Problem #1

Suppose $x$ is the current basic solution with basis $B$ and that the current dual solution $y = c_B A_B^{-1} s$ satisfies $(1 + \epsilon) c_j \geq y A_j$ for all $j$. Then it follows that $\frac{y}{1+\epsilon}$ is dual feasible and, by weak duality, $\frac{yb}{1+\epsilon} \leq cx^*$ where $x^*$ is the optimal solution. Hence, $yb \leq (1 + \epsilon)cx^*$. Finally, note that $yb = c_B A_B^{-1} b = c_B x_B = cx$, so $cx \leq (1 + \epsilon) cx^*$.

Problem #2

We can use the following algorithm:

Initialization: Start with an LP consisting of only slack variables; then, the current (optimal) basic solution is $s_e = u_e$ for all $e \in E$ and $x_P = 0$ for all $P$ (where $s_e, e \in E$ are the slack variables).

Now, given an optimal basis, $B$, compute the dual solution, $z_e$. For every $i$, compute the minimum cost path between $s_i$ and $t_i$ using edge costs $z_e$. If every such minimum cost path has cost at least 1, then STOP—the current solution is optimal. Else, choose a path $P$ with cost less than 1, add $x_P$ to the LP and solve to optimality. Repeat.

Proof of Algorithm:

Note that the dual problem to the original LP is:

$$\begin{align*}
\text{min} & \quad \sum_e u_e z_e \\
\text{s.t.} & \quad \sum_{e \in P} z_e \geq 1 \quad \forall P \in \mathcal{P}, \forall i \\
& \quad z_e \geq 0
\end{align*}$$

The current dual solution is feasible when the sum of the dual variables for every path is at least one. Using the dual variables as costs, if the minimum cost path has cost at least one, then every path does so the current dual solution is feasible. Else, if the minimum cost path has cost less than one, the corresponding reduced cost is positive $(1 - \sum_{e \in P} z_e > 0)$, so it gives us another variable to add to the LP.
Problem #3

(a): Suppose $B$ is the optimal basis for $\lambda = 0$. Then, for arbitrary $\lambda$, $B$ is optimal if and only if:

$$(c + \lambda \bar{c}) - (c_B + \lambda \bar{c}_B)A_B^{-1}A \geq 0$$

Or, equivalently:

$$\lambda \leq \frac{(c - c_BA_B^{-1}A)_j}{(\bar{c}_B A_B^{-1}A - \bar{c})_j} \text{ for all } j \text{ s.t. } \bar{c}_B A_B^{-1}A - \bar{c} > 0$$

Hence, let $a_1$ be the minimum of the right-hand-side over all such $j$ (or let it be infinity if $(\bar{c}_B A_B^{-1}A - \bar{c})_j \leq 0$ for all $j$. Then $B$ will be the optimal basis in $[0, a_1]$.

(b): Next, note that a basis cannot be optimal in two disjoint intervals. To see this, suppose $B$ is optimal for $\lambda_1, \lambda_2$. Then it must be that:

$$(c + \lambda_1 \bar{c}) - (c_B + \lambda_1 \bar{c}_B)A_B^{-1}A \geq 0$$

$$(c + \lambda_2 \bar{c}) - (c_B + \lambda_2 \bar{c}_B)A_B^{-1}A \geq 0$$

Multiplying the first inequality by $\alpha$, the second by $1 - \alpha$ and adding them together (for $0 \leq \alpha \leq 1$) shows that:

$$(c + (\alpha \lambda_1 + (1 - \alpha) \lambda_2) \bar{c}) - (c_B + (\alpha \lambda_1 + (1 - \alpha) \lambda_2) \bar{c}_B)A_B^{-1}A \geq 0$$

Hence, the range over which a basis is optimal is a convex set. Therefore, each basis is optimal over at most one interval and there are finitely many bases (at most $\binom{n}{m}$), so there is a such a finite set as desired.