Multiple fare-class capacity allocation (putting it together)

- $$V_j(s) = \max_{z \in \{0,\dots,s\}} \mathbb{E}\left[P_j \min(z, D_j) + V_{j+1}(s - \min(z, D_j)) \right]$$

- 'Oracle problem' - Suppose D_j was known

$$\bar{V}_j(s) = \mathbb{E}\left[\max_{z \in \{0,\dots,s\}} \left(P_j \min(z,D_j) + \bar{V}_{j+1}(s-\min(z,D_j))\right)\right] = \mathbb{E}\left[\max_{y \in \{(s-D_j)^+,\dots,s\}} \left(P_j(s-y) + \bar{V}_{j+1}(y)\right)\right]$$

- $$\Delta \bar{V}_j(s) = V_j(s+1) - V_j(s) \quad , \quad \bar{V}_j(s|D_j) = \max_{y \in [(s-D_j)^+, s]} \left(P_j(s-y) + \bar{V}_{j+1}(y)\right)$$

- $$x_j^* \triangleq \arg\max_{y \in \{0,1,\dots,\}} \left[\bar{V}_{j+1}(y) - P_j y\right],$$ (Protection level)

- $$\Delta \bar{V}_j(s|D_j) = P_j \mathbb{1}\{D_j \geq s - x_j^* + 1\} + \mathbb{1}\{D_j \leq s - x_j^*\} \left[\Delta \bar{V}_{j+1}(s - D_j)\right]$$

- $$\Delta V_j(s) \geq \Delta V_j(s+1) \, , \, \Delta V_j(s) \geq \Delta V_{j+1}(s)$$

- $$x_1^* \geq x_2^* \geq \dots \geq x_n^* = 0, \quad x_j^* = \arg\min_{y \in \{x_{j+1}^*, x_{j+1}^* + 1, \dots\}} \{\Delta V_{j}(y) < 0\}$$

Closed-form expression for x_j^*, V_j^* (Brumelle & McGill '93)

<u>Thm</u> - Protection levels $\{x_j^*\}_j$ satisfy $\forall j$, x_j^* is smallest s.t.

$$P_j \geq P_n \, \mathbb{P}\left[D_n > x_{n-1}^*, D_n + D_{n-1} > x_{n-2}^*, \dots, D_n + \dots + D_{j+1} > x_j^*\right]$$

(All of above is for discrete capacity alloc^n; all extend naturally for continuous)

Pf - First, we define the events

$$B_j = \{D_n > x^*_{n-1}\} \cap \{D_n + D_{n-1} > x^*_{n-2}\} \cap \ldots \cap \{\sum_{i=j+1}^{n} D_i > x^*_j\}$$

Note i) $B_n \supseteq B_{n-1} \supseteq B_{n-2} \supseteq \ldots$

ii) B_j is a fn of $\{x^*_i\}_{\boxed{i=j}}^{n}$, and depends on $\{D_i\}_{\boxed{i=j+1}}^{n}$

- To prove the theorem, we first define 2 sets of propositions

$Prop_{j,1}$: $\quad P_j \geq P_n \mathbb{P}[B_j]$

$Prop_{j,2}$: $\quad \Delta V_j(s) \geq P_n \mathbb{P}[B_j \cap \{\sum_{i=j}^{n} D_i > s\}] \; \forall s \geq x^*_j$

- We now prove the theorem via induction; in particular we consider the propositions in the order

$$Prop_{n,2} \rightarrow Prop_{n-1,1} \rightarrow Prop_{n-1,2} \rightarrow \ldots \rightarrow Prop_{j,1} \rightarrow Prop_{j,2} \rightarrow \ldots$$

- First, for $j=n$, we know $x^*_n = 0$ and hence

$$V_n(s) = \mathbb{E}[P_n \min\{D_n, s\}] \Rightarrow \Delta V_n(s) = \underbrace{P_n \mathbb{P}[D_n > s]}_{Prop_{n,2}}$$

- Now assume $Prop_{1+j,2}$ is true (and all preceding prop^n's)

We know $\Delta V_j(x^*_j) \leq 0$

$\Rightarrow \quad \Delta V_{j+1}(x^*_j) - P_j \leq 0$, and by $Prop_{1+j,2}$, we have

$\Rightarrow \quad P_j \geq \Delta V_{j+1}(x^*_j) \geq P_n \mathbb{P}[B_j] \qquad (Prop_{j,1})$

Next, for $s > x_j^*$, we have

$$\Delta V_j(s|D_j) = p_j \mathbb{1}\{D_j > s - x_j^*\} + \Delta V_{j+1}(s-D_j)\,\mathbb{1}\{D_j \le s - x_j^*\}$$

Prop $j,1$ Prop $j+1,2$; since $s \geq x_j^* \geq x_{j+1}^*$

$$\geq p_n\left(\mathbb{P}[B_j]\,\mathbb{1}\{D_j > s-x_j^*\} + \mathbb{P}\left[B_{j+1}\cap\left\{\sum_{i=j+1}^n D_i > s-D_j\right\}\right]\mathbb{1}\{D_j\le s-x_j^*\}\right)$$

$$\Rightarrow \Delta V_j(s) \geq p_n\left(\mathbb{P}[\underbrace{B_j\cap\{D_j>s-x_j^*\}}_{C_j}] + \mathbb{P}[\underbrace{B_{j+1}\cap\{\textstyle\sum_{i=j}^n D_i>s\}\cap\{D_j<s-x_j^*\}}_{\hat{C}_j}]\right)$$

Note that C_j and C_{j+1} are mutually exclusive

$$\Rightarrow \Delta V_j(s) \geq p_n\, \mathbb{P}\left[(B_{j}\cap\{D_j>s-x_j^*\})\cup\left(B_{j+1}\cap\{\textstyle\sum_{i=j}^n D_i>s\}\cap\{D_j<s-x_j^*\}\right)\right]$$

Finally, given B_{j+1}, we have 2 cases

1) $D_j > s - x_j^*$: Then $\sum_{i=j+1}^n D_i > x_j^* \Rightarrow \{\sum_{i=j}^n x_i > s\}$

2) $D_j < s - x_j^*$: Then $\sum_{i=j}^n D_i > s \Rightarrow \{\sum_{i=j+1}^n D_i > x_j^*\}$

$$\Rightarrow B_{j+1}\cap\left[\left(\{\textstyle\sum_{i=j+1}^n D_i > x_j^*\}\cap\{D_j>s-x_j^*\}\right)\cup\left(\{\textstyle\sum_{i=j}^n D_i>s\}\cap\{D_j<s-x_j^*\}\right)\right]$$

$$= B_j \cap\left\{\sum_{i=j}^n D_i > s\right\}$$

$$\Rightarrow \Delta V_j(s) \geq p_n\, \mathbb{P}\left[B_j\cap\left\{\sum_{i=j}^n D_i > s\right\}\right]$$

(Prop $j,2$)

This completes the induction

Computing protection levels

Method 1 - Use the DP recursion

- $V_j(s) = \mathbb{E}\left[P_j \min\{D_j, s - x_j^*\} + V_{j+1}\{s - \min\{D_j, (s-x_j^*)^+\}\}\right]$

$$x_j^* = \underset{y \in \{x_{j+1}^*, x_{j+1}^*+1, \dots\}}{\arg\min} \left[y : -P_j + \Delta V_j(y) < 0\right]$$

(Algo 1). - Start with $x_n^* = 0$
- Evaluating $V_j(s)$ takes $O(c)$ time (taking expectation over c terms); need $V_j(s)$ $\forall s \in \{x_j^*, \dots, c\} \Rightarrow O(c)$ evaluations $\Rightarrow O(c^2)$ ops for each stage

$\Rightarrow O(nc^2)$ running time

Method 2 - Use Monte Carlo integration (via Brumelle-McGill)

- $\mathbb{P}[B_j] = \mathbb{P}\left[\sum_{i=j+1}^{n} D_i > x_j^* \mid B_{j+1}\right] \mathbb{P}[B_{j+1}]$

$$\Rightarrow \boxed{\mathbb{P}\left[\sum_{i=j+1}^{n} D_i > x_j^* \mid B_{j+1}\right] = \frac{P_{j+1}}{P_j}} \quad (\text{for continuous } D_j)$$

(Algo 2). - Generate K vectors $S_n = \{d_i^k\}_{k \in [K]}$ from $\{F_i\}$
- For j in $\{n-1, n-2, \dots, 1\}$
  i) Compute 'demand-to-go' $\hat{D}_j^k = \sum_{i=j+1}^{n} d_i^k \;\; \forall k \in S_j$
  ii) Find smallest x_j^* s.t $\dfrac{\sum_{k \in S_j} \mathbb{1}\{\hat{D}_j^k > x_j^*\}}{|S_j|} \geq \dfrac{P_{j+1}}{P_j}$
  iii) Set $S_{j-1} = \{k \in S_j \mid \hat{D}_j^k \geq x_j^*\}$ and repeat
- For K data points, takes $O(nK\log K)$ time

Bounds for $V_n(c)$

Idea: Given $D = (D_1, D_2, \ldots, D_n)$, we want to consider 'sample-wise' bounds for $V_n(c)$

- **Upper bound** – Suppose all demand available simultaneously
  - Consider optimization problem: $\max \sum_{i=1}^{n} p_i x_i$ (LP relaxation)

$$\text{s.t } x_i \le D_i \quad \forall i$$
$$\sum_{i=1}^{n} x_i \le c$$
$$x_i \ge 0 \quad \forall i$$

  - Soln to above problem: $x_k = \min\{D_k, c - \sum_{i=k+1}^{n} D_i\}$

(greedy allocation)

  - $$V_n^U(c) \triangleq \sum_{k=n}^{1} p_k \mathbb{E}\left[\min\{D_k, c - \textstyle\sum_{i=k+1}^{n} D_i\}\right]$$
$$= \sum_{k=n}^{1} (p_k - p_{k-1}) \mathbb{E}\left[\min\{\textstyle\sum_{i=k}^{n} D_i, c\}\right]$$
$$\le \sum_{k=n}^{1} (p_k - p_{k-1}) \min\{\textstyle\sum_{i=k}^{n} \mu_i, c\} \qquad \left(\text{by Jensen's, } \mu_i \triangleq E[D_i]\right)$$

  - 'Fluid' problem: $\max \sum_{i=n}^{1} p_i x_i$

$$\text{s.t } 0 \le x_i \le \mu_i \ \forall i, \quad \sum x_i \le c$$

Fluid opt soln: $V_n^{Fl}(c) = \sum_{k=n}^{1} (p_k - p_{k-1}) \min\left(\sum_{i=k}^{n} \mu_i, c\right)$

Thus $\quad V_n(c) \le V_n^U(c) \le V_n^{Fl}(c)$

- For lower bound: Set all $x_i^* = 0$. (See assgnmnt)