In this lecture, we consider a geometric interpretation to linear programming. Feasible regions are studied from a geometric viewpoint. We will try to provide a slightly different perspective to linear programming than what was done before.

Consider $\mathcal{P}$ to be the region defined by $\mathcal{P} = \{x : Ax \leq b\}$. Suppose, $x$ is two-dimensional and we represent the independent spatial variables as $x_1$ and $x_2$. By now, we all have a notion of vertices in two-dimensional space defined by the set of constraints (straight lines, for this case), $Ax \leq b$. These will be points where at least two lines intersect. The solid dots in the figure below are vertices whereas the hollow dots are not so. Now, let us look at some properties that these vertices satisfy. It is apparent that at any vertex, at least two of the constraints in $Ax \leq b$ are satisfied. However this statement is not sufficient since the hollow dots in the figure below are not vertices. Any vertex should, in fact, satisfy all the remaining constraints also. Summing up this paragraph, we give an (algebraic) mathematical definition of a vertex:

**Algebraic Definition:** A point $x^*$ is a vertex of $Ax \leq b$ if $Ax^* \leq b$ and we can partition the constraints of $A$ into $A^=$ and $A^<$ such that $A^= x^* = b^=$ and $A^< x^* < b^<$ where $A^=$ is of rank $n$.

Now, we will look at a geometric definition of a vertex. If we have three points, $p_1$, $p_2$ and $p_3$ as shown in the figure. What are the properties which are different in these three points, or in other words what exactly makes $p_3$ a vertex. If we perturb the point $p_1$ slightly, the new point still satisfies all the constraints. It is, hence, an interior point. The point $p_2$, on the other hand, satisfies one constraint. and if we perturb this point in two opposite directions, at least one of the perturbed points will satisfy the constraints, $Ax \leq b$. Now consider point $p_3$. It is impossible to perturb this point in two opposite directions while ensuring that both the perturbations satisfy the constraints (for $p_2$, this is possible). Rewriting our intuition mathematically, we arrive at the
following definition:

**Geometric Definition:** A point \( x^* \) is a vertex of \( Ax \leq b \) if it is feasible (i.e. \( Ax^* \leq b \)) and \( \forall y \), if \( x^* + y \) is feasible and \( x^* - y \) is feasible then \( y = 0 \).

Of course, given two “definitions” of a concept, one naturally wishes to establish that they are identical, which is the content of the next theorem.

**Theorem:**
For a point \( \hat{x} \in \mathbb{R}^n \), consider the system \( Ax \leq b \) satisfied by \( \hat{x} \) and let \( A^= \) denote those constraints which are satisfied with equality (i.e. \( a_i \hat{x} = b_i \) for each row of \( A^= \)) and let \( b^= \) denote the corresponding coordinates of \( b \). The following two conditions are equivalent:

(a) The matrix \( A^= \) has rank \( n \).
(b) For each direction \( y \in \mathbb{R}^n \), if \( \hat{x} + y \) and \( \hat{x} - y \) satisfy the constraints, then \( y = 0 \).

A point \( \hat{x} \) satisfying these conditions is called a vertex of the system \( Ax \leq b \).

**Proof.** Let \( P = \{ x : Ax \leq b \} \) and \( \hat{x} \in P \).

- \((a) \Rightarrow (b)\)
  This is the easy direction. Let us suppose we have both \( \hat{x} + y \in P \) and \( \hat{x} - y \in P \), i.e., \( A(\hat{x} + y) \leq b \), \( A(\hat{x} - y) \leq b \).

  In particular, we have
  \[
  A^= (\hat{x} + y) \leq b^=
  \]
  \[
  A^= (\hat{x} - y) \leq b^= 
  \]
  and

  \[
  A^= (\hat{x} + y) = A^= \hat{x} + A^= y = b^= + A^= y \implies A^= y \leq 0 
  \]
  \[
  A^= (\hat{x} - y) = A^= \hat{x} - A^= y = b^= - A^= y \implies -A^= y \leq 0 
  \]

  So \( A^= y = 0 \).

  Since \( A^= \) has rank \( n \), its columns are linearly independent. Therefore \( y = 0 \).

- \((b) \Rightarrow (a)\)
  Suppose not. Then the column rank of \( A^= \) is less than \( n \), which implies \( \exists \hat{y} \) s.t. \( A^= \hat{y} = 0 \) & \( \hat{y} \neq 0 \).

  The first attempt is to prove \( \hat{x} \pm \hat{y} \in P \), but this is actually too much (this is not so obvious now but if you actually try using it, you will see the mistake). Since \( \hat{y} \) is merely a direction, we should scale it since perturbations cannot be of arbitrary magnitude. So we will try to prove \( \hat{x} \pm \epsilon \hat{y} \in P \) for an extremely small \( \epsilon \).

  First, let us consider \( A(\hat{x} + \epsilon \hat{y}) \leq b \), which can be viewed as

  \[
  A^= (\hat{x} + \epsilon \hat{y}) \leq b^=
  \]
  \[
  A^<= (\hat{x} + \epsilon \hat{y}) \leq b^<
  \]

  where \( A^= \hat{x} = b^= \), and \( A^<= \hat{x} < b^< \).
It is easy to verify that \( A(\hat{x} + \epsilon \hat{y}) = A\hat{x} + \epsilon A\hat{y} = b^\geq + \epsilon \cdot 0 = b^\geq \).

We still need \( A^< (\hat{x} + \epsilon \hat{y}) \leq b^< \)

We know that \( A^< \hat{x} < b^< \), and the inequality holds for each component of the resultant product. Let \( A^< \hat{x} = \bar{b}^< \), and \( A^< \hat{y} = \bar{c}^< \). We can then choose \( \epsilon \) sufficiently small so that \( \bar{b}^< + \epsilon \bar{c}^< \leq b^< \) holds for each component. This is possible since the original constraint was a strict 'less than' and we can always find an extremely small number which will satisfy the constraint. The above technique can be applied again to \( \hat{x} - \epsilon \hat{y} \), i.e., we can choose \( \epsilon \) small enough so that both \( \hat{x} + \epsilon \hat{y}, \hat{x} - \epsilon \hat{y} \in \mathcal{P} \). We have found a non-zero \( y \) which satisfies the constraints. Since this is a contradiction to (b), our original assumption is wrong and we indeed have \( A^= \) of rank \( n \) if (b) is true.

In the above lemma, we give two equivalent definitions of vertices of a region given by a set of inequalities. Sometimes, we can actually specify the region itself by the vertices. To advance in this direction, we start with the concept of convexity. A set is said to be convex if points lying on any line segment joining two points in that set lies in the same set. This sentence translates mathematically to:

**Definition:** A set \( Q \) is convex if \( \forall x, y \in Q \), it follows that \( \lambda x + (1 - \lambda) y \in Q \) for all \( 0 \leq \lambda \leq 1 \).

Let \( x_1, \ldots, x_n \) be points in \( \mathbb{R}^m \), and

\[
Q = \{ x : x = \sum_{k=1}^{n} \lambda_k x_k, \sum_{k=1}^{n} \lambda_k = 1, \lambda_k \geq 0, i = 1, \ldots, k \}
\]

Here, \( Q \) consists of all the points that are the convex combinations of \( x_1, \ldots, x_n \). The figure above shows \( Q \) for a particular choice of 5 points in \( \mathbb{R}^2 \).