Problem #1

(a): Let $P_i$ be polytopes as given. Let $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_l\}$ be the vertices of $P_1$ and $P_2$ respectively. We will show $P_1 + P_2 = \text{conv}\{v_i + w_j\}$.

$\subseteq$: Let $x \in P_1 + P_2$. Then there exists $x^1, x^2$ such that $x = x^1 + x^2$ and, since $x^i \in P_i$, we can write $x^1 = \sum_i \lambda_i v_i$ and $x^2 = \sum_j \mu_j w_j$ where $\sum_i \lambda_i = \sum_j \mu_j = 1$ and $\lambda_i, \mu_j \geq 0$. Now, it follows that:

$$x^1 + x^2 = \sum_i \lambda_i v_i + \sum_j \mu_j w_j = \sum_i (\sum_j \mu_j) \lambda_i v_i + \sum_j (\sum_i \lambda_i) \mu_j w_j = \sum_i \sum_j \lambda_i \mu_j (v_i + w_j)$$

Finally, noting that $\lambda_i \mu_j \geq 0$ $\forall i, j$ and $\sum_i \sum_j \lambda_i \mu_j = \sum_i \lambda_i \sum_j \mu_j = 1$, it follows that $x \in \{v_i + w_j\}$.

$\supseteq$: Suppose $x \in \{v_i + w_j\}$. Then $x = \sum_{i,j} \sigma_{ij} (v_i + w_j)$ where $\sigma_{ij} \geq 0$ and $\sum_{i,j} \sigma_{ij} = 1$.

Let $\lambda_i = \sum_j \sigma_{ij}$ and $\mu_j = \sum_i \sigma_{ij}$. Note that $\sum_i \lambda_i = \sum_j \mu_j = 1$ and $\lambda_i, \mu_j \geq 0$.

Now:

$$x = \sum_{i,j} \sigma_{ij} (v_i + w_j) = \sum_i \sum_j \sigma_{ij} v_i + \sum_j \sum_i \sigma_{ij} w_j = \sum_i \lambda_i v_i + \sum_j \mu_j w_j$$

Finally, it follows that the right-hand side terms are in $P_1$ and $P_2$ respectively by definition.

(b): Let $P_1 = \{x \geq 0 : x_1 + x_2 = 1\}$ and $P_2 = \{x \geq 0 : x_1 = 1, x_2 = 1\}$. Then $P_1 + P_2$ is the line segment from $(2,1)$ to $(1,2)$.

Another example is let $P_1 = \{x \geq 0 : x_1 + x_2 = 1\}$ and $P_2 = \{x \geq 0 : 2x_1 + x_2 = 1\}$. Then you can verify that $P_1 + P_2$ has non-empty interior but is not the entire non-negative quadrant.
Problem #2:
Recall the Strong Duality Thm:
\textbf{Thm/} If either the primal or dual LP has a feasible solution, then their optimal objective values are equal.

Consider Farkas’ Lemma:
Exactly one of the following holds:

1. \( \exists x \geq 0 \) such that \( Ax = b \)
2. \( \exists y \) such that \( yA \geq 0, yb < 0 \)

(a) Consider the following primal-dual LP pair:
(P) \( \text{max} \ 0, \ Ax = b, x \geq 0 \)
(D) \( \text{min} \ yb, yA \geq 0 \)

Suppose (1) and (2) hold. Then there exists a primal-feasible \( x \) with objective value 0 and a dual-feasible with objective value \( yb < 0 \), contradicting strong duality (which says the optimal value is 0). Hence, (1) and (2) cannot both hold.
Suppose (1) does not hold. Then the primal is infeasible, so the dual is infeasible or unbounded. Clearly, \( y = 0 \) is feasible, so it must be unbounded. Hence, there exists \( y \) such that \( yA \geq 0, yb < 0 \).

Problem #3

(a): Consider the primal-dual LP pair:
(P) \( \text{min} \ 0, \ Ax = b \)
(D) \( \text{max} \ yb, yA = 0 \)

(a): The proof is almost identical to Problem #2.

(b): Suppose (1) and (2) hold. Let \( (x, y) \) be feasible solutions to (1) and (2). Then \( 1 = yb = yAx = 0x = 0 \) which is impossible.

Suppose (2) doesn’t hold. Consider \( \{y : yA = 0\} =: (Rng(A))^\perp \). Then for all \( y \) in the above set, \( yb = 0 \) (else, by rescaling \( y \), (2) holds). Hence,
\[
b \in ((\text{Rng}(A))^{\perp})^{\perp} = \text{Rng}(A). \text{ Hence, (1) has a solution.}
\]

**Problem #4**

(a) Let \( x, y \in P \) and \( \lambda, \mu \geq 0 \). Then \( A(\lambda x + \mu y) = \lambda Ax + \mu A y \geq 0 \) since each term is non-negative. Hence, \( \lambda x + \mu y \in P \).

(b): We will show if \( x \) is an extreme ray iff the matrix of constraints that hold with equality at \( x \) has rank \( n-1 \).

\( \Rightarrow \): Let \( x \) be given, let \( A^= \) be the matrix of constraints that hold with equality. Suppose \( \text{rank}(A^=) \neq n - 1 \). If the rank is \( n \), then \( A^= x = 0 \) implies \( x = 0 \), so \( x \) is trivially not an extreme ray. Else, assume \( \text{rank}(A^=) < n - 1 \). Then there exists a \( y \) in the null space of \( A^= \) that is linearly independent from \( x \). Clearly, \( A^= y = 0 \) by definition. Hence, choosing \( \epsilon \) small enough, \( A^>(x \pm \epsilon y) \geq 0 \) but since \( x \) and \( y \) are linearly independent, \( y \neq \lambda x \). Hence, \( x \) is not an extreme ray.

\( \Leftarrow \): Suppose \( \text{rank}(A^=) = n - 1 \) and suppose \( y \) is such that \( x + y, x - y \in P \). Then \( A^= (x + y) \geq 0, A^= (x - y) \geq 0 \) and \( A^= x = 0 \) by definition. Hence, it must be that \( A^= y = 0 \). Since \( Ax = 0 \) and the rank of \( A^= \) is \( n - 1 \), the null space of \( A^= \) is a one-dimensional subspace, so it must be that \( y = \lambda x \) for some \( \lambda \).

(c): Note that from (b), each \( \text{rank}(n - 1) \) submatrix of rows corresponds to an extreme ray. Since there are \( m \) rows, there can be at most \( \binom{m}{n-1} \) extreme rays.

(Note that while there are potentially \( \sum_{k=n-1}^{m} \binom{m}{k} \) such matrices, those of more than \( n - 1 \) rows have redundant constraints, hence, their extreme rays correspond with a matrix of \( n - 1 \) rows).