Assortment Optimization

- (Recall) Choice Model. Given $S \subseteq N = \{1, 2, \ldots, n\}$ (products)
  - $\Pi_j(S) = \mathbb{P}[\text{Prod } j \text{ purchased from } S]$
  - LCA choice models - $\exists\, v_i > 0$ s.t

$$\Pi_j(S) = \frac{v_j \mathbb{1}_{\{j \in S\}}}{v_0 + v(S)}, \quad v_0 \equiv \text{'attractiveness' of no purchase}, \quad v(S) \equiv \sum_{i \in S} v_i$$

  - Mixture of LCA (mixed MNL)

$$\Pi_j(S) = \sum_{g \in G} \alpha^g \left(\frac{v_j^g}{v_0^g + v^g(S)} \right), \quad \sum_{g \in G} \alpha^g = 1$$

  - Markov Chain choice model - $\Lambda \equiv \{\lambda_i\}_{i \in N}$, $P \equiv \{\rho_{ij}\}_{i,j \in N^2}$

$\mathbb{P}[\text{purchase } j \in S]$: $$\Pi_j(S) = \lambda_j + \sum_{i \in \bar{S}} \phi_i(S) \rho_{ij} \quad \forall j \in S$$

$\mathbb{P}[\text{consider } j \notin S]$: $$\phi_j(S) = \lambda_j + \sum_{i \in \bar{S}} \phi_i(S) \rho_{ij} \quad \forall j \in \bar{S}$$

$$\Pi_0(S) = 1 - \sum_{j \in S} \Pi_j(S)$$

- The assortment optimization problem
  - Exogenous prices (profits) $p_j \;\forall j \in N$
  - $$R^* = \max_{S \subseteq N} R(S) = \max_{S \subseteq N} \sum_{j \in S} p_j \Pi_j(S)$$

* Assortment Opt under LCA / MNL

- $R(S) = \sum_{j \in S} \frac{p_j v_j}{v_0 + v(S)}$, $R^* = \max_{S \subseteq N} R(S)$

Thm - Let $p_1 \geq p_2 \geq \cdots \geq p_n$

(Nested-by-revenue Sets) $E_0 = \phi$, $E_1 = \{1\}$, $E_2 = \{1,2\}$, $\ldots$, $E_n = N$

Then $\exists\ k^* \in \{0, 1, \ldots, n\}$ s.t $E_{k^*} \in \arg\max_{S \subseteq N} R(S)$

Pf - By definition $R^* \geq \sum_{j \in S} \frac{p_j v_j}{v_0 + v(S)} \quad \forall\ S \subseteq N$

$$\Rightarrow \quad v_0 R^* \geq \sum_{j \in S} v_j (p_j - R^*) \quad \forall\ S \subseteq N$$

$\because$ $\exists$ some $S \subseteq N$ s.t $R(S) = R^*$

$$\Rightarrow \quad \arg\max_{S \subseteq N} R(S) = \arg\max \left\{ \sum_{j \in S} (p_j - R^*) v_j \right\}$$

- Thus we want to find $S \in \arg\max \left\{ \sum_{j \in S} (p_j - R^*) v_j \right\}$

$$\Rightarrow \quad S^* = \{ j \in N \,;\, p_j \geq R^* \}$$

- Now even if we do not know R^*, it is clear that we only need to consider $S \in \{E_0, E_1, \ldots, E_N\}$

LCA with constraints

- Let $x^S \in \{0,1\}^n \equiv$ Indicator of set $S \subseteq N$
- (ie, $x_j^S \equiv \mathbb{1}\{j \in S\}$)
- We now want to solve a constrained assortment opt^n

$$\max_{x \in \{0,1\}^n} \frac{\sum_{j \in N} p_j v_j x_j}{v_0 + \sum_{j \in N} v_j x_j}$$

$$\text{s.t} \quad \sum_{j \in N} a_{ij} x_j \leq b_i \quad \forall\, i \in L$$

$$x_j \in \{0,1\} \quad \forall\, j \in N$$

- Assumption - $A = \{a_{ij}\}$ is totally unimodular, $b_i \in \mathbb{Z}$
  ($\Rightarrow$ extreme points of $\{Ax \leq b\}$ are integral)

Eg - $\sum_{j \in N} x_j \leq C$

- If $N = \underbrace{S_1 \cup S_2 \cup \ldots, S_k}_{\text{Partition}}$, $\sum_{j \in S_i} x_j \in \{b_{S_i}, \ldots, B_{S_i}\}$

- Joint Pricing and assortment opt^n
  - Products $N = \{1, \ldots, n\}$, Prices $P = \{p_1, p_2, \ldots, p_K\}$
  - $v_{ik} \equiv$ attractiveness of product i at price p_k
  - Idea - Create virtual products: $x_{ik} \equiv$ product i at price k
    - Constraint: at most one $x_{ik} = 1$ for every i

- How do we solve constrained MNL pricing?

$\boxed{\text{OPT1}}$: $$\max \sum_{j\in N} \frac{P_j v_j x_j}{v_0 + v^T x}$$

$$\text{s.t.} \quad Ax \leq b$$

$(L \times N) \rightarrow A$

$$x_j \in \{0,1\}$$

$\boxed{\text{OPT2}}$ $$\max \sum_{j\in N} P_j y_j$$

$$\text{s.t.} \quad \sum_{j\in N} y_j + y_0 = 1$$

$$\sum_{j\in N} \frac{a_{ij}}{v_j} y_j \leq \frac{b_i}{v_0} y_0 \quad \forall i \in L$$

$$0 \leq \frac{y_j}{v_j} \leq \frac{y_0}{v_0} \quad \forall j \in N$$

Thm - The above problems have the same optimal objective. Moreover, given a solution to OPT2, we can construct a solution to OPT 1.

Pf - First, as in prev result, we have OPT1 is equiv to

$\boxed{\text{OPT3}}$ $$\max \sum_{j\in N} (P_j - R^*) \frac{v_j}{v_0} x_j \quad , \text{ where } R^* \equiv \text{OPT1 objective}$$

$$\text{s.t.} \quad Ax \leq b, \quad 0 \leq x_j \leq 1$$

This follows from LCA form + total unimodularity of A

Thus we need to show OPT3 $\equiv$ OPT2

- Let $\{y_j^*\}_{j\in N\cup\{0\}}$ be an optimal soln to OPT2
  $\{x_j^*\}_{j\in N\cup\{0\}}$ be an optimal soln to OPT3

By defn, $OPT3(x_j^*) = R^*$

- Now we show $\gamma^* \triangleq OPT2(y_j^*) = R^*$

Let $\hat{y}_j = \dfrac{v_j x_j^*}{v_0 + \sum v_i x_i^*}$, $\hat{y}_0 = 1 - \sum_{j\in N} \hat{y}_j = \dfrac{v_0}{v_0 + \sum v_i x_i^*}$

then $\{\hat{y}_i\}$ satisfies constraints of OPT2

- $\displaystyle\sum_{j\in N} \frac{a_{ij}}{v_j}\hat{y}_j = \sum_{j\in N} \frac{a_{ij}\, x_j^*}{v_0 + \sum v_i x_i^*} \le \frac{b_i}{v_0 + \sum v_i x_i^*} = \frac{b_i \hat{y}_0}{v_0} \;\forall i$

- $\displaystyle\frac{\hat{y}_j}{v_j} = \frac{x_j^*}{v_0 + \sum v_i x_i^*} \le \frac{\hat{y}_0}{v_0} \quad \forall j$

$\Rightarrow \{\hat{y}_i\}$ is feasible for OPT2

$\displaystyle\Rightarrow \gamma^* \geq OPT2(\{\hat{y}_i\}) = \sum_j P_j \frac{x_j^* v_j}{v_0 + \sum v_i x_i^*} = R^*$

- Now suppose $\gamma^* = OPT2(\{y_i^*\}) > R^*$. Note $y_0^* > 0$

Consider $\hat{x}_j = \dfrac{y_j^*/v_j}{y_0^*/v_0}$ - Check that $\hat{x}_j$ is feasible for OPT 3

Then $\displaystyle OPT3(\{\hat{x}_j\}) = \sum_{j\in N} (P_j - R^*)\frac{v_j}{v_0}\hat{x}_j = \frac{1}{y_0^*}\sum_{j\in N} P_j y_j^* - \frac{R^*}{y_0^*}(1-y_0^*)$

$> \gamma^*/y_0^* - \gamma^*(1-y_0^*)/y_0^* = \gamma^*$ $\Rightarrow$ contradiction

* Assortment opt^n for universal approximators

- We now consider assortment optimization for choice models which serve as universal approximators

- First consider the mixture of MNL model

Q: Does this have an 'easy' (poly-time) algorithm for assortment opt^n ?

A: No!

Consider the problem - 2 classes, $\alpha^a + \alpha^b = 1$

- Prices p_i, choice parameters $\{v_i^k\}_{k \in \{a,b\}, \, i \in N}$

Q: Given any K, does there exist $S \subset N$ s.t

$$\text{(2-Class Logit)} \quad \sum_{i \in N} \left(\frac{\alpha^a v_i^a p_i}{v_0^a + v^a(S)} + \frac{\alpha^b v_i^b p_i}{v_0^b + v^b(S)} \right) \geq K \, ?$$

Thm (RSTT 13) - 2-Class Logit is NP complete

(Reduction from set partition)

Eg - (Nested-by-revenue is not optimal)

$\alpha^a = 0.5$, $v^a = (5, 20, 1)$

$\alpha^b = 0.5$, $v^b = (1/5, 10, 10)$

prices $\equiv (8, 4, 3)$

Then opt for class a $\equiv \{1\}$, $R^* = 20/3$

opt for class b $\equiv \{1,2\}$, $R^* = 26/7$

opt for mixture $\equiv \{1,3\}$, $R^* = 4.48$

- What about the Markov Chain choice model

<u>Thm</u> (BGG'16) - Assortment opt under MC model (Λ, P) is equivalent to the following LP

$$
\begin{aligned}
\min \quad & \sum_{i \in N} \lambda_i g_i \\
\text{s.t} \quad & g_i \geq p_i \quad \forall\ i \in N \\
& g_i \geq \sum_{j \in N} P_{ij} g_j \quad \forall\ i \in N \\
& g_i \geq 0
\end{aligned}
$$

- Here $g_i \equiv$ 'optimal expected revenue starting from i'
- Moreover, $g_i = p_i \Rightarrow i \in S^*$, $g_i = \sum P_{ij} g_j \Rightarrow i \in N \setminus S^*$