Mixing Times via Canonical Flows

• Setup - Demands $D(x,y) = \Pi(x)\Pi(y) \neq x,y \in \Omega$

  Capacities $C_{xy} = \Pi(x) \cdot P(y)$

  Flow $f : \bigcup_{x,y} \rightarrow \mathbb{R}_+ \cup \{0\}$ s.t. $\sum_{P\in P_{xy}} f(p) = D_{xy}$

  Paths from $x \rightarrow y$

  $\sum f(p) = D_{xy}$

Cost $\lambda(f) = \max_e \frac{f(e)}{C(e)}$, length $l(f) =$ longest flow-carrying path

• Poincare Constant - $\gamma^* = \inf_{f: \text{non constant}} \left[ \frac{\sum I_{ei}(f)}{\operatorname{Var}_0(f)} \right]$

Flow bound - $t_{\text{mix}}(\varepsilon) \leq \frac{1}{\gamma^*} \left[ 2 \ln(\frac{1}{\varepsilon}) + \ln(\frac{1}{\Pi(3)}) \right]$

$\gamma^* \geq \frac{1}{\lambda(f) l(f)} \quad \text{for lazy, ergodic MC}$

Conversely - $t_{\text{mix}}(\varepsilon) \geq \left( \frac{1}{\gamma^*} - 1 \right) \ln \left( \frac{1}{2\varepsilon} \right)$, $\gamma^* \leq \text{constant \cdot \inf_{p \in P(3)}} \frac{Q(s,s')}{\Pi(s)}$

(Leighton-Rao '88)

For reversible MC

$\frac{\Phi^2}{2} \leq \gamma^* \leq 2\Phi^2$, where $\Phi = \inf_{s, \|s\| \leq 1} \frac{Q(s,s')}{\Pi(s)}$
Eq. LRW on hypercube. Let $N = |\Omega| = 2^n$, $n = \dim$ of hypercube

- $C(u,v) = \frac{1}{N} \cdot \frac{1}{2^n}$, $D(x,y) = \frac{1}{N^2}$

- Consider $D(xy)$ equally split among all shortest paths

  By symmetry, $f(e)$ is equal $\forall e = (u,v)$

  Total flow $\sum_{x \rightarrow y} f(e) = \frac{1}{N^2} \sum_{xy} f(e)$

  $= \frac{1}{N^2} \cdot (N^2 \cdot \frac{n}{2}) = \frac{n}{2} \Rightarrow f(e) = \frac{n}{2} \cdot \frac{1}{Nn}$

  $|E| = Nn$, directed paths

- $P(f) = \frac{\sqrt{2}N}{\sqrt{2}N} = n$, $l(f) = n \Rightarrow t_{mix}(f) \leq n^2 \left( \ln N + \ln 1/\varepsilon \right) = O(N^2)$

- Note: We know $t_{mix}(f) = O(n \log n)$!

  However $1 - \varepsilon = \frac{1}{N} \Rightarrow$ e-value bounds in general give $t_{mix} = O(n^2)$!

  Loss due to ignoring higher order terms (i.e., 13, 14, ...

- Moreover, the best flow bound gives $\gamma^* \geq \frac{1}{N^2}$

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Eq. LRW on line. $|\Omega| = N = n$. $C(u, v) = \frac{1}{4N^2}$, $D(x, y) = \frac{1}{N^2}$

- Let $f$ be flow w unique path $\Rightarrow f(i, i + 1) = i \cdot (N - i) \cdot \frac{1}{N^2} \leq \frac{1}{4}$

  $\Rightarrow P(f) \leq n$, $l(f) = n \Rightarrow t_{mix} \leq n^2 \left( \ln \left( \frac{1}{\varepsilon} \right) + \ln N \right) = O(N^2 \ln N)$

- Hence, true $t_{mix} = O(N^2) \Rightarrow$ only by $\ln N$
General Schema for $O(\text{poly}(n))$ bounds

- Suppose $|\Omega| = N = O(2^n)$, $n = \text{natural dim}$ of $P$
- $P(x,y) \geq \frac{1}{\text{poly}(n)}$, $\Pi(x) = \frac{1}{N} \Rightarrow C(u,v) = \frac{1}{N \text{poly}(n)}$, $D(x,y) = \frac{1}{N^2}$
- $\ell(f) \leq \text{poly}(n)$, $\mu(f) \leq \text{poly}(n) \Rightarrow \lambda_{	ext{mix}} = O(\text{poly}(n))$

$\Rightarrow$ we need $f(e) \leq \frac{\text{poly}(n)}{N}$

- Now since $|E| \approx N, \text{poly}(n)$, $\Sigma f(e) = 1 \Rightarrow \exists e$ s.t $f(e) \geq 1$ $\frac{1}{N \text{poly}(n)}$

$\Rightarrow$ optimizing $f$ can not give better than poly$(n)$!

- Suppose we send $D(x,y)$ along a single path $\beta_{xy}$
  - Let $\mathcal{P}_e = \{(x,y) \mid \beta_{xy} \text{ contain } e \}$
  $\Rightarrow f(e) = |\mathcal{P}_e| \Rightarrow \text{Need } |\mathcal{P}_e| \leq \text{poly}(n), N \forall e$

- Problem - May not know $N \forall e$. E.g: hard-core model

- Solution - Construct injective map $\mathcal{L}_e : \mathcal{P}_e \rightarrow \Omega$
  - Separate $f$ for each $e = (u,v)$
  - Injective $\Rightarrow$ can invert $\mathcal{L}_e(x,y)$, i.e.,
    given $x \in \mathcal{P}_e$ and $e = (u,v)$, if $x \in \mathcal{L}_e(\mathcal{P}_e)$,
    can find unique $(x,y)$ s.t $\mathcal{L}_e((x,y)) = \omega$
Eg - LRW on hypercube

- Let $P_{xy}$ be left-to-right bit fixing path
- Let $e = (u,v)$ be st $u$ and $v$ differ in $i$th position
- Suppose $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ be s.t. $e \in P_{xy}$
- Set $\eta_e(x, y) = x_1 x_2 \ldots x_i y_i+1 y_{i+2} y_{i+3} \ldots y_n$. Given $e$ and $\omega \in \Sigma^*$, clearly can recover $\eta_e^{-1}(\omega)$ if $w \in \text{Royer}(\eta_e)$
- Thus $|P_e| \leq N$ (Prop2) $\Rightarrow f(f) \leq \frac{|P_e|/N^2}{1/2^n} = 2n 

Note - Same as even split $f$ up to constants

Defn - A flow encoding for flow $f$, that uses single paths $P_{xy}$ for all $x, y$, is a set of flows $\eta_e : P_e \rightarrow \Omega$, one for each edge $e = (u, v)$ s.t.
i) $\eta_e$ is injective, ii) $\Pi(x, y) \leq \beta \Pi(x, e) \Pi(y, e(x, y))$ for all $e \in P_e$ (or approx injection), i.e., can store extra info to invert

Prop - If $\exists$ flow encoding $\eta_e \Rightarrow f(f) \leq \beta \max_{(u, v)\in P_e^2} \left[ \frac{1}{f(u, v)} \right]$

Prf - For any $e = (u, v)$,
$$f(e) = \sum_{e \in P_{xy}} \Pi(x) \Pi(y) \leq \beta \Pi(w) \sum_{e \in P_{xy}} \Pi(x, y) \leq \beta \Pi(w)$$

Prop 2

$$\Rightarrow f(f) \leq \beta \max_{e \in P_e} \frac{f(e)}{e} \leq \beta \max_{e \in P_e} \left( \frac{1}{f(e)} \right)$$
Sampling random matchings of $G$

Directed graph $G(V,E)$, $\lambda > 1$

Want to sample matching $M$ w.p. $\Pi(M) = \frac{\lambda^{\sum |M|}}{Z}$

Suppose $M_k = \#$ of $k$-matchings of $G$ = $Z = \sum k^k M_k$

- Computing $Z$ is $\#P$ complete for any $\lambda > 0$!

Markov Chain - Lazy MCMC sampler

i) w.p. $\frac{1}{2}$, no change

ii) Edge, choose edge $(u, v)$ w.a.r.
   - Add if possible (i.e., if $M+e$ is a matching)
   - If $e \in M$, discard $e$ (i.e., $M \rightarrow M - e$) w.p. $\frac{1}{\lambda}$
   - If exactly one of $u$ or $v$ is matched in $M$, (say by edge $e$), then swap (i.e., $M \rightarrow M - e + e'$)

Check - the above chain uniformly samples neighboring matchings of $M$ (i.e., which differ in at most 2 edges),

transitions via the Metropolis rule (Note - $\lambda > 1$)
Flow - Let \( x, y \) = matchings. \( x \otimes y \equiv \text{superposition} \)

- \( x \otimes y \) comprises of
  - Double edges
  - (Even) alternating cycles
  - Alternating paths

- For analysis - Fix a total ordering on all simple paths and even cycles subgraphs of \( G \), and designate a 'start' vertex in each subgraph (endpoint for path, any node on cycle + direction)

- To find \( \beta_{xy} \equiv \text{Construct } x \otimes y \), and 'fix' each alternating path and cycle in order, given by our ordering.

Example:

\[ x \rightarrow \cdots \rightarrow \cdots \rightarrow \cdots \rightarrow y \]