We proved the following result in previous lectures.

**Theorem 1 (Farkas’ Lemma)**  Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^{m \times 1}$. Then exactly one of the following two condition holds:

1. $\exists x \in \mathbb{R}^{n \times 1}$ such that $Ax = b$, $x \geq 0$;
2. $\exists y \in \mathbb{R}^{1 \times m}$ such that $yA \geq 0$, $yb < 0$.

We will get a similar result, which follows from Farkas’ Lemma.

**Theorem 2 (Farkas’ Lemma’)**  Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^{m \times 1}$. Then exactly one of the following two condition holds:

1’. $\exists x \in \mathbb{R}^{n \times 1}$ such that $Ax \leq b$;
2’. $\exists y \in \mathbb{R}^{1 \times m}$ such that $yA = 0$, $yb < 0$, $y \geq 0$.

and the following condition is equivalent to (2’):

2’’. $\exists y \in \mathbb{R}^{1 \times m}$ such that $yA = 0$, $yb = -1$, $y \geq 0$.

**Proof.** First we prove that (2’) $\iff$ (2’’). (2’’) clearly implies (2’). (2’) is homogeneous, s.t. we can always rescale a ’good’ $y$ by any constant $> 0$. Start with $y$ satisfying (2’) and consider $\hat{y} = -\frac{1}{yb}y$, we see that $\hat{y}$ satisfies (2’’).

Then prove not both. Suppose $\hat{x}$ and $\hat{y}$ satisfy both (1’) and (2’). The two conditions imply:

$$0 > \hat{y}b \geq \hat{y}(A\hat{x}) = (\hat{y}A)\hat{x} = 0\hat{x} = 0$$

which is a contradiction, so (1’) and (2’) cannot both hold. Now suppose (2’) does not hold, so (2’’)) does not hold either. Rewrite the system $yA = 0$, $yb = -1$ as:

$$[y \ A \ b] = [0 \ \cdots \ 0 \ -1]$$

Letting superscript $t$ denote matrix transposition, define:

$$\tilde{A} = [A \ b]^t = \begin{bmatrix} A^t \\ b^t \end{bmatrix}, \ \tilde{x} = y^t$$

Then there does not exist $\tilde{x} \geq 0$ such that

$$\tilde{A}\tilde{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}$$
So \(-(2'')\) is a cosmetic rewriting of \(\neg(1)\). Apply the original Farkas’ Lemma in the form \(\neg(1) \Rightarrow (2)\). That is, \(\exists \tilde{y}\) such that

\[
\tilde{y}A \geq 0, \quad \begin{bmatrix} 0 \\ \vdots \\ -1 \end{bmatrix} < 0
\]

Rewrite \(\tilde{y} = \begin{bmatrix} x^t \\ \lambda \end{bmatrix}\). Then

\[
x^t \cdot 0 - \lambda < 0 \quad \Rightarrow \quad \lambda > 0
\]

\[
\Rightarrow Ax \geq -\lambda b \quad \Rightarrow \quad A \left( \frac{-x}{\lambda} \right) \leq b
\]

So \(-\lambda x\) satisfies condition \((1')\) ■

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
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</thead>
<tbody>
<tr>
<td>max (cx)</td>
<td>min (yb)</td>
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<tr>
<td>s.t. (Ax \leq b)</td>
<td>s.t. (yA = c)</td>
</tr>
<tr>
<td>(y \geq 0)</td>
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We already know weak duality. We have the following notation:
If max has no feasible solution \(\iff\) \(\text{opt} = -\infty\).
If min has no feasible solution \(\iff\) \(\text{opt} = +\infty\).

**Theorem 3 (Strong Duality)** If there exists a feasible solution for at least one of the linear programs below, then the values of the two linear programs are equal.

**Proof.** There are three cases:

(Case 1): Let \(\tilde{y}\) be a feasible solution for the dual and suppose \(\tilde{x}\) a feasible solution for the primal. \(\tilde{x}, Ax \leq b\). Apply Farkas’ Lemma’ in the form \(\neg(1') \Rightarrow (2')\). Then there exists \(\tilde{y}\) such that \(\tilde{y}A = 0, \tilde{y}b < 0\), and \(\tilde{y} \geq 0\). Consider the ray defined by \(\tilde{y} + \lambda \tilde{y}, \lambda \geq 0\):

\[
(\tilde{y} + \lambda \tilde{y})A = c + \lambda \cdot 0 = c \quad \text{and} \quad (\tilde{y} + \lambda \tilde{y})b = \tilde{y}b + \lambda \tilde{y}b
\]

Since \(\tilde{y}b < 0\), as \(\lambda \to \infty\), the objective function value of the problem \(\to -\infty\).

(Case 2) Let \(\bar{x}\) be a feasible solution for the primal and suppose \(\bar{y}\) a feasible solution for the dual. We know \(\bar{y}\) solution to a system of = and \(\geq\) variables, so we have a starting point exactly like condition \(\neg(1)\). Apply the original Farkas’ Lemma \(\neg(1) \Rightarrow (2)\) and hence corresponding condition (2) holds.

\(\exists \hat{x}\) such that \(A\hat{x} \geq 0, c\hat{x} < 0\). This means that \(-\hat{x}\) is an improving direction for the primal objective function. Consider \(\bar{x} + \lambda \hat{x}\) where \(\lambda \leq 0\).

\[
A(\bar{x} + \lambda \hat{x}) = A\bar{x} + \lambda A\hat{x} \leq b, \quad c(\bar{x} + \lambda \hat{x}) = c\bar{x} + \lambda c\hat{x} \to +\infty
\]
as $\lambda \to \infty$. So $\bar{x} + \lambda \hat{x}$ stays feasible and has unbounded objective function value.

(Case 3) Let $\bar{x}$ and $\bar{y}$ be feasible solutions to the primal and dual respectively. By weak duality, we know that both optimal values are finite. Let $\gamma$ denote the optimal value of the dual. Suppose that the optimal value of the primal $< \gamma$:

$$\Rightarrow \exists \bar{x} \text{ s.t. } Ax \leq b, \ cx \geq \gamma$$

$$\Leftrightarrow \exists \bar{x} \text{ s.t. } \begin{bmatrix} A \\ -c \end{bmatrix} x \leq \begin{bmatrix} b \\ -\gamma \end{bmatrix}$$

This is a $\neg(1')$ statement, and hence (2') must correspondingly hold. So there exists a row vector $y \geq 0$ and a scalar $\lambda \geq 0$ such that:

$$[y' \ \lambda] \begin{bmatrix} A \\ -c \end{bmatrix} = 0, \quad [y' \ \lambda] \begin{bmatrix} b \\ -\gamma \end{bmatrix} < 0$$

We claim $\lambda \neq 0$. Suppose $\lambda = 0$, then $y' A = 0$, $y' b < 0$, and $y' \geq 0$. That is (2'). Use (2') $\Rightarrow \neg(1')$, which implies that there does not exist $x$ such that $Ax \leq b$. The primal is feasible, so this is a contradiction. Therefore $\lambda > 0$. Expanding out the above matrix equation:

$$y' A - \lambda c = 0 \quad \Rightarrow \quad \left(\frac{y'}{\lambda}\right) A = c$$

Also $\frac{y'}{\lambda} \geq 0$ since $\lambda > 0$. So $\frac{y'}{\lambda}$ is a feasible solution. However, $y' b - \lambda \gamma < 0$, so $\left(\frac{y'}{\lambda}\right) b < \gamma$, which contradicts the optimality of $\gamma$. ■