Mechanism Design - Introduction

- An MDP allowed us to model a sequential decision-making problem with uncertainty over future inputs, which we model using a known stochastic process.

\[ U_0 \xleftarrow{F_0} V_0 \xleftarrow{F_1} V_1 \]

States \( X_0 \rightarrow X_1 \rightarrow \ldots \)

Actions (controller) \( A_0 \rightarrow A_1 \)

- Mechanism design typically deals with static (i.e., one-shot) decisions in settings where inputs are private to strategic agents, who participate in a way that maximizes their selfish interests.

\[ \Theta_1 \xrightarrow{F_1} \Theta_2 \xrightarrow{F_2} \Theta_3 \xrightarrow{F_3} X \]

Private 'Types' (Agents) \( \Theta \)

Mechanism (Principal) \( M \)

- Can be thought of as a three-stage process:
  1. A principal commits to some mechanism (i.e., a protocol for communicating and making decisions).
  2. Next, agents with private types participate in a way which maximizes their personal utilities.
  3. Finally, the principal chooses an outcome (on allocation) at which point the principal and each agent realizes their own utilities (as a function of outcome and types).
Mechanisms and Solution Concepts

- Setting:
  - \( n \) agents \( \{1, 2, \ldots, n\} \)
  - \( X \) = set of feasible allocations
  - Each agent \( i \) has a private type \( \theta_i \in \Theta \)
    - (Bayesian setting) \( \theta_i \sim F_i \), independent known
    - \( \theta_i \) may also be correlated in more complex models
  - Each agent \( i \) has utility function \( u_i : \Theta \times X \rightarrow \mathbb{R} \)
    - we will denote this as \( u_i(\cdot|x_i) \)
  - Agent \( i \) knows \( \theta_i \), and all the \( F_i \)
    - Principal knows all \( F_i \)

- Mechanism:
  - Comprised of 2 components
    1) Collection of Action Sets \( \{A_i : i \in \mathbb{N}\} \), one for each agent
    2) Outcome function \( \phi : A_1 \times A_2 \times \ldots \times A_n \rightarrow X \)

- Agent \( i \) is assumed to only play actions in \( A_i \)
- Agent \( i \)'s strategy \( \sigma_i : \Theta \rightarrow A_i \)
- Strategy profile \( \sigma = \{\sigma_i\} \)

- Solution Concept (Equilibrium):
  - Behavioral assumptions on how agents choose strategies
  - Dominant Strategy Equilibrium (DSE): A strategy profile \( \sigma \) where \( A_i, \sigma_i \) is (weakly) dominant compared to any other \( \sigma_i \), irrespective of \( \sigma_i = \{\sigma_j \} \setminus i \)
  - Nash Equilibrium - Strategy profile \( \sigma \) where \( A_i, \sigma_i \) is a best response to \( \sigma_i \)
  - Bayes-Nash Equilibrium (BNE) - Given common prior \( \{F_i\} \), \( \sigma \) is a BNE if \( A_i, \sigma_i \) is a best response on average, i.e., over \( \sigma_i(\cdot|x_i) \) where \( \theta_i \sim F_i \)
Eg - Allocating with agent $\rightarrow$ principal monetary transfers

- Setting
  - $n$ agents (buyers) + 1 special agent (seller)
  - $X$ = allocation space
  - Agent $i$ has private type $\Theta_i$, which determines her value function $v_i : X \times \Theta_i \rightarrow \mathbb{R}$, where $v_i(x; \Theta_i)$ denotes the value she assigns to an allocation $x$ given her type $\Theta_i$.
  - Buyer $i$ can also be charged a payment $p_i$, resulting in a total utility $u_i(x, p_i; \Theta_i) = v_i(x; \Theta_i) - p_i$.

- Convention
  - $p_i > 0$ ⇒ buyer $i$ pays seller
  - $p_i < 0$ ⇒ buyer $i$ receives money
  - No two buyers can pay each other

- Some examples of allocation spaces
  1) Single item - $X = [0, 1]$, and the allocation vector $x$ satisfies $\sum_i x_i \leq 1$
  2) Multiset (k) items - $X = [0, 1]^k$, $v_i(x; \Theta_i) = \sum_j \Theta_{ij} x_j$
  3) Combinatorial auction = M items, $X =$ Dist over $2^m$ subsets

Note: Here $x_i$ = either fraction of item (divisible) or probability of getting item

Eg - Given $S \subseteq [M]$, $v(S; \Theta_i) = \max_{i \in S} \Theta_{ij}$ (pure substitutes/hit demand)

- $v(S; \Theta_i) = \Theta_{ij} \text{ if } \Theta_{ij} \leq s_j \text{ (pure complements) single minded}

- Unrestricted preferences $\Rightarrow X =$ Partition of $[M]$

- Can model externalities - agent $i$'s utility depends on $\text{agent j's alloc}$

Mechanism

Seller specifies 2 functions (based on all buyer actions)

1) Allocation Function - $X : A_1 \times \ldots \times A_n \rightarrow X^n$, $x_i$ = alloc of agent $i$
2) Payment Function - $P : A_1 \times \ldots \times A_n \rightarrow \mathbb{R}^n$, $p_i$ = payment of agent $i$
From Mechanisms to Optimization

A mechanism \( M = (A, \pi, \phi) \) is said to implement an allocation \( f: \Theta^n \to X \) under a DSE/BNE if \( f(\Theta_1, \ldots, \Theta_n) = \phi(\sigma_1(\Theta_1), \sigma_2(\Theta_2), \ldots, \sigma_n(\Theta_n)) \notin \Theta_i \) under a DSE/BNE strategy profile \( \sigma \).

- A Direct Revelation (DR) mechanism is one where \( A_i = \Theta_i \), i.e., agents are 'asked to reveal' their types.

- A DR mechanism is:
  - Dominant Strategy Incentive Compatible (DSIC) if revealing \( \Theta_i \) (truth-telling) is a DSE.
  - Bayesian Incentive Compatible (BIC) if truth-telling is a BNE.

**Thm (Revelation Principle)**: Any mechanism \( M = (A, \pi, \phi) \) which implements an allocation \( f: \Theta^n \to X \) under a DSE/BNE can be emulated (i.e., same \( f \) implemented) by a DR mechanism which is DSIC/BIC.

**Pf - Simulation argument!**

Because \( \sigma_i(\Theta) \) is a DSE/BNE, if we offer to 'simulate' \( \sigma_i(\Theta) \) when we get \( \Theta_i \) as input, then agents are 'incentivized' to report \( \Theta_i \). Overall mechanism is DSIC/BIC.
**DSIC|BIC Auctions**

- Using the revelation principle, we can now describe mechanisms for the allocation with transfer setting.
  - Recall: Agent $i$ has private type $\Theta_i \in \Theta$ gets allocation $x_i$, payment $p_i$.
  - Value fn $v_i(x_i|\Theta_i)$, utility $u_i(x_i)=v_i(x_i|\Theta_i)-p_i$.
  - By revelation principle, $A_i=\Theta$, i.e., agents asked to report values.

- Mechanism $M =$
  1. Ask agents to report types $\{ti\} \in \Theta$.
  2. Set allocation rule $x = x(t)$.
  3. Payment rule $p = p(t)$.
  4. Agent $i$ gets utility $u_i(x_i)=v_i(x_i|\Theta_i)-p_i$.

- Notation: For any $i$, vector $t=(t_i, t_{-i})$ where $t_{-i} = \{t_j\}_{j \neq i}$.
  - Agent $i$ follows a truth-telling strategy if she reports $t_i=\Theta_i$.

- $M=(x, p)$ is dominant strategy incentive compatible (DSIC) if
  \[ \forall i \in [n], \forall t_i \in \Theta^{t_i}, \forall \Theta_i \in \Theta, \forall t_i \in \Theta, \]
  \[ v_i(x_i(\Theta_i, t_i)|\Theta_i) - p_i(\Theta_i, t_i) \geq v_i(x_i(t_i, t_{-i})|\Theta_i) - p_i(t_i, t_{-i}) \]
  i.e., truth-telling is (weakly) dominant no matter what other agents report.

- Suppose $\Theta_i \sim F_i$, independently for some known prior $\{F_i\}_{i \in [n]}$.
  - $M=(x, p)$ is Bayesian incentive compatible (BIC) if
  \[ \forall i \in [n], \forall \Theta_i \in \Theta, \forall t_i \in \Theta, \]
  \[ E_{\Theta_i}[v_i(x_i(t_i, \Theta_i)|\Theta_i) - p_i(t_i, \Theta_i)] \geq E_{\Theta_i}[v_i(x_i(\Theta_i, t_i)|\Theta_i) - p_i(\Theta_i, t_i)] \]
  i.e., assuming others tell the truth, then truth-telling is (weakly) dominant in expectation (i.e., truth-telling is a BNE).

- $M=(x, p)$ is ex-ante individually rational (IR) if
  \[ E_{\Theta}[v_i(x_i(\Theta_i)|\Theta_i) - p_i(\Theta_i)] \geq 0 \]

- $M=(x, p)$ is ex-post individually rational (IR) if
  \[ E_{\Theta_i}[v_i(x_i(t_i, \Theta_i)|\Theta_i) - p_i(t_i, \Theta_i)] \geq 0 \]
  \[ v_i(x_i(\Theta_i)|\Theta_i) - p_i(\Theta_i) \geq 0 \]
LP formulation of single-item auction

- Objectives:
  1) Revenue (on seller surplus) - \( R(x,p) = \sum_i p_i \)
  2) Social Welfare - \( W(x,p) = \sum_i (v_i(x_i,\theta_i) - p_i) + \sum_i p_i = \sum_i \theta_i(x_i|\theta_i) \)

We get different LPs depending on objective and solution concept.

- Welfare Max under DSE - variables \( \{x_i(\theta), p_i(\theta)\} \forall \theta \in \Theta \)
  Suppose true type vector is \( \hat{\theta} \)

\[
\begin{align*}
\max & \sum_{i \in [n]} x_i(\hat{\theta}) \hat{\theta}_i \\
\text{(DS1c)} & \text{s.t. } x_i(\theta_i, \theta_{-i}) \theta_i - p_i(\theta_i, \theta_{-i}) \geq x_i(t,\theta_{-i}) \theta_i - p_i(t,\theta_{-i}) \quad \forall i, \theta_i, t \\
\text{(ER1R)} & x_i(\theta_i, \theta_{-i}) \theta_i - p_i(\theta_i, \theta_{-i}) \geq 0 \quad \forall i, \theta_i, \theta_{-i} \\
\text{(Implementability)} & \sum_{i \in [n]} x_i(\theta) \leq 1 \\
& x_i(\theta) \geq 0, p_i(\theta) \geq 0 \quad \forall \theta \end{align*}
\]

- Revenue Max under BNE - Suppose \( \Pi_i \sim F_i, 1 \Rightarrow F(\theta) = \prod_{i \in [n]} F_i(\theta_i) \)

\[
\begin{align*}
\max \sum_{\theta \in \Theta} F(\theta) \left[ \sum_{i \in [n]} p_i(\theta) \right] \\
\text{(BNE)} & \text{s.t. } \sum_{\theta_i} F(\theta_i) (x_i(\theta_i, \theta_{-i}) \theta_i - p_i(\theta_i, \theta_{-i})) \geq \sum_{\theta_i} F(\theta_i) (x_i(t,\theta_{-i}) \theta_i - p_i(t,\theta_{-i})) \quad \forall i, \theta_i, t \\
\text{(EI-IR)} & \sum_{\theta_i} F(\theta_i) (x_i(\theta_i, \theta_{-i}) \theta_i - p_i(\theta_i, \theta_{-i})) \geq 0 \quad \forall i, \theta_i \\
\text{(Implementability)} & \sum_{i \in [n]} x_i(\theta) \leq 1 \\
& x_i(\theta) \geq 0, p_i(\theta) \geq 0 \quad \forall i, \theta
\end{align*}
\]

- Notes:
  1) \# of vars = \( 2^m \times n \times |\Theta| \)
  2) Allocation/payment rules are complex menus mapping every type profile vector (\( 1 \times n^t \) in number) to an allocation/payoff for each agent.