Problem Set 4
Mathematical Programming, OR 630
September 21, 2006

Due date: Friday, Sept 29, 2006.

1. Let $A$ be a symmetric $n$ by $n$ matrix, and consider the linear program to minimize $cx$ subject to $Ax \geq b$, $x \geq 0$, where $b$ is the transpose of $c$. Show that if $Ax^* = b$ and $x^* \geq 0$, then $x^*$ is optimal.

2. Consider a pair of primal and dual linear programs in standard form (in our usual notation). Fix some variable $x_j$ in the primal. Suppose that $x_j = 0$ in every optimal solution. Show that there exists an optimal solution $\bar{y}$ such that $\bar{y}A_j < c_j$. (Hint: create an auxiliary LP that that optimizes $x_j$ among all of the original optimal solutions, and apply LP duality.)

3. Consider a pair of primal and dual linear programs in standard form (in our usual notation). Let $x$ be a feasible solution to the primal, and let $N = \{i : x_i = 0\}$. Prove that $x$ is an optimal solution for the primal if and only if the optimal value of the linear program, minimize $cz$ subject to $Az = 0$, where $z_i \geq 0$ for each $i \in N$ is 0.

4. A polytope that has a particularly nice structure is the bipartite matching polytope. Consider a bipartite graph $G = (V_1, V_2, E)$ (that is, the node set is $V_1 \cup V_2$ and each edge has one endpoint in $V_1$ and one endpoint in $V_2$). Suppose $n = |V_1| = |V_2|$. A perfect matching $M$ in $G$ is a subset of $n$ edges such that no two edges share a common endpoint. Suppose that for each perfect matching $M$ we create a point $x^M$ in $R^{|E|}$ where the coordinate $x_e$, for each $e \in E$ is 1 if $e \in M$, and is 0 otherwise. (That is, $x^M$ is the so-called incidence vector of $M$.) Let $P(G)$ be the convex hull of all such points (that is, over all perfect matchings $M$ in $G$).

Consider the polar of $P(G)$, $P(G)^*$, and consider all integer points in the non-negative orthant of $P(G)^*$. What do these points correspond to in terms of the original graph?