \). Via the same reasoning as above, we have:

\[ |u(X_{t,\theta}^{\text{alg}}, \theta) - u(X_{\theta}^{\text{opt}}, \theta)| \leq |u(\bar{X}_{\theta}, \theta) - u(X_{\theta}^{\text{opt}}, \theta)| \leq L_T = 2C_1 \sqrt{\frac{2 \rho_{\text{max}} \log(2T^2|\Theta|/\delta)}{T}}, \]

where the second inequality again follows from Lemma 4.7.

**Hindsight Proportionality:** We conclude by showing the bound on \( \Delta_{\text{prop}} \). We have:

\[ u(B/N, \theta) - u(X_{t,\theta}^{\text{alg}}, \theta) = u(B/N, \theta) - u(X_{\theta}^{\text{opt}}, \theta) + u(X_{\theta}^{\text{opt}}, \theta) - u(X_{t,\theta}^{\text{alg}}, \theta) \leq \max\{1/\sqrt{T}, L_T\}. \]
where the final inequality follows from the fact that $X^{opt}$ satisfies proportionality, by Proposition 4.2, and thus $u(B/N, \theta) - u(X^{opt}, \theta) \leq 0$, and $u(X^{opt}, \theta) - u(X_{t,\theta}^{alg}, \theta) \leq \Delta_{EF} \leq \max\{1/\sqrt{T}, L_T\}$, which we just proved above. Taking the max over $t$ and $\theta$ gives the desired bound on $\Delta_{prop}$. □

5 PERISHABLE RESOURCES

We next show that algorithmic guardrails can be leveraged for resource allocation with perishable resources. Our main result is to show that an algorithm that simply allocates the lower guardrail, a high-probability lower bound on the optimal solution to the EG program, achieves optimal envy and efficiency (with $L_T = T^{-1/2}$). This highlights that the algorithmic principle of guardrail allocations extends to achieving one point on the trade-off curve under perishable resources.

While intuitive, this brief algorithm sketch is underdetermined. As discussed in Section 1, in the setting with perishable resources the algorithm must also decide which of the resources to allocate in every period, rather than simply how many. In this section, we show that, under mild conditions on the perishing process, a carefully constructed guardrail allocation scheme, which intuitively allocates units of resource according to expected perishing date, achieves the optimal envy-efficiency bounds.

5.1 Model

We now consider a setting in which there are $B$ divisible units of a single resource (i.e., $K = 1$). Each unit of resource $b \in [B]$ is associated with a perishing time $T_b$ drawn from a known distribution. We assume items’ perishing times are independent of one another, as well as of the arrival process. The goal is to determine a sequence of allocations $X^{alg} \in \mathbb{R}^{T \times |\Theta|}$ corresponding to the amount of resource to give out to individuals across time. Implicit in this aggregate quantity is the order in which items must be allocated, as discussed above. However, it is easy to show that the optimal objective value of this relaxation is exactly equal to that of a more granular program wherein timing is indeed encoded, under the optimal hindsight timing rule which allocates items according to the known perishing time. We leave the easy proof of this fact to the reader.

As before, we will be interested in designing an algorithm that keeps our notions of counterfactual envy, hindsight envy, and efficiency low with high probability. For arbitrary perishing processes,
we note that such a task is hopeless. Consider for instance a process \((P_t)_{t \in [T]}\) such that all items perish after the first period with constant probability. Then, high efficiency requires allocating the entire budget in the first period, which would result in \(\Omega(T)\) hindsight envy. In such settings, then, hindsight envy ceases to be meaningful, in a sense, and re-defining envy to be forward-looking would be more appropriate. We leave the treatment of such important modeling questions to future work, and instead in this section consider "offset-expiring" processes which are such that \(B/N\), the hindsight optimal allocation without perishing, remains optimal with high probability. In this setting, then, envy remains an achievable task. We present such a condition for this fact to hold almost surely below, and relax it to hold with high probability in Section 5.3.

**Proposition 5.1.** Suppose \(P \leq t \leq N \leq t N \leq t\) for all \(t \in [T]\). Then, \(X^{opt}_{t,\theta} = B/N\) for all \(t \in [T], \theta \in \Theta\).

At a high level, the condition described in Proposition 5.1 states that the total amount that perishes before period \(t\) is no more than what would have been allocated before period \(t\) in the setting without perishing. That is, the optimal fair allocation stays "ahead" of the perishing process.

### 5.2 The Algorithm

We present our algorithm Static-Allocation in Algorithm 2, letting \(\mathcal{A}_t\) denote the set of items allocated in period \(t\), for \(t \in [T]\). Our algorithm is *always cautious*: it considers the minimum fair allocation that can be given out in the worst case (i.e., under a high number of arrivals, and a high degree of perishing), and simply allocates this minimum amount in every period. In this case, then, envy is only ever incurred if the algorithm runs out of budget, in which case agents later on in the time horizon will receive nothing. On the other hand, waste is incurred if the algorithm allocates too little relative to the budget (as in the previous section), or if items perish before they are allocated.

**Algorithm 2: Static-Allocation with Perishable Resources**

**Input:** Budget \(B = B_{alg}^t\), confidence terms with respect to the arrival process \((Conf_{t,t'}^N)_{t,t' \in [T], \theta \in \Theta}\)

**Output:** An allocation \(X_{alg} \in \mathbb{R}^{T \times |\Theta|}\)

Solve for \(X = \frac{B}{\mathbb{E}[N] + Conf_{t,T}} \left(1 - \frac{c}{\sqrt{T}}\right)\) for \(c = 1 + \sqrt{3 \log (2T/\delta)}\) // solve for lower guardrail

**for** \(t = 1, \ldots, T\) **do**

**if** \(B_{alg}^t < \sum_{\theta \in \Theta} N_{t,\theta} X_{alg}^t\) **then** // insufficient budget to allocate lower guardrail

Set \(X_{alg}^t,\theta = \frac{B_{alg}^t}{\sum_{\theta \in \Theta} N_{t,\theta}}\) for each \(\theta \in \Theta\)

**else** // use lower guardrail

Set \(X_{alg}^t,\theta = X_{alg}^t\) for each \(\theta \in \Theta\). Allocate items \(b \in [B_{alg}^t]\) in increasing order of \(\mathbb{E}[T_b]\) (breaking ties arbitrarily).

**end**

Update \(B_{alg}^t = B_{alg}^{t-1} - N_{t-1} X_{alg}^{t-1} - \sum_{b \in [B_{alg}^{t-1}]} I\{T_b = t - 1, b \notin \mathcal{A}_{t-1}\}\)

**return** \(X_{alg}^t\)

### 5.3 Performance Guarantee

Recall, Static-Allocation for the setting with perishable products allocates items according to their expected perishing time. Thus, at a high level, the algorithm makes ‘mistakes’ whenever items perish well before the time at which the algorithm would have allocated it going down the pre-specified order. In order to formalize this intuition, we introduce some additional notation.
For $b \in [B]$, let $t_{b}^{alg}$ denote the time at which the algorithm allocates item $b$, conditional on it having not perished. Moreover, let $\text{Prec}(b) = \sum_{\theta} \mathbb{P}[\exists b' \leq \mathbb{E}[T_{b'}] \leq \mathbb{E}[T_{b}]]$ denote the number of resources whose expected perishing date is before $b$, and $i_{b} = \inf \{t : tX \geq \text{Prec}(b)\}$. That is, $i_{b}$ corresponds to the period in which item $b$ would have been allocated if no item $b'$ such that $\mathbb{E}[T_{b'}] \leq \mathbb{E}[T_{b}]$ perished before the algorithm reached $b$ in the priority list, and $t$ individuals arrived before period $t$ (a lower bound on the total demand before $t$). Let $\eta_{T} = \frac{1}{T} \sum_{b \in [B]} \mathbb{P}(T_{b} < i_{b})$. That is, $\eta_{T}$ represents the average probability a unit of good perished before it would have been allocated in the perfect world described above. We also define $v_{t} = \sum_{b \in [B]} \mathbb{P}(T_{b} < t)$ (i.e., $v_{t}$ is the expected number of units of resource that perish before period $t$). Then, we have the following main result.

**Theorem 5.2.** For $t' > t$, let $\text{CONF}^{N}_{t,t'} = \sqrt{2(t' - t)\rho_{\text{max}}^{3} \log(2T^{2}/\delta)}$, and $\text{CONF}^{P}_{t} = \sqrt{3v_{t} \log(2T/\delta)}$, and suppose the following conditions hold:

1. $\mathbb{E}[P_{<t}] + \text{CONF}^{P}_{t} \leq \frac{\mathbb{E}[N_{t,t'}] - \text{CONF}^{N}_{t,t'}}{\mathbb{E}[N] + \text{CONF}^{N}_{0,t'}} B$ for all $t \in [T]$, and
2. $\eta_{T} \leq \frac{1}{\sqrt{T}}$.

Then with probability at least $1 - 2\delta$, STATIC-ALLOCATION achieves:

$$\Delta_{EF} \leq \frac{1}{\sqrt{T}}, \quad \Delta_{\text{efficiency}} \leq \sqrt{T}, \quad \text{ENVY} = 0, \quad \Delta_{\text{prop}} \leq \frac{1}{\sqrt{T}},$$

where $\leq$ drops poly-logarithmic factors of $T$, $o(1)$ terms, and absolute constants.

We prove the main result, deferring a discussion of the conditions under which our algorithm enjoys its guarantees to Section 5.4. First note that, via a straightforward modification of Lemma 4.4, we have that the event

$$E_{N} = \{\forall t, t' > t \mid N_{t,t'} - \mathbb{E}[N_{t,t'}] \leq \text{CONF}^{N}_{t,t'}\}$$

occurs with probability at least $1 - \delta$. The following lemma similarly establishes that the perishing process is well concentrated around its mean.

**Lemma 5.3 (Concentration on Perishing Process).** Let $E_{p}$ be the event that $|P_{<t} - \mathbb{E}[P_{<t}]| \leq \text{CONF}^{P}_{t}$ for all $t \in [T]$. Then, $E_{p}$ holds with probability at least $1 - \delta$.

We define $E = E_{N} \cap E_{p}$ and note that $\mathbb{P}(E) \geq 1 - 2\delta$. The following lemma states that $X_{t}$ is indeed a good approximation of $X^{opt}$, given $E$.

**Lemma 5.4.** $X_{t,\theta}^{opt} = \frac{B}{N}$ for all $t \in [T], \theta \in \Theta, under good event $E$.

Our final building block establishes that the budget maintained by the algorithm is able to ensure an allocation of $X$ to all future arrivals. Such a fact was also derived in Section 4 for the setting without perishable goods (see Lemma 4.5). This latter lemma, however, cannot be repurposed for the setting with perishable goods, since we must now account for both the uncertainty in arrivals $N_{t}$, in addition the order in which resources perish, and how this relates to the time at which the algorithm sought to allocate them.

**Lemma 5.5.** Under event $E$, $B_{t}^{alg} \geq \mathbb{N}_{x} X_{t}$ for all $t \in [T]$.

With these lemmas in hand, we prove our main result for this section.

**Proof of Theorem 5.2.** By Lemma 5.5, the algorithm never runs out of budget under event $E$, which occurs with probability at least $1 - 2\delta$. In the remainder of the proof, we assume event $E$ holds.
Counterfactual Envy: For all \( t \in [T], \theta \in \Theta \), we have:
\[
    u(X_{t,\theta}^{opt}, \theta) - u(X_{t,\theta}^{alg}, \theta) = X_{t,\theta}^{opt} - X_{t,\theta}^{alg} = \frac{B}{N} - X
\]
\[
    \leq \frac{B}{E[N] - \text{CONF}_{0,T}^{N}} - \frac{B}{E[N] + \text{CONF}_{0,T}^{N}} \cdot \left(1 - \frac{c}{\sqrt{T}}\right)
\]
\[
    = \frac{B}{E[N]} \cdot \frac{1 - \frac{\text{CONF}_{0,T}^{N}}{E[N]}}{1 - \frac{\text{CONF}_{0,T}^{N}}{E[N]}} - \frac{1 - c/\sqrt{T}}{1 + \frac{\text{CONF}_{0,T}^{N}}{E[N]}}
\]
\[
    = \beta_{avg} \cdot \left(1 - \frac{1 - \frac{\text{CONF}_{0,T}^{N}}{E[N]}}{1 + \frac{\text{CONF}_{0,T}^{N}}{E[N]}}\right)
\]
Using the fact that \( \text{CONF}_{0,T}^{N} = \sqrt{2T \max \log(2T^2/\delta)} \) and \( E[N] \in \Theta(T) \), there exists \( c_1, c_2 \in \Theta(1) \) such that
\[
    \left(1 - \frac{\text{CONF}_{0,T}^{N}}{E[N]}\right)^{-1} \leq (1 - c_1/\sqrt{T})^{-1} \leq 1 + 2c_1/\sqrt{T} \quad \text{and} \quad \left(1 + \frac{\text{CONF}_{0,T}^{N}}{E[N]}\right)^{-1} \leq (1 + c_2/\sqrt{T})^{-1} \leq 1 - \frac{1}{2}c_2/\sqrt{T},
\]
where the second inequality for each of these cases holds for large enough \( T \). Plugging this into the above we have that:
\[
    u(X_{t,\theta}^{opt}, \theta) - u(X_{t,\theta}^{alg}, \theta) \leq \beta_{avg} \left(1 + 2c_1/\sqrt{T} - (1 - c/\sqrt{T})(1 - c_2/\sqrt{T})\right)
\]
\[
    \leq \beta_{avg} (2c_1 + c_2 + c)/\sqrt{T}
\]
\[
    \leq 1/\sqrt{T}.
\]

Hindsight Envy: Envy is trivially zero in this setting since every individual is given the same allocation \( X \) under the good event \( \mathcal{E} \) via Lemma 5.5.

Efficiency: We next consider the distance to efficiency bound \( \Delta_{\text{efficiency}} \). Since \( X_{t,\theta}^{opt} = B/N \) for all \( t \in [T], \theta \in \Theta \):
\[
    \Delta_{\text{efficiency}} = \sum_{t \in [T]} \sum_{\theta \in \Theta} N_{t,\theta}X_{t,\theta}^{opt} - \sum_{t \in [T]} \sum_{\theta \in \Theta} N_{t,\theta}X_{t,\theta}^{alg}
\]
\[
    \leq N(B/N - X)
\]
\[
    \leq \frac{N}{\sqrt{T}} \leq \sqrt{T},
\]
where the second inequality follows from similar manipulations as those used in the proof of counterfactual envy.

Proportionality: In this setting, \( \Delta_{\text{prop}} = \Delta_{\text{EF}} \) in this setting since \( X^{opt} = \frac{B}{N} \), so \( \Delta_{\text{prop}} = \Delta_{\text{EF}} \leq \frac{1}{\sqrt{T}} \).

5.4 Examples of Distributions
We next provide examples of perishing and arrival distributions that satisfy the conditions outlined in Theorem 5.2. In the examples that follow, for simplicity we assume the arrival process is such that \( N_t = 1 \) for all \( t \in [T] \). Similar examples can be derived for the more general setting, by defining \( \ell_b \) using the lower bound \( E[N] - \text{CONF}_{0,T}^{N} \) on \( N \), which holds under the good event \( \mathcal{E} \).

We begin our discussion with a natural first model of perishing to consider: that of i.i.d. perishing times. In a sense, such a setting is the worst case for the decision-maker, as the best they can do is
choose which items to allocate uniformly at random, resulting in a high number of ‘mistakes’ in hindsight, unless the vast majority of items perish toward the end of the time horizon. We formalize this intuition below.

**Necessary condition for i.i.d. perishing times.** Recall, \( \bar{t}_b = \inf \{ t : tX \geq \text{Prec}(b) \} \), and \( \eta_T = \frac{1}{T} \sum_{b \in [B]} \mathbb{P}(T_b < \bar{t}_b) \). In this setting, then, standard algebraic manipulations give us that \( \bar{t}_b \geq \mathbb{E}[N] - \text{Conf}_{0,T}^N \), and to have \( \eta_T \leq 1/\sqrt{T} \), the perishing process must satisfy \( \mathbb{P}(T_1 < \mathbb{E}[N] - \text{Conf}_{0,T}^N) \leq \frac{1}{\sqrt{T}} \), i.e., with probability at least \( 1 - 1/\sqrt{T} \), the goods perish within \( \widetilde{O}(\sqrt{T}) \) of the end of the time horizon. Note that this necessary condition fails to hold for one of the most standard models of perishing, in which items perish according to a Geometric distribution with parameter \( 1/T \) (that is, a constant fraction of items perish every day).

The following propositions provide examples of non-i.i.d. perishing processes that satisfy \( \eta_T \leq 1/\sqrt{T} \). For these examples, it is easy to check that the first condition of Theorem 5.2 holds via algebraic manipulations. We leave this verification to the reader.

**Proposition 5.6.** Suppose the perishing process is such that \( 1/\sqrt{T} \) fraction of units have \( T_b \sim \text{Unif}\{1, T + 1\} \), and \( 1 - 1/\sqrt{T} \) fraction of units have \( T_b \sim \text{Unif}\{T - C\sqrt{T}\log T + 1, T + 1\} \). Then, for large enough \( C > 0 \), \( \eta_T \leq 1/\sqrt{T} \).

**Proposition 5.7.** Suppose the perishing process is such that half of the units have \( T_b \sim \text{Unif}\{c_1 T + 1, T + 1\} \), and half of the units have \( T_b \sim \text{Unif}\{c_2 T + 1, T + 1\} \). If \( c_1 > 1/2 \), and \( c_2 \geq \max\{c_1, 3/2 - c_1 + \widetilde{O}(1/\sqrt{T})\} \), then, \( \eta_T \leq 1/\sqrt{T} \).

We leave a further investigation of the necessity of the condition on \( \eta_T \) to future work.

### 6 NUMERICAL RESULTS

We conclude by complementing the theoretical analysis of Guarded-Hope (for Section 4) and Static-Allocation (for Section 5) with an empirical study motivated by data from the Food Bank of the Southern Tier’s (FBST) mobile food pantry program. For each of the models described in Sections 4 and 5, we first describe the data-driven experiments, and then compare the effectiveness of our algorithms to others on the metrics of interest (Definition 3.2). All of the code for the experiments is available at (link omitted for blind review).

#### 6.1 General Utility Functions

Our experiments on resource allocation with general utility functions extend numerical considerations from Sinclair et al. [32] to the more general class of utility functions considered in Section 4, in addition to utility functions of applied interest, that fail to satisfy the assumptions upon which our theoretical results rely.

**6.1.1 Background on FBST.** The FBST operates out of 70 different distribution locations across the Southern Tier of New York State. They receive infrequent shipments of a large amount of resources (hence necessitating a large \( T \) for the number of rounds before a new shipment arrives). The schedule according to which the mobile food pantry visits the different distribution locations is fixed and known in advance. We interpret the total number of rounds \( T \) as the number of distribution locations the mobile food pantry will visit until the arrival of the next shipment of food resources. Each round \( t \) then corresponds to a visit to a specific distribution location where the food pantry decides on an allocation of resources to allocate to the (random) number of individuals congregating at that location.
6.1.2 Model of Arrival Distribution and Utilities. We conducted our experiments using data provided by the FBST in 2019, referred to as Multi-FBST model from Sinclair et al. [32], which we summarize here briefly.

We consider the setting with $K = 5$ resources (corresponding to cereal, pasta, prepared meals, rice, and meat) and three types $\Theta$ (corresponding to vegetarians, omnivores, and "prepared-food
only” individuals). For each experiment, we first pick a random collection of $T$ locations from the 2019 FBST data, and set $N_{t,\theta}$ to be sampled from $\mathcal{F}_t = \mathcal{D}_\theta \text{Normal}(\mu_t, \sigma^2_t)$ where $\mu_t$ and $\sigma^2_t$ are the mean and variance of the demand of the $t$’th selected distribution location. The distribution over the different types (vegetarians, omnivores, and “prepared-food only”) is taken as $\mathcal{D}_\theta = [.25, .3, .45]$ to be the estimated fraction of the FBST’s population with each of those preferences. We consider linear, Leontief and Cobb-Douglas utility functions, given by:

- **Leontief**: $u(x, \theta) = \min_{k \in [K]} \frac{x_k}{w_{\theta,k}}$
- **Cobb-Douglas**: $u(x, \theta) = \prod_{k=1}^{K} w_{\theta,k} x_k^{\alpha_{\theta,k}}$ where $\alpha_{\theta,k} = \frac{1}{K}$.

The weights $w_{\theta,k} = p_k \mathbb{1}\{\text{type } \theta \text{ uses resource } k\}$, and $p_k$ are the historical “prices” used by Feeding America in the non-monetary market used to distribute goods to food banks on a national scale [31]. For example, $w_{\theta,k} = 0$ for the vegetarian type and the meat resource (see Table 1 in the Appendix for the full table of weights).

### 6.1.3 Algorithmic Setup

For each of the experiments, we compare the performance of GUARDED-HOPE with $L_T = T^{-1/2}$ and $T^{-1/3}$ to a STATIC-ALLOC algorithm which always attempts to allocate according to $X$ (also can be interpreted as GUARDED-HOPE initialized with $L_T = 0$). (Note that we abuse naming conventions here: the STATIC-ALLOC we compare against in experiments for general utilities is not the STATIC-ALLOC algorithm defined in Algorithm 2, designed for perishable resources.) For all of the algorithms we use the confidence bounds defined in Lemma 4.4. We also compare against Certainty Equivalence (CE) and Certainty Equivalence + Resolve (RESOLVE CE), which respectively solve the Eisenberg-Gale program at every round $t$ using the current remaining budget and future sizes replaced with their expectation (in the case of the RESOLVE CE), and the initial budget with past sizes as their realizations and future sizes replaced with their expectations (in the case of CE).

### 6.1.4 Simulation Results

For the setup described above, we numerically evaluate the performance of the various policies. In each simulation we set the total budget $B$ to be $\sum_{t,\theta} \mathbb{E}[N_{t,\theta}]$, so that the scaling of $\beta_{\text{avg}} = 1$. We compare the algorithms in terms of the metrics outlined in Definition 3.2 in Section 5.4. Further per-metric comparisons represented as radar plots can be found in Appendix B (and Fig. 4). We make three key observations, especially in contrast to prior experimental results in Sinclair et al. [32].

**Comparison of STATIC-ALLOC vs GUARDED-HOPE.** In Section 5.4 we see that the GUARDED-HOPE algorithms (for varying values of $L_T$) outperform STATIC-ALLOC in terms of efficiency, as our algorithms greedily allocate the upper threshold while ensuring budget compliance. Moreover, under Leontief and Cobb-Douglas utilities, GUARDED-HOPE with parameter $L_T = T^{-1/2}$ achieves lower counterfactual envy than STATIC-ALLOC. This serves in contrast to the results from Sinclair et al. [32] where it was observed that both algorithms achieved similar counterfactual envy. This discrepancy is due to the way sensitivity in the guardrails is propagated under non-linear utilities. In these models we find that the counterfactual envy metric is realized by individuals receiving the upper guardrail allocation (when in linear the difference was the same).

**GUARDED-HOPE for varying $L_T$.** In Section 5.4, GUARDED-HOPE for larger values of $L_T$ achieves better efficiency, but worse performance in terms of $\Delta_{\text{EF}}$. This illustrates the tightness of the upper bounds presented in Theorem 4.3, the lower bound established in Sinclair et al. [32], and empirically validates that the so-called “envy-efficiency” trade-offs are observed in a broad class of utility models.
Empirical Results for Leontief Utilities. While Leontief utilities are utility functions of applied interest, they fail to satisfy the assumptions upon which our theoretical results rely. However, the empirical results in Section 5.4 highlight how the technique of algorithmic guardrails is robust in fair allocation models, regardless of strict monotonicity assumptions on the underlying utilities. See Appendix C for more discussion on this fact.

6.2 Perishing Resources

Here we consider a similar experiment set-up to that described in Section 6.1, and additionally consider the operational questions of allocating resources which perish over time.

6.2.1 Model of Arrival Distribution and Perishing Process. We conducted our experiments using data provided by the FBST referred to as the Multi-FBST model from [32] and described in Section 6.1.2. Since we consider only a single resource, we model individual utilities to be $u(x, \theta) = x$, for simplicity. We also scaled the total budget and arrivals to be exactly equal to $T$ for computational efficiency.

For the perishing distributions, we consider three set-ups (motivated by the examples in Section 5.4):

1. **I.I.D.**: Each resource $b \in [B]$ perishes according to $P(T_b = 1) = \frac{1}{\sqrt{T}}$ and $P(T_b = T) = 1 - \frac{1}{\sqrt{T}}$.

2. **$\sqrt{T}$ Uniforms**: Following Proposition 5.6, we set the first $1/\sqrt{T}$ fraction of resources to perish according to $T_b \sim \text{Unif}(1, T + 1)$, and the remaining resources to perish according to $T_b \sim \text{Unif}(T - \sqrt{T}, T + 1)$.

3. **Constant Uniforms**: Following Proposition 5.7, we set the first $B/2$ resources to perish according to $T_b \sim \text{Unif}(3/4T, T + 1)$ and the remaining resources to perish according to $T_b \sim \text{Unif}(4/5T, T + 1)$.

6.2.2 Algorithmic Setup. For each of the experiments we compare the performance of Static-Allocation (Algorithm 2) to other benchmarks. We include Guarded-Hope with $L_T = T - 1/2$ and $L_T = T^{-1/3}$, which follow the same principle of calculating $X$ and $\overline{X}$ as in Algorithm 1, deciding which threshold to allocate based on uncertainty in the arrivals only, assuming no perishing, and allocates the resources again in order of expected perishing time.

6.2.3 Simulation Results. As before, we let $B = \sum_{t, \theta} E[N_{t, \theta}]$, so that the scaling of $\beta_{avg} = 1$. We make two key observations.

**Comparison of Static-Allocation to Guarded-Hope.** In Figs. 3(a) and 3(b) we observe that in the I.I.D. and $\sqrt{T}$ Uniforms settings, the performance of Guarded-Hope vastly outperforms that of Static-Allocation in terms of efficiency. This highlights how using both an upper and lower algorithmic guardrail improves efficiency with little-to-no trade-off on envy via greedily allocating the upper threshold while ensuring budget compliance.

**Use of Upper and Lower Guardrails.** However, when testing the Constant Uniforms perishing process in Fig. 3(c) we observe instances where the performance of Guarded-Hope for values of $L_T > 0$ is vastly worse than that of Static-Allocation. This occurs since the Guarded-Hope algorithms were frequently running out of budget. This might be due to the fact that naively applying Guarded-Hope in this setting only takes into account uncertainty on the arrivals (instead of simultaneously accounting for both uncertainty in the arrivals and future perishing) when deciding to allocate the upper threshold. This highlights that for these "easy settings" a more nuanced implementation of the guardrails is required.
These experimental results lead to theoretical questions for future work on better understanding how to use two guardrail allocation schemes to achieve a form of “envy-efficiency” tradeoff in settings with perishable resources.
REFERENCES


A OMITTED PROOFS

A.1 Section 3 Proofs

Proof of Proposition 4.1. Consider any feasible solution $X_{t,\theta}$ to the EG program, and let $\bar{X}_{t,\theta} = \frac{\sum_{t'} N_{t',\theta} X_{t',\theta}}{N_\theta}$ for all $t$. That $\{\bar{X}_{t,\theta}\}$ is feasible follows from Linearity. We now show that the EG objective under $\{\bar{X}_{t,\theta}\}$ is no worse than that under $X_{t,\theta}$.

$$\sum_t \sum_\theta N_{t,\theta} \log u(\bar{X}_{t,\theta}, \theta) = \sum_\theta N_\theta \log u\left(\frac{\sum_{t'} N_{t',\theta} X_{t',\theta}}{N_\theta}, \theta\right) \geq \sum_\theta N_\theta \sum_{t'} \frac{N_{t',\theta}}{N_\theta} \log u(X_{t',\theta}, \theta) = \sum_t \sum_\theta N_{t,\theta} \log u(X_{t,\theta}, \theta),$$

where the inequality follows from concavity of the log function and $u$. \hfill \Box

Proof of Proposition 4.2. For concave, one-positively homothetic, and non-decreasing utilities, any solution to the EG program corresponds to a competitive equilibrium in the corresponding Fisher market (cf. Chapter 6 of Nisan et al. [30]). Formally, let $X^*$ denote an optimal allocation for the EG program, and $p^* = (p^*_{k})_{k \in [K]}$ the corresponding optimal dual variables. $(X^*, p^*)$ are said to form a competitive equilibrium if, the following two conditions hold:

1. $\sum_k N_\theta X^*_{\theta,k} = B_k$, for all $k \in [K]$, and
2. For all $\theta \in \Theta$, $X^*_{\theta}$ is an optimal solution to:

$$\max_{X_{\theta} \in \mathbb{R}^K_{+}} \ u(X_{\theta}, \theta) \quad \text{s.t.} \quad \sum_k X_{\theta,k} p^*_k \leq N_\theta,$$

Moreover, by Theorem 2.2 in Varian [33], if $(X^*, p^*)$ form a competitive equilibrium, then under this class of functions $X^*$ is a Pareto-efficient and envy-free allocation.

We conclude by showing that $X^*$ is proportional. Let $\bar{X}$ be such that $\bar{X}_{\theta} = B/N$ for all $\theta \in \Theta$. Clearly, $\bar{X}$ is feasible to the EG program. By concavity of $u$, we have:

$$u(\bar{X}_{\theta}, \theta) \leq u(X^*_{\theta}, \theta) + \nabla u(X^*_{\theta}, \theta)^T (\bar{X}_{t,\theta} - X^*_{\theta}) \Rightarrow u(X^*_{\theta}, \theta) \geq u(\bar{X}_{\theta}, \theta) - \nabla u(X^*_{\theta}, \theta)^T (\bar{X}_{t,\theta} - X^*_{\theta}).$$

Using the fact that $X^*_{\theta}$ is the utility-maximizing allocation for type $\theta \in \Theta$ and concavity of $u$, we have: $\nabla u(X^*_{\theta}, \theta)^T (\bar{X}_{\theta} - X^*_{\theta}) \leq 0$. Substituting in $\bar{X}_{\theta} = B/N$, we obtain the result. \hfill \Box

A.2 Section 4 Proofs

The following proposition relates the constants presented in the statement of the theorem to the guardrails. We defer its proof to the end of the appendix.

Proposition A.1. Under good event $E$, $C_1 \geq \max_{\theta \in \Theta} u(\bar{X}, \theta), C_2 \leq \min_{\theta \in \Theta} ||\bar{X}_\theta||_\infty, C_3 \geq \max_{\theta \in \Theta} ||\bar{X}_\theta||_\infty$.

Proof of Lemma 4.4. Fix $t, t' \in [T], \theta \in \Theta$, and assume WLOG $t' > t$. Recall, for all $r \in [T], \rho_{r,\theta} := |N_{t,\theta} - E[N_{t,\theta}]|$, which implies $N_{t,\theta} \in [E[N_{t,\theta}] - \rho_{t,\theta}, E[N_{t,\theta}] + \rho_{t,\theta}]$. Thus, from a simple application of Hoeffding’s inequality (Theorem D.1):

$$\mathbb{P}(|N_{(t,t'),\theta} - E[N_{(t,t'),\theta}]| \geq \epsilon) \leq 2 \exp\left(-\frac{2\epsilon^2}{\sum_{r \in (t,t')} 4\rho_{r,\theta}^2}\right)$$

(6)
We now consider our desired bound.
\[
\mathbb{P}(|N_{(t,t'),\theta} - \mathbb{E}[N_{(t,t'),\theta}]| \leq \varepsilon \forall t, t', \theta) \geq 1 - \sum_{t,t',\theta} \mathbb{P}(|N_{(t,t'),\theta} - \mathbb{E}[N_{(t,t'),\theta}]| \geq \varepsilon) \\
\geq 1 - \sum_{t,t',\theta} 2 \exp\left(-\frac{2 \varepsilon^2}{\sum_{r \in (t',t)} 4 \rho_{r,\theta}^2}\right) \\
\geq 1 - \sum_{t,t',\theta} 2 \exp\left(-\frac{\varepsilon^2}{2 \rho_{\max}^2 (t' - t)}\right),
\]
where the first inequality follows from a union bound, the second inequality by plugging in Hoeffding’s bound (6), and the final inequality by upper bounding \(\rho_{r,\theta}\) by \(\rho_{\max}\), for all \(r \in (t, t']\).

Solving for \(\varepsilon\) such that \(2 \exp\left(-\frac{\varepsilon^2}{2 \rho_{\max}^2 (t' - t)}\right) = \delta/(T^2|\Theta|)\), we obtain our result. \(\square\)

**Proof of Lemma 4.5.** We show the first statement by induction on \(t\). Given the first statement, the second follows by construction.

**Base Case:** \(t = 1\). We first note that \(B_1^{alg} = B\). Moreover, since \(X = x(\pi_\theta)\), we have that, for all \(k \in [K]\):
\[
\sum_\theta \pi_\theta X_{\theta,k} \leq B_{1,k}^{alg}.
\]
Plugging in the definition of \(\pi_\theta = \mathbb{E}[N_\theta](1 + \max_\theta \frac{\text{CONF}_\theta}{\mathbb{E}[N_\theta]}),\) we obtain:
\[
B_{1,k}^{alg} \geq \sum_\theta \left(\mathbb{E}[N_\theta] \left(1 + \max_\theta \frac{\text{CONF}_\theta}{\mathbb{E}[N_\theta]}\right)\right) X_{\theta,k} \\
\geq \sum_\theta \left(\mathbb{E}[N_\theta] + \text{CONF}_\theta\right) X_{\theta,k} \\
\geq \sum_\theta N_\theta X_{\theta,k},
\]
where the final inequality follows from the definition of event \(E\).

**Step Case:** \(t - 1 \rightarrow t\). We split the analysis into two cases, based on the allocation in round \(t - 1\).

If \(X_{t-1,\theta,k}^{alg} = X_{\theta,k}\), then
\[
B_{t,k}^{alg} = B_{t-1,k}^{alg} - \sum_{\theta \in \Theta} N_{t-1,\theta} X_{\theta,k} \geq \sum_{\theta \in \Theta} N_{\geq t,\theta} X_{\theta,k},
\]
where the last inequality follows from the induction hypothesis.

If \(X_{t-1,\theta,k}^{alg} = \overline{X}_{\theta,k}\), then
\[
B_{t,k}^{alg} = B_{t-1,k}^{alg} - \sum_{\theta \in \Theta} N_{t-1,\theta} X_{t-1,\theta,k} \overset{(a)}{\geq} \sum_{\theta \in \Theta} X_{\theta,k} \left(\mathbb{E}[N_{\geq t,\theta}] + \text{CONF}_{t-1,T,\theta}\right) \overset{(b)}{\geq} \sum_{\theta \in \Theta} N_{\geq t,\theta} X_{\theta,k},
\]
where (a) holds by the condition for allocating \(\overline{X}_{\theta,k}\), and (b) holds under event \(E\). \(\square\)

**Proof of Lemma 4.6.** By definition of \(t_{0,k}\), it must be that \(X_{t_{0,k},\theta,k}^{alg} = X_{\theta,k}\), and \(X_{t,k,\theta,k}^{alg} = \overline{X}_{\theta,k}\) for all \(t > t_{0,k}\). Thus, for all \(t > t_{0,k}\):
\[
\sum_\theta X_{\theta,k} \left(\mathbb{E}[N_{>t,\theta}] + \text{CONF}_{t,T,\theta}\right) \overset{(a)}{\leq} B_{t,k}^{alg} - \sum_{\theta} N_{t,\theta} \overline{X}_{\theta,k} = B_{t_{0,k}}^{alg} - \sum_{\theta} X_{\theta,k} N_{t_{0,k},\theta} - \sum_{\theta} \overline{X}_{\theta,k} \sum_{t' = t_{0,k} + 1}^{t} N_{t',\theta}
\]
where the final inequality is simply used to simplify algebraic manipulations later on. Combining these equations we obtain that for any $t > t_{0,k}$, and (c) follows from the condition in the algorithm for $X_{t,\theta;k}^{alg} = \overline{X}_{\theta;k}$.

Let $N_{(t_{0,k},1),\theta} = \sum_{\theta} \sum_{t'=t_{0,k}+1}^t N_{t',\theta}$. Re-arranging the above inequality, we obtain:

\[
\sum_{\theta} \overline{X}_{\theta;k} N_{(t_{0,k},1),\theta} < \sum_{\theta} (\overline{X}_{\theta;k} - X_{\theta,k}) N_{t_{0,k},\theta} + \sum_{\theta} X_{\theta,k} \mathbb{E}[N_{(t_{0,k},t),\theta}] + \sum_{\theta} X_{\theta,k} (\text{CONF}_{t_{0,k},T,\theta} - \text{CONF}_{t,t',\theta}) \\
\Leftrightarrow \sum_{\theta} \overline{X}_{\theta;k} (N_{(t,\theta),\theta} - \mathbb{E}[N_{(t,\theta),\theta}]) < \sum_{\theta} (\overline{X}_{\theta;k} - X_{\theta,k}) (N_{t_{0,k},\theta} - \mathbb{E}[N_{(t_{0,k},t),\theta}]) \\
+ \sum_{\theta} X_{\theta,k} (\text{CONF}_{t_{0,k},T,\theta} - \text{CONF}_{t,t',\theta})
\]

(Note that $\frac{C_2}{C_1} L_T \leq \overline{X}_{\theta;k} - X_{\theta,k} \leq \frac{C_1}{C_2} L_T$ by Lemma 4.7. Moreover, $|N_{t_{0,k},\theta} - \mathbb{E}[N_{t_{0,k},\theta}]| \leq \rho_{max} \implies N_{t_{0,k},\theta} \leq \mu_{max} + \rho_{max}$. We moreover use the fact that $\mathbb{E}[N_{t,\theta}] \geq 1$ for all $t, \theta$ to obtain the following upper bound on the first term of (7):

\[
\sum_{\theta} N_{(t_{0,k},t),\theta} - \mathbb{E}[N_{(t_{0,k},t),\theta}] \leq \frac{C_3}{C_1} L_T |\Theta| (\rho_{max} + \mu_{max}) - \frac{C_2}{C_1} L_T |\Theta| (t - t_{0,k})
\]

Putting this together with the definition of the confidence terms and the absolute upper bound $C_3$ on $||\overline{X}_{\theta;k}||_\infty$, we obtain:

\[
\sum_{\theta} \overline{X}_{\theta;k} (N_{(t_{0,k},t),\theta} - \mathbb{E}[N_{(t_{0,k},t),\theta}]) < \frac{C_3}{C_1} L_T |\Theta| (\rho_{max} + \mu_{max}) - \frac{C_2}{C_1} L_T |\Theta| (t - t_{0,k}) \\
+ C_3 |\theta| \sqrt{2 \rho_{max}^2 \log(2T^2 |\theta| / \delta) (\sqrt{T - t_{0,k}} - \sqrt{T - t})}
\]

Moreover, given event $\mathcal{E}$, we have the following lower bound on the left-hand side of (7):

\[
\sum_{\theta} \overline{X}_{\theta;k} (N_{(t_{0,k},t),\theta} - \mathbb{E}[N_{(t_{0,k},t),\theta}]) \geq -C_3 |\theta| \sqrt{2 \rho_{max}^2 \log(2T^2 |\theta| / \delta)} (t - t_{0,k}) \\
\geq -C_3 |\theta| \sqrt{2 \rho_{max}^2 \log(2T^2 |\theta| / \delta)} (T - t_{0,k}),
\]

where the final inequality is simply used to simplify algebraic manipulations later on. Combining these equations we obtain that for any $t \geq t_{0,k}$:

\[
-C_3 |\theta| \sqrt{2 \rho_{max}^2 \log(2T^2 |\theta| / \delta)} (T - t_{0,k}) < \frac{C_3}{C_1} L_T |\Theta| (\rho_{max} + \mu_{max}) - \frac{C_2}{C_1} L_T |\Theta| (t - t_{0,k}) \\
+ C_3 |\theta| \sqrt{2 \rho_{max}^2 \log(2T^2 |\theta| / \delta)} (\sqrt{T - t_{0,k}} - \sqrt{T - t})
\]

Let $t = T$ in the above expression. For ease of notation, let $\xi = \sqrt{2 \rho_{max}^2 \log(2T^2 |\theta| / \delta)}$, $\xi = \frac{C_2}{C_1}$, and $\xi = \frac{C_3}{C_1}$. We will show that there exists $c \in \Theta(1)$ such that, for all $t_{0,k} \leq T - cL_T^2$:

\[
0 \geq \xi L_T |\Theta| (\rho_{max} + \mu_{max}) - \xi L_T |\Theta| (T - t_{0,k}) + 2C_3 |\theta| \xi \sqrt{T - t_{0,k}}
\]
a contradiction. Denoting \( a_1 = \bar{\zeta}|\Theta| (\rho_{\text{max}} + \mu_{\text{max}}), a_2 = \bar{\zeta}|\Theta|, \) and \( a_3 = 2C_3|\Theta|\bar{\zeta} \) this reduces to showing:

\[
0 \geq a_1 L_T - a_2 L_T (T - t_{0,k}) + a_3 \sqrt{T - t_{0,k}}.
\]

This is a quadratic equation in terms of \( x = \sqrt{T - t_{0,k}} \) with a zero at:

\[
x = -a_3 - \sqrt{a_3^2 + 4a_2a_1L_T^2} - 2a_2L_T
\]

\[
= \frac{a_3}{2a_2L_T} + \frac{1}{2a_2L_T} \sqrt{a_3^2 + 4a_2a_1L_T^2}.
\]

Using that \( L_T = o(1) \) we note that the quadratic will be non-positive for any value of \( x \geq \frac{a_3}{2a_2L_T} + \frac{1}{2a_2L_T} \sqrt{a_3^2 + 4a_2a_1} = \tilde{c}L_T^{-1}. \) (8)

Taking this and plugging it into \( x = \sqrt{T - t_{0,k}} \) shows that the quadratic is negative so long as \( t_{0,k} \leq T - \bar{c}^2 L_T^{-2}. \)

We conclude by arguing that the right-hand side of (8) is \( \widetilde{O}(1). \) This follows by inspecting each of the individual terms \( a_1, a_2, \) and \( a_3, \) and using the fact that \( L_T = o(1). \) Thus, taking \( \tilde{c} \) in this way provides the construction for our contradiction.

Proof of Lemma 4.7. For ease of notation, let \( y = \max_{\theta} \text{CONF}_{\theta}/\mathbb{E}[N_\theta]. \) Then \( \bar{n}_\theta = \mathbb{E}[N_\theta](1 + y), \)

\( n_\theta = \mathbb{E}[N_\theta](1 - c), \) and so \( \bar{n}_\theta = 1 + \frac{c}{y}n_\theta. \)

The result follows from the following properties of solutions to the offline EG program, which we state below. We defer the proofs of these properties to the end of the appendix.

Lemma A.2. Let \( x((N_\theta)_{\theta \in \Theta}) \) denote optimal primal solutions to the Eisenberg-Gale program (Eq. (EG)) for a given vector of arrivals of each type \( (N_\theta)_{\theta \in \Theta}, \) and fix \( \zeta > 0. \) Then, we have that:

1. Scaling: If \( \bar{N}_\theta = (1 + \zeta)N_\theta \) for every \( \theta \in \Theta \) and \( \zeta \geq 0, \) then:

\[
x((\bar{N}_\theta)_{\theta \in \Theta}) = \frac{x((N_\theta)_{\theta \in \Theta})}{1 + \zeta}
\]

\[
u(x((\bar{N}_\theta)_{\theta \in \Theta})) - u(x((N_\theta)_{\theta \in \Theta})) = \left(1 - \frac{1}{1 + \zeta}\right) u(x((N_\theta)_{\theta \in \Theta}))
\]

2. Monotonicity: If \( N_\theta \leq \bar{N}_\theta \) for every \( \theta \in \Theta \) then

\[
u(x((\bar{N}_\theta)_{\theta \in \Theta})) \leq u(x((N_\theta)_{\theta \in \Theta})) \quad \forall \theta \in \Theta
\]

Given this lemma, we first show Parts 1-3 of the claim. By Part 1 of Lemma A.2, we have that:

\[
u(x((\bar{n}_\theta)_{\theta \in \Theta})) - u(x((n_\theta)_{\theta \in \Theta})) = \left(1 - \frac{1}{1 + y}\right) u(x((n_\theta)_{\theta \in \Theta})) \leq \frac{c + y}{1 + y} C_1 = L_T.
\]

Using Part 1 of Lemma A.2 once again we can lower bound the difference in allocations by

\[
\|x((\bar{n}_\theta)_{\theta \in \Theta}) - x((n_\theta)_{\theta \in \Theta})\|_\infty = \left(1 - \frac{1 - c}{1 + y}\right) \|x((n_\theta)_{\theta \in \Theta})\|_\infty \geq \frac{c + y}{1 + y} C_2 = \frac{C_2}{C_1} L_T.
\]

The argument for the upper bound is identical (where instead we use \( \|x((\bar{N}_\theta)_{\theta \in \Theta})\|_\infty \leq C_3). \)

We now show Parts 4 and 5 of the claim. We have:

\[
N_\theta = \mathbb{E}[N_\theta] + (N_\theta - \mathbb{E}[N_\theta]) = \mathbb{E}[N_\theta]\left(1 + \frac{N_\theta - \mathbb{E}[N_\theta]}{\mathbb{E}[N_\theta]}\right) \quad (9)
\]
Under event $\mathcal{E}$, $\text{CONF}_\theta$ upper bounds $|N_\theta - \mathbb{E}[N_\theta]|$. Plugging this into Eq. (9), we obtain that $N_\theta \leq \mathbb{E}[N_\theta] \left(1 + \max_\theta \frac{\text{CONF}_\theta}{\mathbb{E}[N_\theta]}\right) = \bar{n}_\theta$ given $\mathcal{E}$.

Via symmetric reasoning, $N_\theta \geq \mathbb{E}[N_\theta] \left(1 - \max_\theta \frac{\text{CONF}_\theta}{\mathbb{E}[N_\theta]}\right)$ given $\mathcal{E}$. Simple algebraic manipulations and the assumption that $L_T \geq 2C_1 \max_\theta \frac{\text{CONF}_\theta}{\mathbb{E}[N_\theta]}$ show that $\mathbb{E}[N_\theta] \left(1 - \max_\theta \frac{\text{CONF}_\theta}{\mathbb{E}[N_\theta]}\right) \geq n_\theta$, and thus $N_\theta \geq n_\theta$.

Finally, the fact that $u(x(\bar{n}_\theta)_{\theta \in \Theta}) \leq u(X^*_{\theta}, \theta) \leq u(x(n_\theta)_{\theta \in \Theta}, \theta)$ given $\mathcal{E}$ follows from putting Part 4 of the above claim and the monotonicity property in Lemma A.2 together.

A.2.1 Proofs of Auxiliary Results.

Proof of Proposition A.1. For $C_1$, note that $\bar{X} \leq B$ by the budget constraint, so $\max_{\theta \in \Theta} u(\bar{X}_\theta, \theta) \leq \max_{\theta \in \Theta} u(B, \theta)$, since $u$ is non-decreasing.

For $C_2$ we use that $\bar{X}$ satisfies $B_k = \sum_{\theta} \bar{X}_{\theta,k} n_\theta$ for every $\theta$ by Pareto efficiency of the optimal solutions to the EG program, and the assumption that the utilities are strictly monotone. Using Cauchy-Schwarz and the fact that $n_\theta \leq \mathbb{E}[N_\theta]$ we notice:

$$B_k = \sum_{\theta} n_\theta \bar{X}_{\theta,k} \leq (\max_{\theta} \bar{X}_{\theta,k}) \sum_{\theta} n_\theta = \|\bar{X}_\theta\|_\infty \mathbb{E}[N]$$

$$\implies \|\bar{X}_\theta\|_\infty \geq B_k/\mathbb{E}[N] \geq B_{\min}/\mathbb{E}[N].$$

Lastly, for $C_3$ we note again by the budget constraint that $\max_{\theta \in \Theta} \|\bar{X}_\theta\|_\infty \leq \|B\|_\infty$. □

Proof of Lemma A.2. Let $(p_k)_{k \in [K]}$ and $(\alpha_{\theta,k})_{\theta \in \Theta, k \in [K]}$ respectively denote optimal dual solutions for the EG program for an arbitrary vector of arrivals. The KKT conditions for the EG program are given by:

1. Primal Feasibility: $\sum_{\theta \in \Theta} N_\theta X_{\theta,k} \leq B_k$ for all $k$, $X_{\theta,k} \geq 0$ for all $\theta, k$
2. Dual Feasibility: $p_k \geq 0$ for all $k$, $\alpha_{\theta,k} \geq 0$ for all $\theta, k$
3. Complementary Slackness:

$$\left\{ \begin{array}{l} p_k > 0 \implies \sum_{\theta \in \Theta} N_\theta X_{\theta,k} = B_k \\ \alpha_{\theta,k} > 0 \implies X_{\theta,k} = 0 \end{array} \right.$$  

4. Gradient Condition:

$$-N_\theta \frac{\partial}{\partial x_{\theta,k}} u(X_\theta, \theta) u(X_\theta, \theta) + N_\theta p_k - \alpha_{\theta,k} = 0 \quad \forall \theta \in \Theta, k \in [K].$$

Using the fact that $\alpha_{\theta,k} \geq 0$, with equality if $X_{\theta,k} > 0$ we find that:

$$p_k \geq \frac{\partial}{\partial x_{\theta,k}} u(X_\theta, \theta) u(X_\theta, \theta) \quad \forall \theta \in \Theta, k \in [K].$$  

with equality whenever $X_{\theta,k} > 0$.

We now use these conditions to prove the scaling and monotonicity properties. For ease of notation, we let $X^*$ and $(p^*, \alpha^*)$ denote optimal primal and dual solutions corresponding to arrivals $(N_\theta)_{\theta \in \Theta}$.

**Scaling:** Suppose $\tilde{N}_\theta = (1 + \zeta)N_\theta$ for every $\theta \in \Theta$.

For all $k \in [K]$, $\theta \in \Theta$, define $\tilde{p}_k = (1 + \zeta) p^*_k$, $\tilde{X}_{\theta,k} = X^*_{\theta,k} / (1 + \zeta)$, and $\tilde{\alpha}_{\theta,k} = (1 + \zeta)^2 \alpha^*_{\theta,k}$. Since our constraint set is linear, it suffices to check that $\tilde{X}$, $\tilde{p}$ satisfy the KKT conditions.
Dual feasibility follows from the fact that $\zeta \geq 0$, and primal feasibility follows from:

$$\sum_{\theta} \bar{N}_\theta \tilde{X}_{\theta,k} = \sum_{\theta} (1 + \zeta)N_\theta \frac{X^*_{\theta,k}}{1 + \zeta} \leq B_k$$

by feasibility of $X^*_{\theta,k}$.

Complementary slackness holds since the resource utilizations are equal under both arrival vectors, and $p^*_k > 0 \iff \tilde{p}_k > 0$. A similar argument holds for $\tilde{a}_{\theta,k}$.

Finally, we verify the gradient condition. Since $u$ is one-positively homogeneous, it follows that its partial derivatives are 0-positively homogeneous, and as a result $\frac{\partial u}{\partial X_{\theta,k}}|_{X_\theta = \tilde{X}_\theta} = \frac{\partial u}{\partial X_{\theta,k}}|_{X_\theta = X^*_\theta}$.

Thus, we have that:

$$-\frac{\bar{N}_\theta}{u(\tilde{X}_\theta, \theta)} \frac{\partial}{\partial X_{\theta,k}} u(X_\theta, \theta)|_{X_\theta = \tilde{X}_\theta} + \bar{N}_\theta \tilde{p}_k - \bar{a}_{\theta,k} = \frac{-(1 + \zeta)\bar{N}_\theta}{1 + \zeta} \frac{\partial}{\partial X_{\theta,k}} u(X_\theta, \theta)|_{X_\theta = X^*_\theta} + (1 + \zeta)^2 N_\theta p_k - (1 + \zeta)^2 \tilde{a}_{\theta,k}$$

$$= (1 + \zeta)^2 \left( \frac{-N_\theta}{u(\tilde{X}_\theta, \theta)} \frac{\partial}{\partial X_{\theta,k}} u(X_\theta, \theta)|_{X_\theta = X^*_\theta}}{u(\tilde{X}_\theta, \theta)} + N_\theta p_k - \alpha_{\theta,k} \right)$$

$$= 0$$

by optimality of $X^*_{\theta,k}, p^*_k, \alpha^*_{\theta,k}$.

Plugging optimality of $\tilde{X}_\theta = X^*_{\theta}/(1 + \zeta)$, we obtain:

$$u(X^*_{\theta}, \theta) - u(\tilde{X}_\theta, \theta) = u(X^*_{\theta}, \theta) - u(X^*_{\theta}/(1 + \zeta), \theta) = \left( 1 - \frac{1}{1 + \zeta} \right) u(X^*_{\theta}, \theta),$$

where the final equality follows from the fact that the utility functions are one-homogeneous.

**Monotonicity:** As before, let $X^*$ and $\tilde{X}$ respectively denote optimal solutions to the EG program under $(N_\theta)_{\theta \in \Theta}$ and $(\bar{N}_\theta)_{\theta \in \Theta}$ arrivals, with $\bar{N}_\theta \geq N_\theta$ for all $\theta$, and a strict inequality for at least one type $\theta$. Similarly, let $(p^*_k)_{k \in [K]}$ and $(\tilde{p}_k)_{k \in [K]}$ be the corresponding dual solutions.

Suppose for contradiction that there exists a type $\theta \in \Theta$ such that $u(\tilde{X}_\theta, \theta) > u(X^*_{\theta}, \theta)$. Since $u(\cdot, \theta)$ is non-decreasing, it must be that $\tilde{X}_{\theta,k} > X^*_{\theta,k}$ for some $k \in [K]$. By the KKT condition (10), we have:

$$u(\tilde{X}_\theta, \theta) = \frac{\partial}{\partial X_{\theta,k}} u(X_\theta, \theta)|_{X_\theta = \tilde{X}_\theta} > u(X^*_{\theta}, \theta) \geq \frac{\frac{\partial}{\partial X_{\theta,k}} u(X_\theta, \theta)|_{X_\theta = X^*_\theta}}{p^*_k}.$$

Moreover, since $u$ is concave and $\tilde{X}_{\theta,k} > X^*_{\theta,k}$, $\frac{\partial}{\partial X_{\theta,k}} u(X_\theta, \theta)|_{X_\theta = \tilde{X}_\theta} \leq \frac{\partial}{\partial X_{\theta,k}} u(X_\theta, \theta)|_{X_\theta = X^*_\theta}$. Thus, it must be that $\tilde{p}_k < p^*_k$.

Now, note that, since $\tilde{X}_{\theta,k} > X^*_{\theta,k}$, it must be that $\tilde{X}_{\theta',k} < X^*_{\theta',k}$ for some $\theta' \in \Theta$ since $X^*$ clears the resources $B_k$. Again, using concavity of $u$, this implies that $\frac{\partial}{\partial X_{\theta,k}} u(X_{\theta', \theta'}, \theta')|_{X_{\theta', \theta'} = \tilde{X}_{\theta'}} \geq \frac{\partial}{\partial X_{\theta,k}} u(X_{\theta', \theta'}, \theta')|_{X_{\theta', \theta'} = X^*_{\theta'}}$. Putting this together with the fact that $\tilde{p}_k < p^*_k$, we obtain:

$$\frac{\partial}{\partial X_{\theta', k}} u(X_{\theta', \theta'}, \theta')|_{X_{\theta', \theta'} = \tilde{X}_{\theta'}} \leq \frac{\partial}{\partial X_{\theta', k}} u(X_{\theta', \theta'}, \theta')|_{X_{\theta', \theta'} = X^*_{\theta'}} \frac{\tilde{p}_k}{p^*_k},$$

and by the KKT condition (10), $u(\tilde{X}_{\theta', \theta'}) > u(X^*_{\theta', \theta'})$. 

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Note however that, since \((\tilde{N}_0)_{\theta \in \Theta} \geq (N_0)_{\theta \in \Theta}\), \(\tilde{X}\) is feasible to the EG program under \((N_0)_{\theta \in \Theta}\). Thus, the objective for this latter program under \(\tilde{X}\) strictly improves upon the objective under \(X^*\), which contradicts optimality of \(X^*\). \(\square\)

### A.3 Section 5 Proofs

**Proof of Proposition 5.1.** As before, let \((p_t)_{t \in [T]}\) and \((\alpha_t, \theta)_{t \in [T], \theta \in \Theta}\) denote the dual variables corresponding to the budget constraints and nonnegativity constraints, respectively. Then, by the KKT conditions, we have:

1. \(p_t > 0 \iff \sum_{\tau \geq t} \sum_{\theta} N_{\tau, \theta} X_{\tau, \theta} \leq B - P_{< t}\)
2. \(\alpha_t, \theta > 0 \iff X_{t, \theta} = 0\)
3. \(\frac{1}{X_{t, \theta}} \leq \sum_{\tau \leq t} p_{\tau}, \text{ with equality whenever } X_{t, \theta} > 0.\)

Let \(p_1 = N/B\), and \(p_t = 0\) for all \(t \geq 2\). This construction satisfies properties (1) and (3). It thus suffices to check feasibility, i.e. that \(\sum_{\tau \geq t} \sum_{\theta} N_{\tau, \theta} X_{\tau, \theta} \leq B - P_{< t}\) for all \(t\), for \(X_{t, \theta} = B/N\). This holds if, for all \(t\):

\[
\frac{B}{N} N_{\geq t} \leq B - P_{< t} \iff P_{< t} \leq B \left(1 - \frac{N_{\geq t}}{N}\right) = B \cdot \frac{N_{< t}}{N},
\]

which holds by assumption. \(\square\)

**Proof of Lemma 5.3.** By definition, \(P_{< t} = \sum_{b \in [B]} \mathbb{1}\{T_b < t\}\), and \(\mathbb{E}[P_{< t}] = \sum_{b \in [B]} \mathbb{P}(T_b < t)\). Moreover, for \(b \in [B], \mathbb{1}\{T_b < t\}\) is a Bernoulli random variable with probability of success \(\mathbb{P}(T_b < t)\). Using Theorem D.2 we then have that:

\[
\mathbb{P}(|P_{< t} - v_t| \geq \epsilon v_t) \leq 2 \exp\left( - \frac{v_t \epsilon^2}{3} \right).
\]

Setting the right-hand side equal to \(\delta\) and solving for \(\epsilon\) yields \(\epsilon = \sqrt{\frac{2}{v_t} \log(2/\delta)}\). Thus we have that with probability at least \(1 - \delta\):

\[
|P_{< t} - v_t| \leq v_t \epsilon = \sqrt{3 v_t \log(2/\delta)}.
\]

Relabeling \(\delta\) to \(\delta/T\) and taking a union bound over \(t\) shows the desired result. \(\square\)

**Proof of Lemma 5.4.** Under good event \(E\), we have:

\[
P_{\leq t} \leq \mathbb{E}[P_{\leq t}] + \text{Conf}_t^P \leq \frac{\mathbb{E}[N_{\leq t}] - \text{Conf}_{0,t}^N}{\mathbb{E}[N]} B \leq \frac{N_{\leq t}}{N} B,
\]

where the second inequality follows from Assumption (1) of the Theorem, and the third inequality follows again from the fact that, under good event \(E\), \(\mathbb{E}[N_{\geq t}] - \text{Conf}_{0,t}^N \leq N_{\geq t}\), and \(N \leq \mathbb{E}[N] + \text{Conf}_{0,T}^N\). Then, by Proposition 5.1, \(X^{opt}\) is optimal given \(E\). \(\square\)

**Proof of Lemma 5.5.** We prove the claim by induction.

**Base case:** \(t = 1\). We have:

\[
B_{alg}^1 = B = \frac{B}{\mathbb{E}[N] + \text{Conf}_{0,T}^N} \geq \frac{B}{\mathbb{E}[N] + \text{Conf}_{0,T}^N} \cdot \text{by event } E
\]

\[
\geq \frac{N}{\mathbb{E}[N] + \text{Conf}_{0,T}^N} \cdot \text{by definition of } X
\]
Inductive step: $t - 1 \rightarrow t$. We have:

$$B_{t}^{\text{alg}} = B_{t-1}^{\text{alg}} - N_{t-1}X_{t-1}^{\text{alg}} - \sum_{b \in [B_{t-1}^{\text{alg}}]} 1 \{T_{b} = t - 1, b \notin \mathcal{A}_{t-1}\},$$

where the final term corresponds to the set of unallocated items that perished at $t - 1$. Now, by the inductive hypothesis, at $t - 1$ the algorithm had enough budget remaining to underline $X$, so we have:

$$B_{t}^{\text{alg}} = B_{t-1}^{\text{alg}} - N_{t-1}X - \sum_{b \in [B]} 1 \{T_{b} = t - 1, b \notin \mathcal{A}_{t-1}\}.$$

Rolling this out, we obtain:

$$B_{t}^{\text{alg}} = B - N_{\leq t}X - \sum_{b \in [B]} \sum_{t' < t} 1 \{T_{b} = t', b \notin \mathcal{A}_{t} \cup \mathcal{A}_{t'} \forall \tau < t'\}$$

$$= B - N_{\leq t}X - \sum_{b \in [B]} 1 \{T_{b} < t_{b}^{\text{alg}}\},$$

by definition of $t_{b}^{\text{alg}}$.

The following lemma allows us to bound the third term, as a function of $t_{b}^{\text{alg}}$. We defer its proof to the end of this section.

**Lemma A.3.** $t_{b}^{\text{alg}} \leq \tilde{t}_{b}$, almost surely.

We then have:

$$B_{t}^{\text{alg}} \geq B - N_{\leq t}X - \sum_{b \in [B]} 1 \{T_{b} < \tilde{t}_{b}\}.$$ 

We seek to show that, given $\mathcal{E}$,

$$B - N_{\leq t}X - \sum_{b \in [B]} 1 \{T_{b} < \tilde{t}_{b}\} \geq N_{\geq t}X,$$

which is equivalent to showing

$$B - \sum_{b \in [B]} 1 \{T_{b} < \tilde{t}_{b}\} \geq NX.$$

However, $\sum_{b \in [B]} 1 \{T_{b} < \tilde{t}_{b}\} = P_{< \tilde{t}_{b}}$ by definition and so under the good event $\mathcal{E}$ we know that:

$$\sum_{b \in B} 1 \{T_{b} < \tilde{t}_{b}\} \leq \sum_{b \in [B]} \mathbb{P}(T_{b} < \tilde{t}_{b}) + \sqrt{3 \sum_{b \in [B]} \mathbb{P}(T_{b} < \tilde{t}_{b}) \log(2T/\delta)}$$

$$\leq \left(1 + \sqrt{3 \log(2T/\delta)}\right) \sum_{b \in [B]} \mathbb{P}(T_{b} < \tilde{t}_{b}).$$

Similarly, under $\mathcal{E}$ we have that $NX \leq (\mathbb{E}[N] + \text{Conf}_{0,T}^{N}X) = B(1 - \frac{c}{\sqrt{T}})$. Thus it suffices to show that:

$$B - \left(1 + \sqrt{3 \log(2T/\delta)}\right) \sum_{b \in [B]} \mathbb{P}(T_{b} < \tilde{t}_{b}) \geq B(1 - \frac{c}{\sqrt{T}}).$$
\[
\Rightarrow \frac{1}{B} \sum_{b \in [B]} \mathbb{P}(T_b < \tilde{t}_b) = \eta_T \leq \frac{c}{1 + \sqrt{3\log(2T/\delta)}} \frac{1}{\sqrt{T}}
\]

which, by taking \( c \geq 1 + \sqrt{3\log(2T/\delta)} \), proves the result. \( \square \)

### A.3.1 Examples of perishing processes satisfying \( \eta_T \leq 1/\sqrt{T} \)

**Proof of Proposition 5.6.** Let \( \tilde{t}^{(1)} \) and \( \tilde{t}^{(2)} \) respectively denote the upper bound on allocation time for the first and second half of the units. Then, \( \tilde{t}^{(1)} = \frac{B/\sqrt{T}(\mathbb{E}[N] + \text{Conf}_0)}{B/(1-\sqrt{T})} \leq \frac{1}{\sqrt{T}}(\mathbb{E}[N] + \text{Conf}_0)(1 + 2c/\sqrt{T}) \), for large enough \( T \). Similarly, \( \tilde{t}^{(2)} \leq (\mathbb{E}[N] + \text{Conf}_0)(1 + 2c/\sqrt{T}) \).

Then, we have:

\[
\frac{1}{B} \sum_{b \in [B]} \mathbb{P}(T_b < \tilde{t}_b) \leq \frac{1}{\sqrt{T}}(\mathbb{E}[N] + \text{Conf}_0)(1 + 2c/\sqrt{T}) - 1
\]

\[
+ \left(1 - \frac{1}{\sqrt{T}}\right) (\mathbb{E}[N] + \text{Conf}_0)(1 + 2c/\sqrt{T}) - (T - C\sqrt{T}\log T)
\]

Thus, it suffices to show that:

\[
\frac{1}{\sqrt{T}}(\mathbb{E}[N] + \text{Conf}_0)(1 + 2c/\sqrt{T}) + \left(1 - \frac{1}{\sqrt{T}}\right)(\mathbb{E}[N] + \text{Conf}_0)(1 + 2c/\sqrt{T}) - (T - C\sqrt{T}\log T) \leq 1,
\]

which holds for large enough \( C > 0 \). \( \square \)

**Proof of Proposition 5.7.** Let \( \tilde{t}^{(1)} \) and \( \tilde{t}^{(2)} \) respectively denote the upper bound on allocation time for the first and second half of the units. Then, \( \tilde{t}^{(1)} = \frac{B/2(\mathbb{E}[N] + \text{Conf}_0)}{B/(1-\sqrt{T})} \leq \frac{1}{2} (\mathbb{E}[N] + \text{Conf}_0)(1 + 2c/\sqrt{T}) \), for large enough \( T \). Similarly, \( \tilde{t}^{(2)} \leq (\mathbb{E}[N] + \text{Conf}_0)(1 + 2c/\sqrt{T}) \).

Then, we have:

\[
\frac{1}{B} \sum_{b \in [B]} \mathbb{P}(T_b < \tilde{t}_b) \leq \frac{1}{2} \left(\mathbb{E}[N] + \text{Conf}_0\right)(1 + 2c/\sqrt{T}) - c_1 T
\]

\[
+ \frac{1}{2} \left(\mathbb{E}[N] + \text{Conf}_0\right)(1 + 2c/\sqrt{T}) - c_2 T
\]

Using the fact that \( c_1 < c_2 \), it suffices to show that:

\[
\frac{1}{2} \left(\mathbb{E}[N] + \text{Conf}_0\right)(1 + 2c/\sqrt{T}) - c_1 T + \left(\mathbb{E}[N] + \text{Conf}_0\right)(1 + 2c/\sqrt{T}) - c_2 T \leq 2(1 - c_1) \sqrt{T}
\]

\[
\iff c_2 \geq \frac{1}{T} \left(\frac{3}{2} \left(\mathbb{E}[N] + \text{Conf}_0\right)(1 + 2c/\sqrt{T}) - c_1 T - 2(1 - c_1) \sqrt{T}\right).
\]

which is satisfied by \( c_2 \geq \frac{1}{T} - c_1 + \tilde{O}(1/\sqrt{T}) \). \( \square \)

### A.3.2 Proofs of Auxiliary Results

**Proof of Lemma A.3.** The claim follows from the following two facts. Firstly, that under good event \( E_t \), \( \mathbb{E}[N_{\leq T}] - \text{Conf}_0 \leq N_{\leq T} \), and as a result \( \tilde{t}_b \) is an upper bound on the time at which \( b \) would have been allocated under the true arrival sequence \( (N_{t,\theta})_{t \in \mathbb{T}, \theta \in \Theta} \), if no item \( b' \) ranked above \( b \) in the schedule perished beforehand. Secondly, during the algorithm’s execution, there
may have been an item \( b' \) with \( \mathbb{E}[T_{b'}] \leq \mathbb{E}[T_b] \) which perished before it was meant to be allocated, in which case item \( b \) “moved up” in the rank of items to be allocated next.

\[ \Box \]

B EXPERIMENT DETAILS

Table 1. Weights \( w_k \) for the different products considered in the Multi-FBST experiments. Here we use the weights taken from the historical prices used in the market mechanism to distribute food resources to food pantries across the United States [31].

<table>
<thead>
<tr>
<th>Resource</th>
<th>Cereal</th>
<th>Pasta</th>
<th>Prepared Meals</th>
<th>Rice</th>
<th>Meat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weights (type ( \theta = \text{omnivore} ))</td>
<td>3.9</td>
<td>3</td>
<td>2.8</td>
<td>2.7</td>
<td>1.9</td>
</tr>
<tr>
<td>Weights (type ( \theta = \text{vegetarian} ))</td>
<td>3.9</td>
<td>3</td>
<td>0</td>
<td>2.7</td>
<td>0</td>
</tr>
<tr>
<td>Weights (type ( \theta = \text{prepared-only} ))</td>
<td>3.9</td>
<td>3</td>
<td>2.8</td>
<td>2.7</td>
<td>0</td>
</tr>
</tbody>
</table>

Metrics Included: In the line plots we include three plots of the following:

- \( \mathbb{E} [\Delta_{EF}] = \mathbb{E} \left[ \| u(X_{t,\theta}^{op,t} , \theta) - u(X_{t,\theta}^{alg,t} , \theta) \|_\infty \right] \), the expected maximum difference between utility individuals receive for the two allocations as we scale the number of rounds \( T \) (Definition 3.2)
• $\mathbb{E}[\Delta_{\text{efficiency}}] = \mathbb{E}\left[\sum_k B_k - \sum_{t,\theta} N_{t,\theta} X_{t,\theta,k}^{\text{alg}}\right]$, the expected leftover resources as we scale the number of rounds $T$ (Definition 3.2)
• $\mathbb{E}[\text{ENVY}] = \mathbb{E}\left[\max_{t,\tau,\rho,\theta} u(\mathbf{X}_{t,\theta}^{\text{alg}}, \theta) - u(\mathbf{X}_{t,\rho}^{\text{alg}}, \theta)\right]$, the expected hindsight envy as we scale the number of rounds $T$ (Definition 3.2)

In the radar plots (see Fig. 4) we include:
• $\mathbb{E}[\Delta_{\text{EF}}] = \mathbb{E}\left[\|u(\mathbf{X}_{t,\theta}^{\text{opt}}, \theta) - u(\mathbf{X}_{t,\theta}^{\text{alg}}, \theta)\|_{\infty}\right]$, the expected maximum difference between utility individuals receive for the two allocations (Definition 3.2)
• $\mathbb{E}[\Delta_{\text{efficiency}}] = \mathbb{E}\left[\sum_k B_k - \sum_{t,\theta} N_{t,\theta} X_{t,\theta,k}^{\text{alg}}\right]$, the expected leftover resources (Definition 3.2)
• $\mathbb{E}[\text{ENVY}]$, the expected maximum envy between any two agents (Definition 3.2)
• $\mathbb{E}[\Delta_{\text{prop}}]$, the expected leftover envy between an agent and equal allocation (Definition 3.2)

**Experiment Setup**: Each experiment was run with 200 iterations where the relevant plots are taking the mean of the related quantities. In all experiments the budget $B = \sum_{t,\theta} \mathbb{E}\left[N_{t,\theta}\right]$ so that $\beta_{\text{avg}}$ scales as a constant as we vary the number of rounds $T$. All randomness is dictated by a seed set at the start of each simulation for verifying results.

**Computing Infrastructure**: The experiments were conducted on a personal computer with an AMD Ryzen 5 3600 6-Core 3.60 GHz processor and 16.0GB of RAM. No GPUs were harmed in these experiments.

C LEONTIEF UTILITIES

In this section, leveraging the observation that $\epsilon$-perturbed Leontief utilities satisfy the requirements for our main results of Section 4, we obtain bounds on ($\Delta_{\text{EF}}, \Delta_{\text{efficiency}}, \text{ENVY}$) for Leontief utilities, given by:

$$u(x, \theta) = \min\{x_i / w_{\theta,k}\} \quad \forall \theta \in \Theta.$$ 

Let $u_{\epsilon}(x, \theta) = u(x, \theta) + \epsilon\langle w_{\theta}, x \rangle$ for some $\epsilon > 0$. We refer to $u(x, \theta)$ as the nominal utility function and $u_{\epsilon}(x, \theta)$ as the perturbed utility function. Finally, we introduce the following notions of $\epsilon$-perturbed fairness.

**Definition C.1 (Perturbed Counterfactual Envy, Hindsight Envy, and Efficiency).** Given individuals with types $\Theta$, sizes $(N_{t,\theta})_{t \in [T], \theta \in \Theta}$, and resource budgets $(B_k)_{k \in [K]}$, for any online allocation $(\mathbf{X}_{t,\theta}^{\text{alg}})_{t \in [T], \theta \in \Theta} \in \mathbb{R}^K$, we define:

- **Counterfactual Envy**: The counterfactual distance of $\mathbf{X}^{\text{alg}}$ to envy-freeness as

  $$\Delta_{\text{EF}}(\epsilon_1, \epsilon_2) = \max_{t \in [T], \theta \in \Theta} \|u_{\epsilon_1}(\mathbf{X}_{t,\theta}^{\text{alg}}, \theta) - u_{\epsilon_2}(\mathbf{X}_{t,\theta}^{\text{opt}(\epsilon_2)}, \theta)\|_{\infty}$$

  where $\mathbf{X}^{\text{opt}(\epsilon_2)}$ is the solution to (EG) with true values $(N_{t,\theta})_{t \in [T], \theta \in \Theta}$ and utilities $u_{\epsilon_2}$.

- **Hindsight Envy**: The hindsight distance of $\mathbf{X}^{\text{alg}}$ to envy-freeness as

  $$\text{ENVY}(\epsilon) = \max_{t, \tau \in [T], \theta, \rho \in \Theta} u_{\epsilon}(\mathbf{X}_{t,\theta}^{\text{alg}}, \theta) - u_{\epsilon}(\mathbf{X}_{t,\rho}^{\text{alg}}, \theta).$$

- **Efficiency**: The distance to efficiency as

  $$\Delta_{\text{efficiency}} = \sum_{k \in K} \left(B_k - \sum_{t \in [T]} \sum_{\theta \in \Theta} N_{t,\theta} X_{t,\theta,k}^{\text{alg}}\right)$$
Given these definitions, obtaining guarantees with respect to the nominal Leontief utilities is equivalent to obtaining $\epsilon$-perturbed fairness guarantees on $\Delta_{\text{EF}}(0, \epsilon), \text{Envy}(0),$ and $\Delta_{\text{efficiency}}$ for some $\epsilon > 0$. Note that we use $X^{\text{opt}(\epsilon)}$ as the benchmark allocation in this setting, as in general solutions to EG under nominal utilities may not be fair without strict monotonicity.

Given this setup, the following theorem shows that we are able to obtain the desired bounds on the nominal utilities by running GUARDED-HOPE on the perturbed utilities, for an appropriately chosen value of $\epsilon$.

**Theorem C.2.** Fix $L_T = o(1)$, and let GUARDED-HOPE be initialized with utility functions $u_\epsilon$ where $\epsilon = L_T$, and

$$
\text{CONF}_{t', t, \theta} = \sqrt{2(t' - t)\rho_{\max}^2 \log(2T^2\Theta/\delta)} \quad \forall t' > t \in [T], \theta \in \Theta
$$

$$
\bar{n}_\theta = \mathbb{E}[N_\theta] \left(1 + \max_{\theta} \frac{\text{CONF}_{0, T, \theta}}{\mathbb{E}[N_\theta]}\right) \quad \forall \theta \in \Theta
$$

$$
\bar{n}_\theta = \mathbb{E}[N_\theta] (1 - c) \quad \forall \theta \in \Theta, c = \frac{L_T}{C_1} \left(1 + \max_{\theta} \frac{\text{CONF}_{0, T, \theta}}{\mathbb{E}[N_\theta]}\right) - \max_{\theta} \frac{\text{CONF}_{0, T, \theta}}{\mathbb{E}[N_\theta]}.
$$

Then, with probability at least $1 - \delta$, GUARDED-HOPE achieves:

$$
\text{Envy}(0) \leq L_T \quad \Delta_{\text{EF}}(0, L_T) \leq \max\{1/\sqrt{T}, L_T\} \quad \Delta_{\text{efficiency}} \leq \min\{\sqrt{T}, 1/L_T\}
$$

where $\leq$ drops poly-logarithmic factors of $T$, $o(1)$ terms, and absolute constants.

**Proof of Theorem C.2.** The proof follows from Theorem 4.3 by relating $(\Delta_{\text{EF}}(\epsilon_1, \epsilon_2), \text{Envy}(\epsilon), \Delta_{\text{efficiency}})$ to their quantities under the nominal utilities. In particular, we first show the following lemma:

**Lemma C.3.** For any allocation $X^{\text{alg}}$ we have that:

$$
\text{Envy}(0) \leq \text{Envy}(\epsilon) + \epsilon \|B\|_\infty \|w'_0\|_1
$$

$$
\Delta_{\text{EF}}(0, \epsilon) \leq \Delta_{\text{EF}}(\epsilon, \epsilon) + \epsilon \|w'_0\|_1.
$$

**Proof of Lemma C.3.** We start with $\text{Envy}(0)$. Fix all $t, t', \theta, \theta'$, we have:

$$
u(X_{t', \theta'}, \theta) - u(X_{t, \theta}, \theta) = u_e(X_{t', \theta'}, \theta) - u_e(X_{t, \theta}, \theta) - \epsilon \langle w'_{\theta'}, X_{t', \theta'} \rangle + \epsilon \langle w'_{\theta}, X_{t, \theta} \rangle \leq u_e(X_{t', \theta'}, \theta) - u_e(X_{t, \theta}, \theta) + \epsilon \langle X_{t, \theta}, w'_0 \rangle \leq u_e(X_{t', \theta'}, \theta) - u_e(X_{t, \theta}, \theta) + \epsilon \|w'_0\|_1 \|B\|_\infty \leq \text{Envy}(\epsilon) + \epsilon \|w'_0\|_1 \|B\|_\infty,$$

where the final inequality uses the fact that the allocations are upper bounded by $B$, by the feasibility constraint. The bound on $\text{Envy}(0)$ follows by taking the max over $t, t', \theta, \theta'$.

We next consider $\Delta_{\text{EF}}(0, \epsilon)$. Then, for $t, \theta$, we have that:

$$
u(X_{t, \theta}, \theta) - u(X_{t, \theta}, \theta) = u_e(X_{t, \theta}, \theta) - u_e(X_{t, \theta}, \theta) + \epsilon \langle X_{t, \theta}^{\text{opt}(\epsilon)}, X_{t, \theta}, w'_0 \rangle \leq u_e(X_{t, \theta}, \theta) - u_e(X_{t, \theta}, \theta) + \epsilon \|B\|_\infty \|w'_0\|_1 \leq \Delta_{\text{EF}}(\epsilon, \epsilon) + \epsilon \|B\|_\infty \|w'_0\|_1.$$

The result then follows again by taking the max over $t$ and $\theta$. □

Using Lemma C.3 we are able to show the claim. Running GUARDED-HOPE on the perturbed utilities, with $\epsilon = L_T$, we have the following upper bounds on $(\text{Envy}(L_T), \Delta_{\text{EF}}(L_T, L_T), \Delta_{\text{efficiency}}), \ldots$
Theorem 4.3, with probability at least $1 - \delta$:

$$\text{Envy}(L_T) \leq L_T \quad \Delta_{\text{EF}} \leq \max\{1/\sqrt{T}, L_T\}.$$  

Plugging this into Lemma C.3, we obtain:

$$\text{Envy}(0) \leq \text{Envy}(L_T) + L_T\|B\|_\infty\|w'_\theta\|_1 \leq L_T + L_T\|B\|_\infty\|w'_\theta\|_1 \leq \max\{1/\sqrt{T}, L_T\}.$$  

We conclude the proof of the theorem by noting that the bounds on $\Delta_{\text{efficiency}}$ and $\Delta_{\text{prop}}$ follow trivially from Theorem 4.3 (since these are statements about the allocations themselves, rather than the utilities).

\[\square\]

### D Useful Lemmas

We use the following standard theorems throughout the proof. See, e.g. [35] for proofs and further discussion.

**Theorem D.1** (Hoeffding’s Inequality). Let $X_1, \ldots, X_n$ be independent random variables such that $a_i \leq X_i \leq b_i$ almost surely and set $S_n = \sum_i X_i$. Then we have that for all $t > 0$:

$$\mathbb{P}(|S_n - \mathbb{E}[S_n]| \geq t) \leq 2 \exp\left(-\frac{2t^2}{\sum_i (b_i - a_i)^2}\right).$$  

**Theorem D.2** (Chernoff Bound for Sum of Independent Bernoulli Random Variables). Consider a sequence of Bernoulli random variables $(X_i)_{i \in [N]}$, independently distributed with probability of success $p_i \in (0, 1)$. Let $X = \sum_i X_i$, and let $\mu = \mathbb{E}[X] = \sum_i p_i$. Then for all $0 < \delta < 1$:

$$\mathbb{P}(|X - \mu| \geq \delta \mu) \leq 2 \exp\left(-\frac{\mu \delta^2}{3}\right).$$