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Conditioning on an extreme component: Model consistency and regular variation on cones

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Multivariate extreme value theory assumes a multivariate domain of attraction condition for the distribution of a random vector necessitating that each component satisfies a marginal domain of attraction condition. [12] and [11] developed an approximation to the joint distribution of the random vector by conditioning that one of the components be extreme. Prior papers left unresolved the consistency of different models obtained by conditioning on different components being extreme and we provide understanding of this issue. We also clarify the relationship between these conditional distributions, multivariate extreme value theory, and standard regular variation on cones of the form $[0,\infty] \times (0,\infty]$.

Keywords: regular variation, domain of attraction, asymptotic independence, conditioned limit theory.

1. Introduction

Classical multivariate extreme value theory (abbreviated as MEVT) captures the extremal dependence structure between components under a robust multivariate domain of attraction condition which requires that each marginal distribution belongs to the domain of attraction of some univariate extreme value distribution. Extremal dependence has been well-studied both in the case of asymptotic dependence [6, 7, 19, 23] and asymptotic independence [2, 5, 15, 21, 22]. An innovative approach was provided by [12], who approximated multivariate distributions by assuming only one of the components was in an extreme value domain of attraction and that this component was extreme. The approach allowed a variety of examples of different types of asymptotic dependence and asymptotic independence. Their statistical ideas were given a more mathematical framework by [11] after some slight changes in assumptions to make the theory more probabilistically viable.

[11] considered a bivariate random vector (X, Y) where the distribution of Y is in the domain of attraction of an extreme value distribution G_γ, where for $\gamma \in \mathbb{R}$,

$$G_\gamma(x) = \exp\{-(1+\gamma x)^{-1/\gamma}\}, \qquad 1+\gamma x > 0. \tag{1.1}$$

For $\gamma = 0$, the distribution function is interpreted as $G_0(x) = e^{-e^{-x}}$, $x \in \mathbb{R}$. Instead of conditioning on Y being large, their theory was developed under the equivalent assumption of the existence of a vague limit for the modified joint distribution of a suitably scaled and centered (X, Y). The precise description of this vague limit is provided in Definition 1.1. This condition differs from classical MEVT in the sense that only one of the marginal distributions is assumed to be in the domain of attraction of an extreme value distribution.

Section 2 studies consistency issues raised in [12] for such conditional models. In practice one has a choice of variable to condition on being large and potentially different models are therefore possible. We show that if conditional approximations are possible no matter which variable is chosen as the conditioning variable, then in fact the joint distribution is in a classical multivariate domain of attraction of an extreme value law and no new theory is required. Section 3 gives the relationship between multivariate extreme value theory and conditioned limit theory and explains when the CEVM can be reduced to standard regular variation on a cone. In Section 4 we provide conditions under which the CEVM can be extended to classical MEVT. Section 5 presents some illuminating examples to show the features of conditional models and the last section gives some proofs deferred from earlier sections.

1.1. Notation

We list below commonly used notation. References are provided for further reading.

$\mathbb{R}_+^d$	$[0,\infty)^d$. Also denote similarly $\overline{\mathbb{R}}_+^d = [0,\infty]^d, \overline{\mathbb{R}}^d = [-\infty,\infty]^d$.
$\mathbb{E}^*$	A nice subset of the compactified finite dimensional Euclidean space. Often denoted $\mathbb{E}$ with different subscripts and superscripts as required.
$\mathcal{E}^*$	The Borel $\sigma-$field of the subspace $\mathbb{E}^*$.
$\mathbb{M}_+(\mathbb{E}^*)$	The class of Radon measures on Borel subsets of $\mathbb{E}^*$.
$f^{\leftarrow}$	The left continuous inverse of a monotone function f.
	For an increasing function $f^{\leftarrow}(x) = \inf\{y : f(y) \geq x\}$.
	For a decreasing function $f^{\leftarrow}(x) = \inf\{y : f(y) \leq x\}$.
RV_ρ	The class of regularly varying functions with index ρ; see [1, 6, 10, 23, 24].
Π	Π-varying class of functions; see [1, 23].
$\mathbb{E}^{(\gamma)}$	$\{x : 1 + \gamma x > 0\}$ for $\gamma \in \mathbb{R}$.
$\overline{\mathbb{E}}^{(\gamma)}$	The closure on the right of the interval $\mathbb{E}^{(\gamma)}$.
$\overline{\overline{\mathbb{E}}}^{(\gamma)}$	The closure on both sides of the interval $\mathbb{E}^{(\gamma)}$.
$\mathbb{E}^{(\lambda,\gamma)}$	$\overline{\overline{\mathbb{E}}}^{(\lambda)} \times \overline{\overline{\mathbb{E}}}^{(\gamma)} \setminus \{(-\frac{1}{\lambda}, -\frac{1}{\gamma})\}$.

$\mathbb{E}$	Usually $[0,\infty]^2 \setminus \{\mathbf{0}\}$.
$\mathbb{E}_0$	Usually $(0,\infty]^2$.
$\mathbb{E}_\sqcap$	$[0,\infty]\times(0,\infty]$.
$\mathbb{E}_\sqsupset$	$(0,\infty]\times[0,\infty]$.
$\xrightarrow{v}$	Vague convergence of measures; see [13, 18].
G_γ	An extreme value distribution given by (1.1), with parameter $\gamma\in\mathbb{R}$.
$D(G_\gamma)$	The domain of attraction of the extreme value distribution G_γ; i.e., the set of F's satisfying (1.7). For $\gamma>0$, $F\in D(G_\gamma)$ is equivalent to $1-F\in RV_{-1/\gamma}$.

1.2. Model setup and basic assumptions

The basic model assumptions for our discussion are as follows [11]:

Definition 1.1 (Conditional extreme value model). *Suppose* $(X,Y)\in\mathbb{R}^2$ *is a random vector and there exist functions* $\alpha(t)>0, a(t)>0$, $\beta(t), b(t)\in\mathbb{R}$, *a constant* $\gamma\in\mathbb{R}$ *and a non-null Radon measure* μ *on Borel subsets of* $[-\infty,\infty]\times\overline{\mathbb{E}}^{(\gamma)}$ *such that,*

$$[a]\quad t\mathbf{P}\Big(\Big(\frac{X-\beta(t)}{\alpha(t)},\frac{Y-b(t)}{a(t)}\Big)\in\cdot\Big)\xrightarrow{v}\mu(\cdot)\ \text{in}\ \mathbb{M}_+([-\infty,\infty]\times\overline{\mathbb{E}}^{(\gamma)}). \tag{1.2}$$

$$[b]\quad \mu([-\infty,x]\times(y,\infty])\quad \text{is not a degenerate distribution in } x, \tag{1.3}$$

$$[c]\quad \mu([-\infty,x]\times(y,\infty])<\infty. \tag{1.4}$$

$$[d]\quad H(x):=\mu([-\infty,x]\times(0,\infty])\ \text{is a probability distribution.} \tag{1.5}$$

We say that (X,Y) *follows a* conditional extreme value model *(abbreviated as CEVM) if conditions* (1.2)–(1.5) *are satisfied. We write* $(X,Y)\in CEVM(\alpha,\beta,a,b,\gamma)$.

A non-null Radon measure $\mu(\cdot)$ satisfies the *conditional non-degeneracy conditions* if both of (1.3) and (1.4) hold. Conditions (1.2), (1.3) and (1.4) imply that for continuity points (x,y) of $\mu(\cdot)$,

$$\mathbf{P}\bigg(\frac{X-\beta(t)}{\alpha(t)}\le x\,\bigg|\,Y>b(t)\bigg)\to H(x)=\mu([-\infty,x]\times(0,\infty]),\qquad (t\to\infty) \tag{1.6}$$

i.e., a conditioned limit holds. Hence the name conditional extreme value model. By taking the marginal of Y in (1.2), we note that if $Y\sim F$, then $F\in D(G_\gamma)$ for some $\gamma\in\mathbb{R}$, as defined in (1.1); that is, as $t\to\infty$,

$$t(1-F(a(t)y+b(t)))=t\mathbf{P}\bigg(\frac{Y-b(t)}{a(t)}>y\bigg)\to(1+\gamma y)^{-1/\gamma},\quad 1+\gamma y>0. \tag{1.7}$$

Under the above assumptions, a convergence to types argument [11] yields properties of the normalizing and centering functions: there exists functions $\psi_1(\cdot), \psi_2(\cdot)$ such that

$$\lim_{t\to\infty} \frac{\alpha(tc)}{\alpha(t)} = \psi_1(c), \qquad \lim_{t\to\infty} \frac{\beta(tc)-\beta(t)}{\alpha(t)} = \psi_2(c). \tag{1.8}$$

This implies that $\psi_1(c) = c^\rho$ for some $\rho \in \mathbb{R}$ [6, Theorem B.1.3]. Either $\psi_2 \equiv 0$ or $\psi_2(c) = k\frac{c^\rho - 1}{\rho}$ for some $c \neq 0$ [6, Theorem B.2.1]. We refer often to these properties.

2. Consistency of conditional extreme value models

Suppose $(X, Y) \in \mathbb{R}^2$ satisfy conditions (1.2)–(1.5). Hence $(X, Y) \in \text{CEVM}(\alpha, \beta, a, b, \gamma)$ with $F \in D(G_\gamma)$ where $Y \sim F$. Assume also (Y, X) satisfy conditions (1.2)–(1.5), i.e., $(Y, X) \in CEVM(c, d, \chi, \phi, \lambda)$ for some $\chi(t) > 0, c(t) > 0, \phi(t), d(t) \in \mathbb{R}, \lambda \in \mathbb{R}$ with $G \in D(G_\lambda)$ where $X \sim G$. What are the implications of these assumptions for the joint distribution of (X, Y)? We show that these two assumptions imply that (X, Y) is in a domain of attraction of some multivariate extreme value distribution.

Recall the notation, $\mathbb{E}^{(\gamma)} = \{x \in \mathbb{R} : 1 + \gamma x > 0\}$ for $\gamma \in \mathbb{R}$ and $\overline{\mathbb{E}}^{(\gamma)}$ is the right closure of $\mathbb{E}^{(\gamma)}$, i.e.,

$$\overline{\mathbb{E}}^{(\gamma)} = \begin{cases} (-\frac{1}{\gamma}, \infty] & \gamma > 0 \\ (-\infty, \infty] & \gamma = 0 \\ (-\infty, -\frac{1}{\gamma}] & \gamma < 0, \end{cases} \tag{2.1}$$

and $\overline{\overline{\mathbb{E}}}^{(\gamma)}$ is the closure of $\mathbb{E}^{(\gamma)}$ on both sides and $\mathbb{E}^{(\lambda,\gamma)} := \overline{\overline{\mathbb{E}}}^{(\lambda)} \times \overline{\overline{\mathbb{E}}}^{(\gamma)} \setminus \{(-\frac{1}{\lambda}, -\frac{1}{\gamma})\}$.

Theorem 2.1. *Suppose we have a bivariate random vector $(X, Y) \in \mathbb{R}^2$ and non-negative functions $\alpha(\cdot), a(\cdot), \chi(\cdot), c(\cdot)$ and real valued functions $\beta(\cdot), b(\cdot), \phi(\cdot), d(\cdot)$ such that*

$$t\mathbf{P}\Big[\Big(\frac{X-\beta(t)}{\alpha(t)}, \frac{Y-b(t)}{a(t)}\Big) \in \cdot\Big] \xrightarrow{v} \mu(\cdot) \text{ in } \mathbb{M}_+\big([-\infty, \infty] \times \overline{\mathbb{E}}^{(\gamma)}\big), \tag{2.2}$$

$$t\mathbf{P}\Big[\Big(\frac{X-\phi(t)}{\chi(t)}, \frac{Y-d(t)}{c(t)}\Big) \in \cdot\Big] \xrightarrow{v} \nu(\cdot) \text{ in } \mathbb{M}_+\big(\overline{\mathbb{E}}^{(\lambda)} \times [-\infty, \infty]\big) \tag{2.3}$$

for some $\lambda, \gamma \in \mathbb{R}$, where both μ and ν satisfy the appropriate conditional non-degeneracy conditions corresponding to (1.3) *and* (1.4)*. Then (X, Y) is in the domain of attraction of a multivariate extreme value distribution on $\mathbb{E}^{(\lambda,\gamma)}$ in the following sense:*

$$t\mathbf{P}\Big[\Big(\frac{X-\phi(t)}{\chi(t)}, \frac{Y-b(t)}{a(t)}\Big) \in \cdot\Big] \xrightarrow{v} (\mu \diamond \nu)(\cdot) \text{ in } \mathbb{M}_+\big(\mathbb{E}^{(\lambda,\gamma)}\big)$$

where $(\mu \diamond \nu)(\cdot)$ is a non-null Radon measure on $\mathbb{E}^{(\lambda,\gamma)}$.

The proof is in Section 6. The significant feature of Theorem 2.1 is that we do not need any other condition on the normalizing functions. The convergences (2.2)and (2.3) imply that $\alpha = O(\chi)$ and $c = O(a)$ as $t \to \infty$. If either $\alpha = o(\chi)$ or $c = o(a)$ then we have asymptotic independence and the existence of hidden regular variation.

Consistency: standard regularly varying case. We were led to Theorem 2.1 by considering the special case of regular variation where (X, Y) satisfies $(X, Y) \in$ CEVM$(\alpha(t) = t, \beta(t) = 0, a(t) = t, b(t) = 0, \gamma = 1)$, and $(Y, X) \in$ CEVM$(\alpha(t) = t, \beta(t) = 0, a(t) = t, b(t) = 0, \gamma = 1)$ and the vague convergence in (1.2) is regular variation on the cone $\mathbb{E}_\sqcap = [0, \infty] \times (0, \infty]$ ([22, page 173], [4, 21]).

For this case, if

$$t\mathbf{P}\Big[\Big(\frac{X}{t}, \frac{Y}{t}\Big) \in \cdot\Big] \xrightarrow{v} \mu(\cdot) \text{ in } \mathbb{M}_+\big(\mathbb{E}_\sqcap = [0, \infty] \times (0, \infty]\big), \tag{2.4}$$

$$t\mathbf{P}\Big[\Big(\frac{X}{t}, \frac{Y}{t}\Big) \in \cdot\Big] \xrightarrow{v} \nu(\cdot) \text{ in } \mathbb{M}_+\big(\mathbb{E}_\sqsupset = (0, \infty] \times [0, \infty]\big). \tag{2.5}$$

where μ and ν satisfy appropriate conditional non-degeneracy conditions corresponding to (1.3) and (1.4), then (X, Y) is standard regularly varying in $\mathbb{E} := [0, \infty]^2 \setminus \{\mathbf{0}\}$; i.e.,

$$t\mathbf{P}\Big[\Big(\frac{X}{t}, \frac{Y}{t}\Big) \in \cdot\Big] \xrightarrow{v} (\mu \diamond \nu)(\cdot) \qquad \text{in } \mathbb{M}_+\big(\mathbb{E}\big), \tag{2.6}$$

where $(\mu \diamond \nu)$ is a Radon measure on $\mathbb{E}$ such that

$$(\mu \diamond \nu)|_{\mathbb{E}_\sqcap}(\cdot) = \mu(\cdot) \quad \text{on } \mathbb{E}_\sqcap \qquad \text{and} \qquad (\mu \diamond \nu)|_{\mathbb{E}_\sqsupset}(\cdot) = \nu(\cdot) \quad \text{on } \mathbb{E}_\sqsupset.$$

Consistency: the absolutely continuous case. Calculations become more explicit when (X, Y) has a joint density. This case is often of most interest to statisticians.

Assume g_ρ is the density of a univariate extreme value distribution G_ρ with shape parameter $\rho \in \mathbb{R}$ in (1.1) and that $(X, Y) \in \mathbb{R}^2$ is a bivariate random vector satisfying

1. (X, Y) has a density $f_{X,Y}(x, y)$.
2. The marginal densities f_X, f_Y satisfy (as $t \to \infty$):

$$t\chi(t) f_X(c(t)x + d(t)) \to g_\lambda(x), x \in \mathbb{E}^{(\lambda)}, \tag{2.7}$$

$$ta(t) f_Y(a(t)y + b(t)) \to g_\gamma(y), y \in \mathbb{E}^{(\gamma)}. \tag{2.8}$$

3. The joint density satisfies (as $t \to \infty$):

$$t\alpha(t)a(t) f_{X,Y}(\alpha(t)x + \beta(t), a(t)y + b(t)) \to g_1(x, y) \in L^1([-\infty, \infty] \times \overline{\mathbb{E}}^{(\gamma)}), \tag{2.9}$$

$$t\chi(t)d(t) f_{X,Y}(\chi(t)x + \phi(t), c(t)y + d(t)) \to g_2(x, y) \in L^1(\overline{\mathbb{E}}^{(\lambda)} \times [-\infty, \infty]), \tag{2.10}$$

where $g_1(x, y), g_2(x, y) \geq 0$ are non-trivial, 0 outside of $[-\infty, \infty] \times \overline{\mathbb{E}}^{(\gamma)}$ and $\overline{\mathbb{E}}^{(\lambda)} \times [-\infty, \infty]$ respectively.

Then

$$t\mathbf{P}\Big[\Big(\frac{X-\beta(t)}{\alpha(t)},\frac{Y-b(t)}{a(t)}\Big)\in\cdot\Big]\xrightarrow{v}\mu(\cdot)\text{ in }\mathbb{M}_+\big(\mathbb{E}^{(\lambda,\gamma)}\big),\tag{2.11}$$

for some non-degenerate Radon measure μ on $\mathbb{E}^{(\lambda,\gamma)}$.

***Example* 1.** Suppose (X,Y) is a bivariate random variable with joint density

$$f_{X,Y}(x,y)=\frac{4x}{(x^2+y)^3}+\frac{4y}{(x+y^2)^3},\quad x\geq 1,y\geq 1.$$

The conditions required for the absolutely continuous result hold: As $t\to\infty$,

$$\begin{aligned}
&t^2f_X(tx)\to\frac{2}{x^2},\quad t^2f_Y(ty)\to\frac{2}{y^2},\quad x,y>0,\\
&t^{5/2}f_{X,Y}(tx,\sqrt{t}y)\to\frac{4y}{(x+y^2)^3}=:g_1(x,y)\in L_1(\mathbb{E}_\sqsupset)\\
&t^{5/2}f_{X,Y}(\sqrt{t}x,ty)\to\frac{4x}{(x^2+y)^3}=:g_1(x,y)\in L_1(\mathbb{E}_\sqcap).
\end{aligned}$$

These statements lead to the analogues of (1.2) on different cones:

$$\begin{aligned}
&t\mathbf{P}(\frac{X}{\sqrt{t}}\leq x,\frac{Y}{t}>y)\to\frac{1}{y}-\frac{1}{y+x^2},\quad x\geq 0,y>0,\\
&t\mathbf{P}(\frac{X}{t}>x,\frac{Y}{\sqrt{t}}\leq y)\to\frac{1}{x}-\frac{1}{x+y^2},\quad x>0,y\geq 0,\\
&t\mathbf{P}\big((\frac{X}{t},\frac{Y}{t})\in([0,x]\times[0,y])^c\big)\to\frac{1}{x}+\frac{1}{y},\quad x>0,y>0.
\end{aligned}$$

Consistency in d-dimensions, $d>2$. Suppose we have a d-dimensional vector $\boldsymbol{X}:=(X_1,X_2,\dots,X_d)$ where a multivariate CEVM holds (with a definition similar to Definition 1.1) with a limit holding for whichever X_i we consider to be extreme. Then the distribution of $(X_1,X_2,\dots,X_d)$ belongs to the domain of attraction of a d-dimensional extreme value distribution. However, when $d>2$, it is possible to study the CEVM model by assuming subsets of components are extreme in various senses and we are still considering what is sensible to assume and whether detection of the best subset to condition on being extreme is statistically possible. Results which hold for the CEVM model in the bivariate case do not naturally extend to this more complex setting.

3. The CEVM and standard regular variation

As remarked after Theorem 2.1, questions about the general conditional model are effectively analyzed by starting with standard regular variation on the cones ($\mathbb{E}_\sqsupset$ or $\mathbb{E}_\sqcap$). A relevent issue, therefore, is whether standardization of the conditional extreme value model is always possible. A partial answer is in [11, Section 2.4] and we now consider this issue in more detail. We start by making precise what we mean by *standardization*.

3.1. Standardization

Standardization is the process of marginally transforming a random vector $\boldsymbol{X}$ into a different vector $\boldsymbol{Z}^*$, $\boldsymbol{X} \mapsto \boldsymbol{Z}^*$, so that the distribution of $\boldsymbol{Z}^*$ is standard regularly varying on a cone $\mathbb{E}^*$; that is, for some Radon measure $\mu^*(\cdot)$

$$t\mathbf{P}\Big[\frac{\boldsymbol{Z}^*}{t} \in \cdot\,\Big] \xrightarrow{v} \mu^*(\cdot), \qquad \text{in } \mathbb{M}_+(\mathbb{E}^*). \tag{3.1}$$

Depending on the cone, this says one or more components of $\boldsymbol{Z}^*$ are asymptotically Pareto. For classical multivariate extreme value theory, each component is asymptotically Pareto and $\mathbb{E}^* = \mathbb{E} = [\mathbf{0}, \boldsymbol{\infty}]\backslash\{\mathbf{0}\}$. The technique is used in classical multivariate extreme value theory to characterize multivariate domains of attraction and dates at least to [7]. See also [6, 16, 17, 22],[23, Chapter 5]. Standardization is analogous to the copula transformation but is better suited to studying limit relations [14].

In Cartesian coordinates, the limit measure in (3.1) has the scaling property:

$$\mu^*(c\,\cdot\,) = c^{-1}\mu^*(\cdot), \quad c > 0. \tag{3.2}$$

This scaling in Cartesian coordinates translates to a product limit when expressed in polar coordinates. An angular measure exists allowing characterization of limits:

$$\mu^*\{\boldsymbol{x} : \|\boldsymbol{x}\| > r, \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \in \Lambda\} = r^{-1}S(\Lambda),$$

for Borel subsets Λ of the unit sphere in $\mathbb{E}^*$.

In classical multivariate extreme value theory, S is a finite measure which we may take to be a probability measure without loss of generality. However, when $\mathbb{E}^* = \mathbb{E}_\sqcap$, S is NOT necessarily finite because absence of the horizontal axis boundary in $\mathbb{E}_\sqcap$ implies the unit sphere is not compact.

Here is an explicit description of *standardization*. Suppose $\boldsymbol{X} = (X_1, X_2, \dots, X_d)$ is a random vector in $\mathbb{R}^d$ which satisfies:

$$t\mathbf{P}\Big[\Big(\frac{X_1 - \beta_1(t)}{\alpha_1(t)}, \frac{X_2 - \beta_2(t)}{\alpha_2(t)}, \dots, \frac{X_d - \beta_d(t)}{\alpha_d(t)}\Big) \in \cdot\,\Big] \xrightarrow{v} \mu(\cdot) \qquad \text{on } \mathbb{M}_+(\mathfrak{D}), \tag{3.3}$$

for some $\mathfrak{D} \subset \overline{\mathbb{R}}^d$, $\alpha_i(t) > 0, \beta_i(t) \in \mathbb{R}$, for $i = 1, \dots, d$. Suppose we have $\boldsymbol{f} = (f_1, \dots, f_d)$ such that, for $i = 1, \dots, d$:

(a) f_i : Range of $X_i \to (0, \infty)$,
(b) f_i is monotone,
(c) $\nexists K > 0$ such that $|f_i| \le K$.

Then $\boldsymbol{f}$ *standardizes* $\boldsymbol{X}$ if $\boldsymbol{Z}^* = \boldsymbol{f}(\boldsymbol{X}) = (f_i(X_i),\, i = 1, \dots, d)$ satisfies (3.1). Call $\boldsymbol{f}$ the *standardizing function* and say (3.1) is the *standardization* of (3.3).

For the conditional model defined in Definition 1.1 in Section 1.2 where F, the distribution of Y, satisfies $F \in D(G_\gamma)$, we can always use

$$b(\cdot) = \Big(\frac{1}{1 - F}\Big)^{\leftarrow}(\cdot)$$

to standardize Y and $Y^* = b^{\leftarrow}(Y)$ is the standardization of Y. See [11].

3.2. When can the conditional extreme value model be standardized?

Suppose (X, Y) satisfies Definition 1.1 and, in particular, (1.2) holds. Standardization in (1.2) is possible *unless* $(\psi_1, \psi_2) = (1, 0)$ which is equivalent to the limit measure being a product measure [11]. We show the converse is also true. Consequently, when the limit measure is not a product measure, we can always reduce to standard regular variation on the cone $\mathbb{E}_{\sqcap}$, and conversely, we can think of the general conditional model as a transformation of standard regular variation on $\mathbb{E}_{\sqcap}$.

We begin with an initial result about the impossibility of the limit measure being a product when we have standardized convergence on $\mathbb{E}_{\sqcap}$.

Lemma 3.1. *Suppose (X, Y) is standard regularly varying on the cone $\mathbb{E}_{\sqcap}$, such that,*

$$t\mathbf{P}\Big[\Big(\frac{X}{t}, \frac{Y}{t}\Big) \in \cdot\Big] \xrightarrow{v} \mu(\cdot) \qquad \text{in } \mathbb{M}_+\big(\mathbb{E}_{\sqcap}\big) \tag{3.4}$$

for some non-null Radon measure $\mu(\cdot)$ on $\mathbb{E}_{\sqcap}$, satisfying the conditional non-degeneracy conditions as in (1.3) *and* (1.4)*. Then $\mu(\cdot)$ cannot be a product measure.*

Proof. If μ is a product measure we have

$$\mu\big([0, x] \times (y, \infty]\big) = G(x)y^{-1} \qquad \text{for } x \geq 0, y > 0 \tag{3.5}$$

for some finite distribution function G on $[0, \infty)$. Now (3.4) implies that μ is homogeneous of order -1, i.e.,

$$\mu(c\Lambda) = c^{-1}\mu(\Lambda), \qquad \forall c > 0, \tag{3.6}$$

where Λ is a Borel subset of $\mathbb{E}_{\sqcap}$. Therefore using (3.5)

$$\mu\big(c([0, x] \times (y, \infty])\big) = \mu\big([0, cx] \times (cy, \infty]\big) = G(cx) \times \frac{1}{cy} = c^{-1}G(cx)y^{-1}.$$

Moreover, using (3.5) and (3.6), $\mu\big(c([0, x] \times (y, \infty])\big) = c^{-1}G(x)y^{-1}$, and therefore, $G(cx) = G(x)$, $\forall c > 0$, $x > 0$. Hence for fixed $y \in \mathbb{E}^{(\gamma)}, c > 0, x > 0$,

$$\mu\big([0, cx] \times (y, \infty]\big) = G(cx)y^{-1} = G(x)y^{-1} = \mu\big([0, x] \times (y, \infty]\big).$$

Thus, μ becomes a degenerate distribution in x, contradicting our conditional non-degeneracy assumptions and consequently $\mu(\cdot)$ cannot be a product measure. $\square$

According to Lemma 3.1, a limit measure in standard regular variation on $\mathbb{E}_\sqcap$ satisfying the conditional non-degeneracy conditions cannot be a product measure. Suppose we have a general CEVM as in Definition 1.1 and the limit measure is a product. We show this CEVM cannot be *standardized* to regular variation on some cone $\mathfrak{C} \subset \mathbb{E}$ ($\mathfrak{C} = \mathbb{E}_\sqcap$ for our case). Recall that when Definition 1.1 holds, Y can always be standardized so in the following we assume Y^* is the standardized Y and we only consider when X can be the standardized.

Theorem 3.2. *Suppose $X \in \mathbb{R}, Y^* > 0$ are random variables, such that for functions $\alpha(\cdot) > 0$, $\beta(\cdot) \in \mathbb{R}$, we have as $t \to \infty$,*

$$t\mathbf{P}\Big[\Big(\frac{X-\beta(t)}{\alpha(t)}, \frac{Y^*}{t}\Big) \in \cdot\Big] \xrightarrow{v} G \times \nu_1(\cdot) \qquad \text{in } \mathbb{M}_+\big([-\infty,\infty] \times (0,\infty]\big), \tag{3.7}$$

where $\nu_1(x,\infty] = x^{-1}$, $x > 0$, and G is some finite, non-degenerate distibution on $\mathbb{R}$. Then there does not exist a standardizing function, $f(\cdot)$: Range of $X \mapsto (0,\infty)$, in the sense of the discussion after (3.3), *such that*

$$t\mathbf{P}\Big[\Big(\frac{f(X)}{t}, \frac{Y^*}{t}\Big) \in \cdot\Big] \xrightarrow{v} \mu(\cdot) \qquad \text{in } \mathbb{M}_+\big(\mathbb{E}_\sqcap\big), \tag{3.8}$$

where μ satisfies the conditional non-degeneracy conditions.

Proof. Note that Y^* is already standardized here. Suppose there exists a standardization function $f(\cdot)$ such that (3.8) holds. Without loss of generality assume $f(\cdot)$ to be non-decreasing. This implies that for $\mu-$ continuity points (x,y) we have,

$$t\mathbf{P}\Big[\frac{f(X)}{t} \le x, \frac{Y^*}{t} > y\Big] \to \mu\big((-\infty,x] \times (y,\infty]\big) \qquad (t \to \infty)$$

which is equivalent to

$$t\mathbf{P}\Big(\frac{X-\beta(t)}{\alpha(t)} \le \frac{f^{\leftarrow}(xt)-\beta(t)}{\alpha(t)}, \frac{Y^*}{t} > y\Big] \to \mu\big((-\infty,x] \times (y,\infty]\big), \qquad (t \to \infty). \tag{3.9}$$

Since $\mu((-\infty,x] \times (y,\infty]) < \infty$ and is non-degenerate in x, we have as $t \to \infty$ that

$$\frac{f^{\leftarrow}(xt)-\beta(t)}{\alpha(t)} \to h(x) \tag{3.10}$$

for some non-decreasing function $h(\cdot)$ which has at least two points of increase. Thus (3.9) and (3.10) imply that $\mu((-\infty,x] \times (y,\infty]) = G(h(x)) \times y^{-1}$. Hence $\mu(\cdot)$ is a product measure which by Lemma 3.1 is not possible. □

Summary. We now summarize the information about when standardization is possible and the relationship of this to the limit measure being a product. A proof is provided in Section 6. Statistical methods for detecting that a CEVM is appropriate and whether the limit measure is a product are given in [3].

1. Suppose (X,Y) satisfy Definition 1.1 so that limits in (1.8) hold. If $(\psi_1,\psi_2)\neq(1,0)$, then there exists a standardization function $\boldsymbol{f}=(f_1,f_2)$ such that $(X^*,Y^*)=(f_1(X),f_2(Y))$ is standard regularly varying on $\mathbb{E}_\sqcap$; that is
$$t\mathbf{P}\Big[\Big(\frac{f_1(X)}{t},\frac{f_2(Y)}{t}\Big)\in\cdot\Big]\xrightarrow{v}\mu^{**}(\cdot)\qquad\text{in } \mathbb{M}_+\big(\mathbb{E}_\sqcap\big),$$
and μ^{**} is a non-null Radon measure satisfying the conditional non-degeneracy conditions.
2. Conversely, suppose we have a bivariate random vector $(X^*,Y^*)\in\mathbb{R}_+^2$ satisfying
$$t\mathbf{P}\Big[\Big(\frac{X^*}{t},\frac{Y^*}{t}\Big)\in\cdot\Big]\xrightarrow{v}\mu^{**}(\cdot)\qquad\text{in } \mathbb{M}_+\big(\mathbb{E}_\sqcap\big),$$
where μ^{**} is a non-null Radon measure, satisfying the conditional non-degeneracy conditions. Consider functions $\alpha(\cdot)>0$, $\beta(\cdot)\in\mathbb{R}$ such that (1.8) holds with $(\psi_1,\psi_2)\neq(1,0)$. Then there exist functions $a(\cdot)>0$, $b(\cdot)\in\mathbb{R}$ satisfying (1.7) and $\lambda(\cdot)\in\mathbb{R}$, $\gamma\in\mathbb{R}$ such that
$$t\mathbf{P}\Big[\Big(\frac{\lambda(X^*)-\beta(t)}{\alpha(t)},\frac{b(Y^*)-b(t)}{a(t)}\Big)\in\cdot\ \Big]\xrightarrow{v}\tilde{\mu}(\cdot)\text{ in } \mathbb{M}_+\big([-\infty,\infty]\times\overline{\mathbb{E}}^{(\gamma)}\big),\quad(3.11)$$
where $\tilde{\mu}$ is a non-null Radon measure in $[-\infty,\infty]\times\overline{\mathbb{E}}^{(\gamma)}$ satisfying the conditional non-degeneracy conditions and $b(Y^*)\in D(G_\gamma)$.

***Remark* 3.1.** Suppose for a non-degenerate probability distribution H, we have in $\mathbb{M}_+([-\infty,\infty]\times(0,\infty])$,
$$t\mathbf{P}\Big[\Big(\frac{X-\beta(t)}{\alpha(t)},\frac{Y^*}{t}\Big)\in\cdot\Big]\xrightarrow{v}H\times\nu_1(\cdot).$$
Define $X^*=\big((X-\beta(Y^*))Y^*\big)/\alpha(Y^*)$. Then for continuity points (x,y) of the limit, in $\mathbb{M}_+([-\infty,\infty]\times(0,\infty])$,
$$t\mathbf{P}\Big(\frac{X^*}{t}\le x,\frac{Y^*}{t}>y\Big)\to\int_0^{1/y}H(xv)dv.$$
The limit measure is homogeneous of order -1 and thus, a transformation of (X,Y^*) to a standard regularly varying pair exists even when we have a limit measure which is a product. Note this transformation is not in the sense of the discussion after (3.3) and is more complex than just a marginal transformation.

3.3. A characterization of regular variation on $\mathbb{E}_\sqcap$

The CEVM model with limit measure which is not a product can always be standardized to give regular variation on $\mathbb{E}_\sqcap$ so we would like useful characterizations of such regular variation. Standard regular variation on $\mathbb{E}$ was characterized by de Haan [9] in terms of one dimensional regular variation of max linear combinations and [20] provides a characterization of hidden regular variation in $\mathbb{E}$ and $\mathbb{E}_0$ in terms of max and min linear combinations of the random vector. Here are comparable results for $\mathbb{E}_\sqcap$.

Proposition 3.3. *Suppose $(X,Y) \in \mathbb{R}^2$ is a random vector and $\mathbf{P}(X=0)=0$. Then the following are equivalent:*

1. *(X,Y) is standard multivariate regularly varying on $\mathbb{E}_\sqcap$ with limit measure satisfying the non-degeneracy conditions* (1.3) *and* (1.4)*.*
2. *For all $a \in (0,\infty]$ we have*

$$\lim_{t\to\infty} t\mathbf{P}\Big(t^{-1}\min(aX,Y) > y\Big) = c(a)y^{-1}, \qquad y>0, \tag{3.12}$$

for some non-constant, non-decreasing function $c:(0,\infty] \to (0,\infty)$.

Proof. Since $\mathbf{P}(X=0)=0$ we have $\mathbf{P}(X>0)=1$.

(2) $\Rightarrow$ (1) : Assume that (3.12) for some function $c:(0,\infty]\to(0,\infty)$. For $x \geq 0, y>0$,

$$\begin{aligned} t\mathbf{P}\Big(\frac{X}{t} \leq x, \frac{Y}{t} > y\Big) &= t\mathbf{P}\Big(\frac{Y}{t} > y\Big) - t\mathbf{P}\Big(\frac{X}{t} > x, \frac{Y}{t} > y\Big) \\ &= t\mathbf{P}\Big(X>0, Y>ty\Big) - t\mathbf{P}\Big((y/x)X > ty, Y > ty\Big) \\ &= t\mathbf{P}\Big(\min(a_1X,Y) > ty\Big) - t\mathbf{P}\Big(\min((y/x)X,Y) > ty\Big) \end{aligned}$$

(where $a_1 = \infty$)

$$\to c(\infty)y^{-1} - c(y/x)y^{-1} =: \mu([0,x]\times(y,\infty]).$$

Since $c(\cdot)$ is non-decreasing and non-constant, μ is a non-null Radon measure on $\mathbb{E}_\sqcap$ and we have our result. The non-degeneracy of μ follows from the fact that $c(\cdot)$ is a non-constant function.

(1) $\Rightarrow$ (2) : Now assume that (X,Y) is standard multivariate regularly varying on $\mathbb{E}_\sqcap$. Hence there exists a non-degenerate Radon measure μ on $\mathbb{E}_\sqcap$ such that

$$\lim_{t\to\infty} t\mathbf{P}\Big(\frac{X}{t} \leq x, \frac{Y}{t} > y\Big) = \mu([0,x]\times(y,\infty]),$$

and for any $a \in (0,\infty]$,

$$\begin{aligned} t\mathbf{P}\Big(\frac{\min(aX,Y)}{t} > y\Big) &= t\mathbf{P}\Big(\frac{X}{t} > \frac{y}{a}, \frac{Y}{t} > y\Big) \to \mu((\frac{y}{a},\infty]\times(y,\infty]) \\ &= y^{-1}\mu((\frac{1}{a},\infty]\times(1,\infty]) =: c(a)y^{-1}, \end{aligned}$$

by defining $c(a) = \mu((a{-}1,\infty]\times(1,\infty])$ and using the homogeneity property (3.2). Note that the conditional non-degeneracy of μ implies that c is non-constant and non-decreasing. Hence we have the result. □

The condition $\mathbf{P}(X=0)$ can be removed if we assume as $t \to \infty$, $t\mathbf{P}(Y/t > y) \to y^{-1}$.

3.4. Polar co-ordinates

Section 3.2 shows that when the limit measure is not a product measure, we can transform (X, Y) to (X^*, Y^*) such that

$$\mathbf{P}\Big[\Big(\frac{X^*}{t}, \frac{Y^*}{t}\Big) \in \cdot\Big] \xrightarrow{v} \mu^{**}(\cdot) \qquad \text{in } \mathbb{M}_+(\mathbb{E}_\sqcap) \tag{3.13}$$

Hence μ^{**} satisfies (3.2) and when written in polar coordinates has a spectral form [11, Section 3.2]). We summarize some useful facts. For convenience take the norm $\|(x,y)\| = |x|+|y|$, $(x,y) \in \mathbb{R}^2$, although any other norm would also suffice. A standard homogeneity argument [23, Chapter 5] yields, for $r > 0$ and Λ a Borel subset of $[0,1)$,

$$\begin{aligned}&\mu^{**}\Big\{(x,y) \in [0,\infty] \times (0,\infty] : x+y > r, \frac{x}{x+y} \in \Lambda\Big\}\\ &=r^{-1}\mu^{**}\Big\{(x,y) \in [0,\infty] \times (0,\infty] : x+y > 1, \frac{x}{x+y} \in \Lambda\Big\} =: r^{-1}S(\Lambda).\end{aligned} \tag{3.14}$$

where S is a Radon measure on $[0,1)$. For $x > 0, y > 0$, we get from (3.14),

$$\mu^{**}\big([0,x] \times (y,\infty]\big) = y^{-1}\int_0^{x/(x+y)} (1-w)S(dw) - x^{-1}\int_0^{x/(x+y)} wS(dw). \tag{3.15}$$

S need not be a finite measure on $[0,1)$ but to guarantee that

$$H^{**}(x) := \mu^{**}\big([0,x] \times (1,\infty]\big) \tag{3.16}$$

is a probability measure, we can see by taking $x \to \infty$ in (3.15) that we need

$$\int_0^1 (1-w)S(dw) = 1. \tag{3.17}$$

Conclusion: The class of limits μ^{**} in (3.13) or conditional limits

$$H^{**}(x) = \lim_{t\to\infty} P\Big[\frac{X^*}{t} \le x | Y^* > t\Big]$$

is indexed by Radon measures S on $[0,1)$ satisfying the integrability condition (3.17).

Example* 2** (Finite angular measure).*** If S is uniform on $[0,1)$, $S(dw) = 2dw$, then (3.17) is satisfied and we have

$$\mu^{**}([0,x] \times (y,\infty]) = \frac{x}{y(x+y)}.$$

Putting $y = 1$ we get that $H^{**}(x) = 1 - (1+x)^{-1}$, for $x > 0$, which is a Pareto distribution.

***Example* 3** (Infinite angular measure)**.** The infinite measure $S(dw) = (1-w)^{-1}dw$ also satisfies equation (3.17) and we have

$$\mu^{**}([0,x]\times(y,\infty]) = \frac{1}{y} + \frac{1}{x}\log(1 - \frac{x}{x+y}).$$

Putting $y = 1$ yields $H^{**}(x) = 1 - x^{-1}\log(1+x),\quad x > 0$ and H^{**} is a continuously increasing probability distribution function. One way to get a class of infinite angular measures satisfying (3.17) is to take $S(dw) = \frac{1}{1-w}F(dw)$ for probability measures $F(\cdot)$ on $[0,1)$.

4. Extending the CEVM to a multivariate extreme value model

The CEVM assumes the existence of a vague limit in a subset of the Euclidean space which is smaller than that in case of classical MEVT. This section considers when can we extend a CEVM to a MEVT model. Any extension of the CEVM to MEVT will require X to also have a distribution in a domain of attraction so this will be assumed. The first Proposition provides a sufficient condition for such an extension.

Proposition 4.1. *Suppose* (X,Y) *satisfy Definition 1.1 and in particular* (1.2)–(1.5)*. Also assume that* $X \in D(G_\lambda)$ *for some* $\lambda \in \mathbb{R}$*; i.e., there exists functions* $\chi(t) > 0, \phi(t) \in \mathbb{R}$ *such that for* $x \in \mathbb{E}^{(\lambda)}$,

$$t\mathbf{P}\Big(\frac{X-\phi(t)}{\chi(t)} > x\Big) \to (1+\lambda x)^{-1/\lambda}, \qquad 1+\lambda x > 0.$$

If $\lim\limits_{t\to\infty} \alpha(t)/\chi(t)$ *exists finite and both* $\lim_{t\to\infty}\beta(t)$, $\lim_{t\to\infty}\phi(t)$ *exist* ($\le \infty$) *and are equal then* (X,Y) *is in the domain of attraction of a multivariate extreme value distribution on* $\mathbb{E}^{(\lambda,\gamma)}$*; that is, for a Radon measure* $(\mu\diamond\nu)(\cdot)$ *on* $\mathbb{E}^{(\lambda,\gamma)}$,

$$t\mathbf{P}\Big[\Big(\frac{X-\phi(t)}{\chi(t)}, \frac{Y-b(t)}{a(t)}\Big) \in \cdot\Big] \xrightarrow{v} (\mu\diamond\nu)(\cdot) \quad \text{in } \mathbb{M}_+\big(\mathbb{E}^{(\lambda,\gamma)}\big)$$

Proof. For $\lambda > 0$, the proof is a consequence of cases 1 and 2 of Theorem 2.1. The other cases can be proved similarly. □

We now discuss extension of CEVM to MEVT after first standardizing (X,Y) to (X^*,Y^*) which is regularly varying on $\mathbb{E}_\sqcap$. We consider extending regular variation on $\mathbb{E}_\sqcap$ to an *asymptotically tail equivalent* regular variation on $\mathbb{E}$, a notion we define next.

Definition 4.1 (Tail equivalence in multivariate regular variation [15]). *Suppose* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are* $\mathbb{R}_+^d$*-valued random vectors. Then* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are tail equivalent on a cone* $\mathfrak{C} \subset \overline{\mathbb{R}}_+^d$ *if there exists a scaling function* $b(t) \uparrow \infty$ *such that*

$$t\mathbf{P}\Big(\frac{\boldsymbol{X}}{b(t)} \in \cdot\Big) \xrightarrow{v} \nu(\cdot) \qquad \text{and} \qquad t\mathbf{P}\Big(\frac{\boldsymbol{Y}}{b(t)} \in \cdot\Big) \xrightarrow{v} c\nu(\cdot)$$

in $M_+(\mathfrak{C})$ *for some* $c > 0$ *and non-null Radon measure* ν *on* $\mathfrak{C}$. *We write* $X \overset{te(\mathfrak{C})}{\sim} Y$.

Proposition 4.2. *Suppose* $(X^*, Y^*) \in \mathbb{R}^2$ *is standard regularly varying on the cone* $\mathbb{E}_\sqcap$ *with limit measure* $\nu_\sqcap$ *and angular measure* $S_\sqcap$ *on* $[0,1)$. *The following are equivalent:*

1. $S_\sqcap$ *is finite on* $[0,1)$.
2. *There exists a random vector* $(X^\#, Y^\#)$ *defined on* $\mathbb{E} = [0,\infty]^2 \setminus \{\mathbf{0}\}$ *such that*

$$(X^\#, Y^\#) \overset{te(\mathbb{E}_\sqcap)}{\sim} (X^*, Y^*)$$

and $(X^\#, Y^\#)$ *is multivariate regularly varying on* $\mathbb{E}$ *with limit measure* ν *such that* $\nu|_{\mathbb{E}_\sqcap} = \nu_\sqcap$.

Proof. Consider each implication separately.

(1) $\Rightarrow$ (2): Define the polar coordinate transformation $(R, \Theta) = (X^* + Y^*, \frac{X^*}{X^*+Y^*})$. From Section 3.4 and (3.14), for $r > 0$, and Λ a Borel subset of $[0,1)$, as $t \to \infty$,

$$t\mathbf{P}\Big[\frac{R}{t} > r, \Theta \in \Lambda\Big] \to r^{-1}S_\sqcap(\Lambda) = \nu_\sqcap\{(x,y) \in \mathbb{E}_\sqcap : x + y > r, \frac{x}{x+y} \in \Lambda\}.$$

Since $S_\sqcap$ is finite on [0,1), the distribution of Θ is finite on $[0,1)$. Assume $S_\sqcap[0,1) = 1$ so that it is a probability measure and extend the measure $S_\sqcap$ to $[0,1]$ by putting $S_\sqcap(\{1\}) = 0$. Let us define R_0 and Θ_0 such that they are independent, Θ_0 has distribution given by the extended $S_\sqcap$ on [0,1] and R_0 has the standard Pareto distribution. Define $(X^\#, Y^\#) = (R_0\Theta_0, R_0(1-\Theta_0))$, and $(X^\#, Y^\#)$ is regularly varying on $\mathbb{E}$ with standard scaling and limit measure ν where $\nu|_{\mathbb{E}_\sqcap} = \nu_\sqcap$.

(2) $\Rightarrow$ (1): Referring to (3.14) note that $S_\sqcap\big([0,1)\big) = \nu_\sqcap\{(x,y) \in \mathbb{E}_\sqcap : x + y > 1\}$. Since $(X^\#, Y^\#)$ is regularly varying on $\mathbb{E}$, we have

$$t\mathbf{P}(\frac{X^\# + Y^\#}{t} > 1) \to \nu\{(x,y) \in \mathbb{E}_\sqcap : x + y > 1\} < \infty.$$

However,

$$\nu\{(x,y) \in \mathbb{E}_\sqcap : x+y > 1\} = \nu_\sqcap\{(x,y) \in \mathbb{E}_\sqcap : x + y > 1\} = S_\sqcap\big([0,1)\big).$$

Hence $S_\sqcap$ is finite on $[0,1)$. $\square$

5. Examples

This section presents examples that illustrate how the CEVM differs from the usual multivariate extreme value model. The initial example emphasizes that different normalizations are required for different cones. Cf. Example 5.1 in [15].

***Example* 4.** Let X, Z be i.i.d. *Pareto*(1) random variables. Define $Y = X^2 \wedge Z^2$ and observe as $t \to \infty$:

(i) In $\mathbb{M}_+(\mathbb{E})$,

$$t\mathbf{P}\Big[\Big(\frac{X}{t}, \frac{Y}{t}\Big) \in ([0,x] \times [0,y])^c\Big] \to \frac{1}{x} + \frac{1}{y}, \quad x \vee y > 0. \tag{5.1}$$

(ii) In $\mathbb{M}_+(\mathbb{E}_0)$, for $\frac{1}{2} < \alpha < 1$,

$$t\mathbf{P}\Big[\Big(\frac{X}{t^\alpha}, \frac{Y}{t^{2(1-\alpha)}}\Big) \in (x,\infty] \times (y,\infty]\Big] \to \frac{1}{x\sqrt{y}}, \quad x \wedge y > 0,$$

or in standard form,

$$t\mathbf{P}\Big[\Big(\frac{X^{1/\alpha}}{t}, \frac{Y^{1/2(1-\alpha)}}{t}\Big) \in (x,\infty] \times (y,\infty]\Big] \to \frac{1}{x^\alpha y^{1-\alpha}}, \quad x \wedge y > 0. \tag{5.2}$$

(iii) In $\mathbb{M}_+(\mathbb{E}_\sqcap)$, the limit is not a product measure, and in standard form

$$t\mathbf{P}\Big[\Big(\frac{X^2}{t}, \frac{Y}{t}\Big) \in [0,x] \times (y,\infty]\Big] \to \frac{1}{y} - \frac{1}{\sqrt{y}} \times \frac{1}{\sqrt{x} \vee \sqrt{y}}, \quad x \geq 0, y > 0. \tag{5.3}$$

(iv) In $\mathbb{M}_+(\mathbb{E}_\sqsupset)$, again, the limit is not a product measure, and in standard form

$$t\mathbf{P}\Big[\Big(\frac{X}{t}, \frac{Y^{1/2}}{t}\Big) \in (x,\infty] \times [0,y]\Big] \to \frac{1}{x} - \frac{1}{x \vee y}, \quad x > 0, y \geq 0. \tag{5.4}$$

These results can be expressed in polar co-ordinates by using the transformation $(r,\theta) : (x,y) \mapsto (x+y, \frac{x}{x+y})$ (Section 3.4). Note that the absolute value of the Jacobian of the inverse transformation here is $|J| = r$. Hence,

$$f_{R,\Theta}(r,\theta) = r f_{X,Y}(r\theta, r(1-\theta)).$$

We revisit the different cones and analyze the angular measure.

(i) The angular measure has a point mass at 0 and 1,

$$S(d\theta) = \delta_{\{0\}}(d\theta) + \delta_{\{1\}}(d\theta).$$

(ii) The limit measure in standard form is $\mu((x,\infty]\times(y,\infty]) = \frac{1}{x^{\alpha}y^{1-\alpha}}$, $x\wedge y>0$ and $\frac{1}{2}<\alpha<1$. Hence,

$$\mu'(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\mu((x,\infty]\times(y,\infty]) = \frac{\alpha(1-\alpha)}{x^{\alpha+1}y^{2-\alpha}}.$$

Taking the polar coordinate transformation

$$\mu'_{R,\Theta}(r,\theta) = r\frac{\alpha(1-\alpha)}{(r\theta)^{\alpha+1}(r(1-\theta))^{2-\alpha}} = r^{-2}\frac{\alpha(1-\alpha)}{\theta^{\alpha+1}(1-\theta)^{2-\alpha}}.$$

The right side is a product, as expected. Thus the angular measure has density

$$S(d\theta) = \frac{\alpha(1-\alpha)}{\theta^{\alpha+1}(1-\theta)^{2-\alpha}}\mathbf{1}_{\{0<\theta<1\}}d\theta.$$

(iii) The limit measure in standard form is

$$\mu([0,x]\times(y,\infty]) = \frac{1}{y} - \frac{1}{\sqrt{y}}\times\frac{1}{\sqrt{x}\vee\sqrt{y}}, \quad x\geq 0, y>0,$$

which is equivalent to

$$\mu((x,\infty]\times(y,\infty]) = \frac{1}{\sqrt{y}}\times\frac{1}{\sqrt{x}\vee\sqrt{y}}, \quad x\geq 0, y>0.$$

Hence, for $x>y>0$

$$\mu'(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\mu((x,\infty]\times(y,\infty]) = \frac{1}{4}\frac{1}{x^{3/2}y^{3/2}}.$$

Taking the polar coordinate transformation, we get for $\theta>1/2$,

$$\mu'_{R,\Theta}(r,\theta) = r\frac{1}{4}\frac{1}{(r\theta)^{3/2}(r(1-\theta))^{3/2}} = \frac{1}{4}r^{-2}\frac{1}{\theta^{3/2}(1-\theta)^{3/2}},$$

the density of a product measure. For $x\leq y$ the density does not exist and we have a point mass at $\theta=\frac{1}{2}$ whose weight can be calculated using (3.17). Thus the angular measure has density,

$$S(d\theta) = (2-\sqrt{3})\delta_{\{1/2\}}(d\theta) + \frac{1}{4}\theta^{-3/2}(1-\theta)^{-3/2}\mathbf{1}_{\{1/2<\theta<1\}}d\theta.$$

(iv) The angular measure has a point mass at $\frac{1}{2}$,

$$S(d\theta) = 2\delta_{\{1/2\}}(d\theta).$$

***Example* 5.** Suppose in Definition 1.1 we have functions $\alpha(t) > 0, \beta(t) \in \mathbb{R}$ satisfying (1.8) where $\alpha(\cdot) \in RV_\rho$ for some $\rho \neq 0$ and $\psi_2(c) \equiv 0$. Refer to [11, Remark 2, page 545]. In such a case, the limit measure μ satisfies:

$$\mu([-\infty, x] \times (y, \infty]) = y^{-1}\mu([-\infty, \frac{x}{y^\rho}] \times (1, \infty]) = y^{-1}H\big(\frac{x}{y^\rho}\big) \tag{5.5}$$

for $x \in \mathbb{R}$ and $y > 0$, where $H(\cdot)$ is a proper non-degenerate distribution. The following is an example of such a limit measure.

Assume $0 < \rho < 1$ and suppose $X \sim Pareto(\rho)$ and $Z \sim Pareto(1-\rho)$ are independent. Define $Y = X \wedge Z$ and we have for $x \geq y > 0$ and t large,

$$\begin{aligned} t\mathbf{P}\Big[\Big(\frac{X}{t}, \frac{Y}{t}\Big) \in [0, x] \times (y, \infty]\Big] &= t\mathbf{P}\Big(\frac{X}{t} \leq x, \frac{X}{t} > y, \frac{Z}{t} > y\Big) = \frac{1}{y^{1-\rho}}\Big(\frac{1}{y^\rho} - \frac{1}{x^\rho}\Big), \\ &= \frac{1}{y}\Big(1 - \frac{y^\rho}{x^\rho}\Big) =: \mu^{**}([-\infty, x] \times (y, \infty]). \end{aligned}$$

Now as in case 1 of the summary following Theorem 3.2, we have for $x \geq y^\rho > 0$,

$$\begin{aligned} t\mathbf{P}\Big[\Big(\frac{\alpha(X)}{\alpha(t)}, \frac{Y}{t}\Big) \in [0, x] \times (y, \infty]\Big] &\to \mu^{**}([0, x^{1/\rho}] \times (y, \infty]) = \frac{1}{y}\Big(1 - \frac{y^\rho}{x}\Big) \\ &=: \mu([0, x] \times (y, \infty]). \end{aligned}$$

If we take $H(\cdot)$ to be $Pareto(1)$, then we have the limit measure for $x \geq 0, y > 0$,

$$\mu([-\infty, x] \times (y, \infty]) \ := \frac{1}{y}H\big(\frac{x}{y^\rho}\big).$$

***Example* 6.** This example gives a class of limit distributions on $\mathbb{E}_\sqcap$ indexed by probability distributions on $[0, \infty]$. Suppose R is a Pareto random variable on $[1, \infty)$ with parameter 1 and ξ is a random variable with distribution $G(\cdot)$ on $[0, \infty]$. Assume that ξ and R are independent and define $(X, Y) = (R\xi, R)$. Then for $y > 0, x \geq 0$ and $ty > 1$,

$$\begin{aligned} t\mathbf{P}\Big[\frac{X}{t} \leq x, \frac{Y}{t} > y\Big] &= t\mathbf{P}\Big[\frac{R\xi}{t} \leq x, \frac{R}{t} > y\Big] = t\int_{ty}^{\infty} \mathbf{P}\big[\xi \leq \frac{tx}{r}\big]r^{-2}dr \\ &= \int_y^\infty \mathbf{P}\big[\xi \leq \frac{x}{s}\big]s^{-2}ds = \int_y^\infty G\big(\frac{x}{s}\big)s^{-2}ds = \frac{1}{x}\int_0^{x/y} G(s)ds = \mu\big([0, x] \times (y, \infty]\big). \end{aligned}$$

This can be expressed in polar co-ordinates. The angular measure $S(\cdot)$ on $\mathbb{E}_\sqcap$ is

$$S([0, \eta]) = \mu\{(u, v) : u + v > 1, \frac{u}{u + v} \leq \xi\}, \quad 0 \leq \eta < 1.$$

Hence we have

$$t\mathbf{P}\Big[\frac{X + Y}{t} > 1, \frac{X}{X + Y} \leq \eta\Big] = t\mathbf{P}\Big[\frac{R\xi + R}{t} > 1, \frac{R\xi}{R\xi + R} \leq \eta\Big]$$

$$= t\mathbf{P}\Big[\frac{R(1+\xi)}{t} > 1, \xi \le \frac{\eta}{1-\eta}\Big] = t \int\limits_{0\le s\le \frac{\eta}{1-\eta}} \mathbf{P}\Big[\frac{R}{t}(1+s) > 1\Big] G(ds)$$

$$= t \int\limits_{0\le s\le \frac{\eta}{1-\eta}} \Big(\frac{t}{1+s} \vee 1\Big)^{-1} G(ds) = \int\limits_{0\le s\le \frac{\eta}{1-\eta}} (1+s)G(ds),$$

for $t > 1/1-\eta$. But the left side goes to $\mu\{(u,v) : u+v > 1, \frac{y}{u+v} \le \xi\} = S([0,\eta])$ as $t \to \infty$ and thus,

$$S([0,\eta]) = \int\limits_{0\le s\le \frac{\eta}{1-\eta}} (1+s)G(ds), \qquad 0 \le \eta < 1.$$

Hence S is a finite angular measure if and only if G has first moment.

6. Proofs

In this section we provide proofs of some of the results given in the previous sections.

6.1. Proof of Theorem 2.1.

Assume $\lambda > 0, \gamma > 0$; other cases can be dealt with similarly. From (2.2) and (2.3) respectively we get

$$t\mathbf{P}\Big(\frac{Y-b(t)}{a(t)} > y\Big) \to (1+\gamma y)^{-1/\gamma}, \qquad 1+\gamma y > 0, \tag{6.1}$$

$$t\mathbf{P}\Big(\frac{X-\phi(t)}{\chi(t)} > x\Big) \to (1+\lambda x)^{-1/\lambda}, \qquad 1+\lambda x > 0. \tag{6.2}$$

Hence for $(x,y) \in \mathbb{E}^{(\lambda)} \times \mathbb{E}^{(\gamma)}$,which are continuity points of the limit measures μ and ν,

$$\begin{aligned} Q_t(x,y) :&= t\mathbf{P}\Big[\Big(\frac{X-\phi(t)}{\chi(t)}, \frac{Y-b(t)}{a(t)}\Big) \in \big([-\infty,x] \times [-\infty,y]\big)^c\Big] \\ &= t\mathbf{P}\Big[\frac{X-\phi(t)}{\chi(t)} > x\Big] + t\mathbf{P}\Big[\frac{Y-b(t)}{a(t)} > y\Big] - t\mathbf{P}\Big[\frac{X-\phi(t)}{\chi(t)} > x, \frac{Y-b(t)}{a(t)} > y\Big] \\ &= A_t(x) + B_t(y) + C_t(x,y) \qquad \text{(say)}. \end{aligned} \tag{6.3}$$

It suffices to show $Q_t(x,y)$ has a limit and the limit is non-degenerate in (x,y) (using a generalized version of [22, Lemma 6.1]). As $t \to \infty$ we have the limits for $A_t(x)$ and $B_t(y)$ from (6.2) and (6.1). Clearly $0 \le C_t(x,y) \le \min(A_t(x), B_t(y))$ and these inequalities hold for any limit of Q_t as well.

From [11, Proposition 1], there exist functions $\psi_1(\cdot), \psi_2(\cdot), \psi_3(\cdot), \psi_4(\cdot)$ such that

$$\lim_{t\to\infty}\frac{\alpha(tz)}{\alpha(t)} = \psi_1(z) = z^{\rho_1}, \qquad \lim_{t\to\infty}\frac{\beta(tz)-\beta(t)}{\alpha(t)} = \psi_2(z), \tag{6.4}$$

$$\lim_{t\to\infty}\frac{c(tz)}{c(t)} = \psi_3(z) = z^{\rho_2}, \qquad \lim_{t\to\infty}\frac{d(tz)-d(t)}{c(t)} = \psi_4(z). \tag{6.5}$$

for $z > 0$ and ρ_1, ρ_2 real. Temporarily assume ρ_1 and ρ_2 positive. Either $\psi_2(z) = 0$, which implies $\lim_{t\to\infty}\beta(t)/\alpha(t) = 0$ (from [1, Theorem 3.1.12 a,c]), or $\psi_2(z) = k(z^{\rho_1}-1)/\rho_1$ for $k \neq 0$, which means $\lim_{t\to\infty}\beta(t)/\alpha(t) = k/\rho_1$ ([6, Proposition B.2.2]). Hence allowing the constant k to be zero as well, we can write both cases as $\lim_{t\to\infty}\beta(t)/\alpha(t) = k_1/\rho_1$ for some $k_1 \in \mathbb{R}$. Similarly we have $\lim_{t\to\infty} d(t)/c(t) = k_2/\rho_2$ for some $k_2 \in \mathbb{R}$.

Additionally, marginal domain of attraction conditions for X, Y yield ($z > 0$, $w > 0$),

$$\lim_{t\to\infty}\frac{b(tz)-b(t)}{a(t)} = \frac{z^\gamma - 1}{\gamma}, \quad \lim_{t\to\infty}\frac{\phi(tw)-\phi(t)}{\chi(t)} = \frac{w^\lambda-1}{\lambda}, \tag{6.6}$$

which imply

$$\lim_{t\to\infty}\frac{a(tz)}{a(t)} = z^\gamma, \quad \lim_{t\to\infty}\frac{\chi(tw)}{\chi(t)} = w^\lambda. \tag{6.7}$$

Observe that

$$\begin{aligned} C_t(x,y) &= t\mathbf{P}\Big[\frac{X-\phi(t)}{\chi(t)} > x, \frac{Y-b(t)}{a(t)} > y\Big] \\ &= t\mathbf{P}\Big[\frac{X-\beta(t)}{\alpha(t)} > \Big(x + \frac{\phi(t)}{\chi(t)}\Big)\frac{\chi(t)}{\alpha(t)} - \frac{\beta(t)}{\alpha(t)}, \frac{Y-b(t)}{a(t)} > y\Big], \end{aligned} \tag{6.8}$$

and also

$$C_t(x,y) = t\mathbf{P}\Big[\frac{X-\phi(t)}{\chi(t)} > x, \frac{Y-d(t)}{c(t)} > \Big(y + \frac{b(t)}{a(t)}\Big)\frac{a(t)}{c(t)} - \frac{d(t)}{c(t)}.\Big] \tag{6.9}$$

From [6, Proposition B.2.2] we have that

$$\frac{b(t)}{a(t)} \to \frac{1}{\gamma} \quad \text{and} \quad \frac{\phi(t)}{\chi(t)} \to \frac{1}{\lambda}. \tag{6.10}$$

We analyze $C_t(x,y)$ for the different cases. First we will show that at least one of the limits $\lim\limits_{t\to\infty}\frac{\chi(t)}{\alpha(t)}$ and $\lim\limits_{t\to\infty}\frac{a(t)}{c(t)}$ has to exist. Suppose both do not exist. We have for $(x,y) \in \mathbb{E}^{(\lambda)} \times \mathbb{E}^{(\gamma)}$, which are continuity points of the limit measures μ and ν,

$$t\mathbf{P}\Big[\frac{X-\beta(t)}{\alpha(t)} > x, \frac{Y-b(t)}{a(t)} > y\Big] \to \mu((x,\infty]\times(y,\infty]), \tag{6.11}$$

$$t\mathbf{P}\Big[\frac{X-\phi(t)}{\chi(t)} > x, \frac{Y-d(t)}{c(t)} > y\Big] \to \nu((x,\infty]\times(y,\infty]). \tag{6.12}$$

Now (6.11) implies

$$t\mathbf{P}\Big[\frac{X-\phi(t)}{\chi(t)}\frac{\chi(t)}{\alpha(t)}+\frac{\phi(t)-\beta(t)}{\alpha(t)}>x,\frac{Y-d(t)}{c(t)}\frac{c(t)}{a(t)}+\frac{d(t)-b(t)}{a(t)}>y\Big] \\ \to \mu\big((x,\infty]\times(y,\infty]\big),$$

which is equivalent to

$$t\mathbf{P}\Big[\frac{X-\phi(t)}{\chi(t)}>\frac{\alpha(t)}{\chi(t)}\big(x-\frac{\phi(t)-\beta(t)}{\alpha(t)}\big),\frac{Y-d(t)}{c(t)}>\frac{a(t)}{c(t)}\big(y-\frac{d(t)-b(t)}{a(t)}\big)\Big] \\ \to \mu\big((x,\infty]\times(y,\infty]\big).$$

From (6.12) we also have that the left side of the previous line has a limit

$$t\mathbf{P}\Big[\frac{X-\phi(t)}{\chi(t)}>\frac{\alpha(t)}{\chi(t)}\big(x-\frac{\phi(t)-\beta(t)}{\alpha(t)}\big),\frac{Y-d(t)}{c(t)}>\frac{a(t)}{c(t)}\big(y-\frac{d(t)-b(t)}{a(t)}\big)\Big] \\ \to \nu\big((f(x),\infty]\times(g(y),\infty]\big)$$

for some $(f(x),g(y))$, assumed to be a continuity point of the limit ν, iff as $t\to\infty$, the following two limits hold,

$$\frac{\alpha(t)}{\chi(t)}\Big(x-\frac{\phi(t)-\beta(t)}{\alpha(t)}\Big)\to f(x), \tag{6.13}$$

$$\frac{a(t)}{c(t)}\Big(y-\frac{d(t)-b(t)}{a(t)}\Big)\to g(y). \tag{6.14}$$

For ν to be non-degenerate f and g should be non-constant and we also have $\mu\big((x,\infty]\times(y,\infty]\big)=\nu\big((f(x),\infty]\times(g(y),\infty]\big)$. Considering (6.13) and (6.14) we can see that the limit as $t\to\infty$ exists iff $\lim_{t\to\infty}a(t)/c(t)$ and $\lim_{t\to\infty}\chi(t)/\alpha(t)$ exists.

We conclude $\lim_{t\to\infty}\chi(t)/\alpha(t)\in[0,\infty]$ and consider the following cases.

- **Case 1:** $\lim_{t\to\infty}\chi(t)/\alpha(t)=\infty$. Consider (6.8) and note

$$\Big(x+\frac{\phi(t)}{\chi(t)}\Big)\frac{\chi(t)}{\alpha(t)}-\frac{\beta(t)}{\alpha(t)}\to\Big(x+\frac{1}{\lambda}\Big)\times\infty-\frac{k_1}{\rho_1}=\infty,$$

which entails $\lim_{t\to\infty}C_t(x,y)=\mu(\{\infty\}\times(y,\infty])=0$. Hence

$$\lim_{t\to\infty}Q_t(x,y)=(1+\lambda x)^{-1/\lambda}+(1+\gamma y)^{-1/\gamma}.$$

- **Case 2:** $\lim_{t\to\infty}\chi(t)/\alpha(t)=M\in(0,\infty)$. From (6.8), we have

$$\Big(x+\frac{\phi(t)}{\chi(t)}\Big)\frac{\chi(t)}{\alpha(t)}-\frac{\beta(t)}{\alpha(t)}\to\Big(x+\frac{1}{\lambda}\Big)\times M-\frac{k_1}{\rho_1}=f(x),\quad\text{(say)}.$$

Therefore

$$\lim_{t\to\infty}C_t(x,y)=\mu((f(x),\infty]\times(y,\infty])\le(1+\lambda y)^{-1/\lambda}$$

with strict inequality holding for some x because of the non-degeneracy condition (1.3) for μ. Hence

$$\lim_{t\to\infty} Q_t(x,y) = ((1+\lambda x)^{-1/\lambda} + (1+\gamma y)^{-1/\gamma} - \mu((f(x),\infty]\times(y,\infty]).$$

- **Case 3:** $\lim_{t\to\infty}\chi(t)/\alpha(t) = 0$. In this case (6.8) leads to a degenerate limit in x for $C_t(x,y)$ and putting $M_1 = k/\rho_1$ we get

$$\lim_{t\to\infty} C_t(x,y) = \mu((M_1,\infty]\times(y,\infty]) =: f_1(y) \le (1+\gamma y)^{-1/\gamma}.$$

So consider (6.9).

1. If $\lim_{t\to\infty} a(t)/c(t)$ exists in $(0,\infty]$, then we can use a similar technique as in case 1 or case 2 to obtain a non-degenerate limit for $Q_t(x,y)$.
2. If $\lim_{t\to\infty} a(t)/c(t) = 0$, then for some $M_2\in\mathbb{R}$,

$$\lim_{t\to\infty} C_t(x,y) = \nu((x,\infty]\times(M_2,\infty] =: f_2(x) \le (1+\lambda x)^{-1/\lambda}$$

Therefore we have for any $(x,y)\in\mathbb{E}^{(\lambda)}\times\mathbb{E}^{(\gamma)}$,which are continuity points of the limit measures μ and ν,

$$f_1(y) = \mu((M_1,\infty]\times(y,\infty]) = \nu((x,\infty]\times(M_2,\infty]) = f_2(x).$$

It is easy to check now that for any $(x,y)\in\mathbb{E}^{(\lambda)}\times\mathbb{E}^{(\gamma)}$, which are continuity points of the limit measures μ and ν, we have $f_1(y) = f_2(x) = 0$. Hence $C_t(x,y)\to 0$ and thus $Q_t(x,y)$ has a non-degenerate limit.

This proves the result.

6.2. Proof of the Summary following Theorem 3.2

(1) This part has been dealt with in [11, Section 2.4].

(2) First simplify the problem. For (x,y) a continuity point of $\mu(\cdot)$,

$$t\mathbf{P}\Big[\frac{\lambda(X^*)-\beta(t)}{\alpha(t)} \le x, \frac{b(Y^*)-b(t)}{a(t)} > y\Big] \to \tilde\mu\big([-\infty,x]\times(y,\infty]\big) \qquad (t\to\infty)$$

is equivalent, as $t\to\infty$, to

$$t\mathbf{P}\Big(\frac{\lambda(X^*)-\beta(t)}{\alpha(t)} \le x, \frac{Y^*}{t} > y\Big) \to \tilde\mu\big([-\infty,x]\times(h(y),\infty]\big) =: \mu^*\big([-\infty,x]\times(y,\infty]\big) \tag{6.15}$$

where

$$h(y) = \begin{cases}(1+\gamma y)^{\frac{1}{\gamma}} & \gamma\neq 0\\ e^y & \gamma = 0.\end{cases} \tag{6.16}$$

Hence (3.11) is equivalent to

$$t\mathbf{P}\Big[\Big(\frac{\lambda(X^*)-\beta(t)}{\alpha(t)},\frac{Y^*}{t}\Big)\in\cdot\Big]\xrightarrow{v}\mu^*(\cdot)$$

and μ^* is a non-null Radon measure on $[-\infty,\infty]\times\bar{\mathbb{E}}^{(\gamma)}$ satisfying the conditional non-degeneracy conditions. Hence our proof will show the existence of $\lambda(\cdot)$ satisfying (6.15). Now note that (1.8) implies that $\alpha(\cdot)\in RV_\rho$ for some $\rho\in\mathbb{R}$ and $\psi_1(x)=x^\rho$ [23, page 14]. The function $\psi_2(\cdot)$ may be identically equal to 0, or

$$\psi_2(x)=\begin{cases}k(x^\rho-1)/\rho, & \text{if } \rho\neq 0, x>0\\ k\log x & \text{if } \rho=0, x>0\end{cases} \tag{6.17}$$

for $k\neq 0$ [6, page 373]. We have assumed that $(\psi_1,\psi_2)\neq(1,0)$. We will consider three cases: $\rho>0,\rho=0,\rho<0$.

Case 1 : $\rho>0$. First suppose $\psi_2\equiv 0$. Since $\alpha(\cdot)\in RV_\rho$, there exists $\tilde\alpha(\cdot)\in RV_\rho$ which is ultimately differentiable and strictly increasing and $\alpha\sim\tilde\alpha$ [6, page 366]. Thus $\tilde\alpha^{\leftarrow}$ exists. Additionally, we have from [1, Theorem 3.1.12(a)], that $\beta(t)/\alpha(t)\to 0$. Hence we have for $x>0$, as $t\to\infty$,

$$\frac{\tilde\alpha(tx)+\beta(t)}{\alpha(t)}=\frac{\tilde\alpha(tx)}{\tilde\alpha(t)}\cdot\frac{\tilde\alpha(t)}{\alpha(t)}+\frac{\beta(t)}{\alpha(t)}\to x^\rho,$$

and inverting we get for $z>0$

$$\frac{\tilde\alpha^{\leftarrow}\big(\alpha(t)z+\beta(t)\big)}{t}\to z^{1/\rho}\qquad(t\to\infty).$$

Thus we have,

$$t\mathbf{P}\Big[\frac{\tilde\alpha(X^*)-\beta(t)}{\alpha(t)}\le x,\frac{Y^*}{t}>y\Big]=t\mathbf{P}\Big[\frac{X^*}{t}\le\frac{\tilde\alpha^{\leftarrow}(\alpha(t)x+\beta(t))}{t},\frac{Y^*}{t}>y\Big]$$
$$\to\mu^{**}\big([0,x^{1/\rho}]\times(y,\infty]\big).$$

Set $\lambda(\cdot)=\tilde\alpha(\cdot)$ and this defines $\tilde\mu$.

Next suppose $\psi_2\neq 0$. Therefore

$$\psi_2(x)=\lim_{t\to\infty}\frac{\beta(tx)-\beta(t)}{\alpha(t)}=k(x^\rho-1)/\rho;$$

that is, $\beta(\cdot)\in RV_\rho$ and $k>0$. There exists $\tilde\beta$ which is ultimately differentiable and strictly increasing and $\tilde\beta\sim\beta$ [6, page 366]. Thus $\tilde\beta^{\leftarrow}$ exists. Then we have for $x>0$, as $t\to\infty$,

$$\frac{\tilde\beta(tx)-\beta(t)}{\alpha(t)}=\frac{\tilde\beta(tx)-\beta(tx)}{\alpha(t)}+\frac{\beta(tx)-\beta(t)}{\alpha(t)}$$

$$= \frac{\tilde\beta(tx)-\beta(tx)}{\beta(tx)}\frac{\beta(tx)}{\alpha(tx)}\frac{\alpha(tx)}{\alpha(t)} + \frac{\beta(tx)-\beta(t)}{\alpha(t)}$$
$$\to (1-1)\cdot\frac{1}{\rho}\cdot x^\rho + k\frac{x^\rho-1}{\rho} = k\frac{x^\rho-1}{\rho}.$$

Inverting, we get as $t\to\infty$,

$$\frac{\tilde\beta^{\leftarrow}\big(\alpha(t)x+\beta(t)\big)}{t} \to (1+\frac{\rho x}{k})^{1/\rho}.$$

Thus we have,

$$t\mathbf{P}\Big[\frac{\tilde\beta(X^*)-\beta(t)}{\alpha(t)}\le x, \frac{Y^*}{t}>y\Big] = t\mathbf{P}\Big[\frac{X^*}{t}\le\frac{\tilde\beta^{\leftarrow}(\alpha(t)x+\beta(t))}{t}, \frac{Y^*}{t}>y\Big]$$
$$\to \mu^{**}\big([0,(1+\frac{\rho x}{k})^{1/\rho}]\times(y,\infty]\big).$$

Here we can set $\lambda(\cdot)=\tilde\beta(\cdot)$ and this defines $\tilde\mu$.

Case 2: $\rho=0$. We have $\psi_1(x)=1, \psi_2(x)=k\log x$ for $x>0$ and some $k\in\mathbb{R}$. By assumption, $(\psi_1,\psi_2)\neq(1,0)$ and hence $k\neq 0$. First assume that $k>0$, which means $\beta\in\Pi_+(\alpha)$. There exists $\tilde\beta(\cdot)$ which is continuous, strictly increasing and $\beta-\tilde\beta=o(\alpha)$ [8, page 1031]. If $\beta(\infty)=\tilde\beta(\infty)=\infty$, then for $x>0$,

$$\frac{\tilde\beta(tx)-\beta(t)}{\alpha(t)} = \frac{\tilde\beta(tx)-\beta(tx)}{\alpha(tx)}\frac{\alpha(tx)}{\alpha(t)} + \frac{\beta(tx)-\beta(t)}{\alpha(t)} \to 0+k\log x,$$

and inverting, we get for $z\in\mathbb{R}$, as $t\to\infty$, $\tilde\beta^{\leftarrow}\big(\alpha(t)z+\beta(t)\big)/t\to\exp\{z/k\}$. Thus we have,

$$t\mathbf{P}\Big(\frac{\tilde\beta(X^*)-\beta(t)}{\alpha(t)}\le x, \frac{Y^*}{t}>y\Big) = t\mathbf{P}\Big(\frac{X^*}{t}\le\frac{\tilde\beta^{\leftarrow}(\alpha(t)x+\beta(t))}{t}, \frac{Y^*}{t}>y\Big)$$
$$\to \mu\big([0,e^{k/x}]\times(y,\infty]\big).$$

If $\beta(\infty)=\tilde\beta(\infty)=B<\infty$, define

$$\beta^*(t)=\frac{1}{B-\tilde\beta(t)}, \qquad \alpha^*(t)=\frac{\alpha(t)}{(B-\tilde\beta(t))^2}$$

and we have that $\beta^*\in\Pi_+(\alpha^*)$, $\beta^*(t)\to\infty$ and $\frac{B-\tilde\beta(t)}{\alpha(t)}\to\infty$ [10, page 25]. Hence we have reduced the problem to the previous case which implies,

$$t\mathbf{P}\Big(\frac{\beta^*(X^*)-\beta^*(t)}{\alpha^*(t)}\le x, \frac{Y^*}{t}>y\Big)\to\mu\big([0,e^{k/x}]\times(y,\infty]\big),$$

or equivalently,

$$t\mathbf{P}\Big(\frac{\tilde\beta(X^*)-\tilde\beta(t)}{\alpha(t)} \le \frac{x}{1+\frac{\alpha(t)x}{B-\tilde\beta(t)}}, \frac{Y^*}{t} > y\Big) \to \mu\big([0,e^{k/x}]\times(y,\infty]\big)$$

and since $B-\tilde\beta(t)/\alpha(t)\to\infty$ implies $\alpha(t)/B-\tilde\beta(t)\to 0$, we can write

$$t\mathbf{P}\Big(\frac{\tilde\beta(X^*)-\tilde\beta(t)}{\alpha(t)} \le x, \frac{Y^*}{t} > y\Big) \to \mu\big([0,e^{k/x}]\times(y,\infty]\big),$$

which implies since since $\beta-\tilde\beta = o(\alpha)$,

$$t\mathbf{P}\Big(\frac{\tilde\beta(X^*)-\beta(t)}{\alpha(t)} \le x, \frac{Y^*}{t} > y\Big) \to \mu\big([0,e^{k/x}]\times(y,\infty]\big)$$

and we have produced the required transformation $\lambda(\cdot)=\tilde\beta(\cdot)$.

The case for which $k<0$; i.e., $\beta\in\Pi_-(\alpha)$ can be proved similarly.

Case 3: $\rho<0$. This case is similar to the case for $\rho>0$ and is omitted.

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