Perfect Sampling - We will see two methods of generating perfect samples $X \sim \mathcal{T}$ for given Markov chain $P$.

1) Strong stationary times
2) Coupling from the past

Example - (Top-to-random shuffle): Given $n$ cards

- take top card and insert it u.a.r. at position $1, 2, \ldots, n$

- Random walk on $S_n \equiv$ Symmetric group

- $\sigma_t = \text{Permutation at time } t$ (i.e., set of permutations of $\{1, 2, \ldots, n\}$)

- Let $T_{\text{top}} = \min \{ t \geq 0 \mid \sigma_t(1) = \sigma_t(n) \}$
  (i.e., first time bottom card comes to top plus 1)

- Proposition: If at time $t$, all cards below bottom card, then these are perfectly shuffled. Hence $\sigma_{T_{\text{top}}} \equiv \text{Uniform in } S_n$

Proof: True at $t=0$. Suppose it's true at time $t$, then in $\sigma_t$, either # of cards below bottom card is same, or new card inserted at random position below original bottomed.

- Mixing time $\equiv$ Coupon collector
Stopping time $T$ is a:

i) Stationary time if $P_x \left[ X_t = 1|y \right] = T(y)$ $\forall x, y$

ii) Strong stationary time if $P_x \left[ T=t, X_t = y \right] = P_x \left[ T=t \right] T(y)$

(i.e., $X_t$ has drift $T(t)$ and $T \perp \perp X_t$) or

$P_x \left[ x \in \mathbb{R} \right] T(t) = T(y)$

Lemma - Let $X_t \sim MC \left( \Omega, \mathbb{P} \right)$ with stationary drift $T(t)$. If $Z_{st}$ is a strong stopping time for $P$, then

$$d(t) = \max_{x \in \Omega} \max_{A \in \mathcal{F}} \left( P_x^{(z_{st})} - T(A) \right) \leq \max_{x \in \Omega} P_x \left[ Z_{st} \geq t \right]$$

Proof - For any $x \in \Omega$, $d(t) = \max_{A \in \mathcal{F}} \max_{x, y} \left\{ P_x^{(z_{st})} - T(A) \right\}$. Now

$P_x \left[ x_{t \in A} \right] = P_x \left[ x_{t \in A}, Z_{st} = t \right] + \sum_{t' \leq t} P_x \left[ x_{t \in A}, Z_{st} = t' \right]$  

$= P_x \left[ x_{t \in A}, Z_{st} = t \right] P_x \left[ Z_{st} = t \right] + T(A) \sum_{t' \leq t} P_x \left[ Z_{st} = t' \right]$  

$= P_x \left[ x_{t \in A} \right] P_x \left[ Z_{st} = t \right] + T(A) \sum_{t' \leq t} P_x \left[ Z_{st} = t' \right]$  

$\Rightarrow P_x \left[ x_{t \in A} \right] = T(A) \sum_{t' \leq t} P_x \left[ Z_{st} = t' \right] \left( P_x^{(z_{st})} - T(A) \right)$

This holds for $\forall x \in \Omega$, $A \in \mathcal{F}$

$\Rightarrow d(t) \leq \max_{x \in \Omega} P_x \left[ Z_{st} \geq t \right]$
Note - Independence of $T_t$ and $X_{T_t}$ is critical!

Eg - For RW on $n$-cycle, let $Z_t = Bin \left( \frac{t}{n} \right)$. If $Z = 0$, then $X_t = 0$; else, run RW from 0 and let $Z = \text{cover time}$.

Eg - For RW on hypercube $[0,1]^n$, consider following chain

- Pick $I_t$ van from $\{0,1,2,...,n^3\}$
- Flip $X_t(I_t)$ wp $\frac{1}{2}$

Claim - $T_{st} = \text{First time } \{I_t, I_{t+1},..., I_{t+3}\} \subseteq \{0,1,2,...,n^3\}$

- Thus $T_{st} = \text{Coupon collector time}$

- $P[Z > \Theta(n \ln n + cn)] \leq e^{-c}$

- $\text{mix}(\epsilon) \leq n(\ln(n) + \ln(\frac{1}{\epsilon}))$

- Identical to coupling time of random walks.

Eg - Transposition Shuffle (Pick $L_t, R_t$ van with replacement & swap)

- Earlier we showed $\text{mix}(\epsilon) \leq \frac{6n^2}{\epsilon^2}$ via mixing arguments

- SS time (Broder) - Start with no marked card

(see 8.2 in [9])

- (In round $t$, mark $R_t$ if unmarked) and ($L_t$ marked OR $L_t = R_t$)

- $E[T_{st}] = \sum_{k=0}^{n^2} \frac{n^2}{(kn+1)n^2} = 2n(\ln n + O(1))$, $Var(T_{st}) = O(n^2)$

- By Chebyshev, $\text{mix}(\epsilon) \leq n \ln n (2 + O(\sqrt{\epsilon}))$
Random Mapping Representation of a MC

- Given MC \((\Omega, P)\), and an \(\Lambda\)-valued random variable \(Z\) satisfying, a random mapping representation is a fn \(f: \Omega \times \Lambda \to \Omega\) s.t.

\[
P[f(x|\Lambda) = y] = P(x, y)
\]

- **Eg** - i) RW on n-cycle - \(Z \sim \pm 1\) wp \(\frac{1}{2}\)

\[
f(x|Z) = (x + Z) \mod n
\]

ii) lazy RW on hypercube - \(Z \sim \mathcal{U}([0, 1]); \text{Ber}(1/2)\)

\[
f(x|Z) \equiv \text{flip } x(i) \text{ if } F = 1, \text{ else no change}
\]

- **Thm**: Every \(P\) on finite \(\Omega\) has a random mapping representation

**Pf** - Choose arbitrary ordering \(\Omega = \{x_1, x_2, \ldots, x_n\}\)

- Generate \(Z \sim \mathcal{U}([0, 1])\)
- Define \(F_{j,k} = \sum_{i=1}^k P(x_i, x_k)\)
- Set \(f(x_j|Z) = x_k\) when \(F_{j,k-1} < Z \leq F_{j,k}\)
Some obs on random mapping representations

- Not unique
- \( \{ f(x|Z) \}_x \) forms a grand coupling (i.e., \( f(x|Z), f(y|Z) \) is a coupling for)

- If \( |\Omega|=n \), then sufficient to consider a discrete \( \mathbb{Z} \) with \( \leq n^2 \) values
  (Corresponding to breakpoints \( F_{j,k} \), \( \forall (j,k) \in \mathbb{Z}^2 \))

- If \( (x_t, y_t) \sim (f(x_t|Z_t), f(y_t|Z_t)) \), then
  \( \text{Couple related } f \text{ to } \max_{x,y} \| f(x|Z) - f(y|Z) \|_{TV} \)
  (or more generally \( \| f(f(... f(x|Z) | Z) ... | Z) - f(f(... f(y|Z) | Z) ... | Z) \|_h \))

- Henceforth write \( f \circ f(x) = f(f(x|Z) | Z) \)

- Compositional of random funs

- Can obtain a SS time from a random function representation (with \( Z \in \Lambda, |\Lambda| \leq n^2 \)) by
  sampling \( Z_1, Z_2, ..., Z_{\ell} \) until we hit every
  \( \Lambda \) value in \( \Lambda \) (i.e., coupon collector on \( \Lambda \))

- Problem: Difficult to specify \( \Lambda \) in general
Example: MC \((\Omega, P)\) s.t. \(\alpha = \sum_{y} \min_{x} (P_{x,y}) > 0\) (i.e., a sum over states of min transition prob)
- Related to the strong Doeblin condition

- Can write \(P = \alpha 1\theta + (1-\alpha) \Theta\), where \(\Theta\) = distribution over \(\Omega\) and \(\Theta\) = stochastic matrix

- Natural induced coupling - up \(\alpha\), sample \(X_t, X_0 \sim \Theta\)
- else, advance \((X_t, Y_t)\) using \(\Theta\)

This is a random mapping representation!

- \(P[\text{couple} > t] \leq \alpha(1-\alpha)^t\)
- \(\Rightarrow t_{\text{mix}}(\epsilon) \leq \frac{1}{\epsilon} \ln(1-\alpha)\)

- To generate a perfect sample from \(\Pi\)
  - Sample \(X_0 \sim \Theta\), \(Z \sim \text{Geom}(d)\)
  - Return \(X(2-1)\)

\(\Pi^T P = \underbrace{\alpha \Pi^T \theta}_{\theta} + (1-\alpha) \Pi^T \Theta = \Pi\)

\(\Rightarrow \theta^T \Pi = (I - (1-\alpha) \Theta)^{-1} d \Theta = \sum_{k=0}^{\alpha} (1-\alpha)^k d \Theta^k \Theta\)

\(= E [\Theta^{Z-1} \Theta]\)