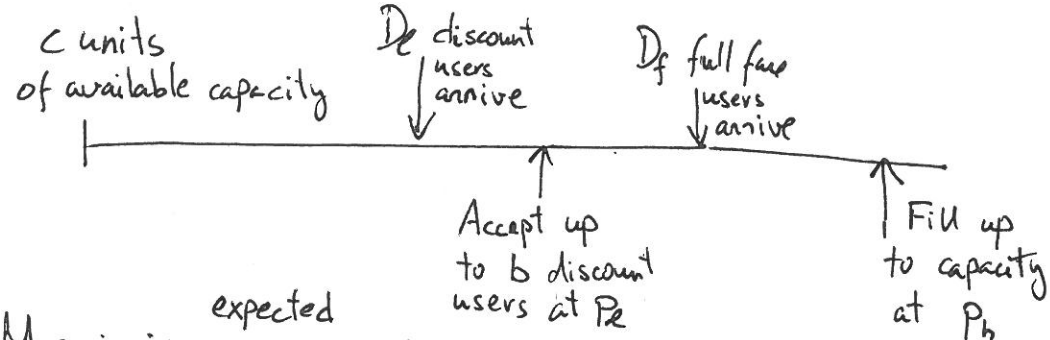
Single-leg Two Fare-class Capacity Allocation

- Fixed capacity C
- 2 fare classes $\quad p_\ell < p_h \quad$ $\boxed{\text{exogenous fares}}$

  p_ℓ: discount customers 'early booking' $\quad$ p_h: full fare customers

- Demand for each fare class $\equiv$ $\left.\begin{array}{l} D_h \sim F_h \\ D_\ell \sim F_\ell \end{array}\right\}$ $\boxed{\text{known distn}}$
- Control - Protection level x for full fare
  - Booking limit b for discount fare $\quad$ (Equivalent as $x+b=C$)
- Dynamics .



- Objective - Maximize expected revenue
  - $R(b, D_\ell, D_h) = p_\ell \cdot \min\{b, D_\ell\} + p_h \min\left\{D_f, \max\begin{pmatrix} C-b, \\ C-D_\ell \end{pmatrix}\right\}$
  - want to choose b to maximize $\mathbb{E}[R(b, D_\ell, D_h)]$

Heuristic derivation of Littlewood's rule (Littlewood 1972)

- Marginal analysis - Suppose we have $y > 0$ units left, and a discount customer arrives.
  - Accept $\Rightarrow \Delta$revenue $= p_\ell$, Reject $\Rightarrow \Delta$revenue $= p_h \mathbb{P}[D_h \geq y]$

  $\Rightarrow$ OPT protection level $y^* = \max\left\{ y \in \mathbb{N} \;\middle|\; \mathbb{P}[D_h \geq y] > \frac{p_\ell}{p_h} \right\}$

- Formal proof 1 * Assume D_e, D_h are continuous ②

* Leibniz rule:
$$\frac{d}{d\alpha}\int_{\phi(\alpha)}^{\psi(\alpha)} F(x,\alpha)\,dx = \left(\frac{d\psi(\alpha)}{d\alpha}\right) F(\psi(\alpha),\alpha) - \left(\frac{d\phi(\alpha)}{d\alpha}\right) F(\phi(\alpha),\alpha) + \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial F(x,\alpha)}{\partial\alpha}\,dx$$

* $$\mathbb{E}\left[R(b,D_\ell,D_h)\right] = P_e\,\mathbb{E}\left[\min(b,D_e)\right] + P_h\,\mathbb{E}\left[\min\left(c-\min(b,D_e),D_h\right)\right]$$
$$= \underbrace{P_e\int_{-\infty}^{b} y\,f_e(y)\,dy}_{\sigma_1(b)} + \underbrace{P_e\int_{b}^{\infty} b\,f_e(y)\,dy}_{\sigma_2(b)}$$
$$+ P_h \underbrace{\int_{-\infty}^{b} \mathbb{E}\left[\min(c-y,D_h)\right] f_e(y)\,dy}_{H_1(b)} + \underbrace{\int_{b}^{\infty} \mathbb{E}\left[\min(c-b,D_h)\right] f_e(y)\,dy}_{H_2(b)}$$

* First-order condition - $\dfrac{d\,\mathbb{E}\left[R(b,D_\ell,D_h)\right]}{db} = 0$

- $\dfrac{d\sigma_1(b)}{db} = b f_e(b)$, $\dfrac{d\sigma_2(b)}{db} = \displaystyle\int_b^{\infty} f_e(y)\,dy - b f_e(b)$

$\dfrac{dH_1(b)}{db} = \mathbb{E}\left[\min(c-b,D_h)\right]$, $\dfrac{dH_2(b)}{db} = -\mathbb{E}\left[\min(c-b,D_h)\right] + \displaystyle\int_b^{\infty} \frac{d}{db}\mathbb{E}\left[\min(c-b,D_h)\right] f_e(y)\,dy$

$$\Rightarrow \frac{d\,\mathbb{E}[R(b)]}{db} = P_e \underbrace{\int_b^{\infty} f_e(y)\,dy}_{\mathbb{P}[D_e \geq b]} + P_h \int_b^{\infty} \frac{d}{db}\mathbb{E}\left[\min(c-b,D_h)\right] f_e(y)\,dy$$

$$* \quad \mathbb{E}\left[\min(c-b, D_h)\right] = \int_{-\infty}^{c-b} x f_h(x)\,dx + \int_{c-b}^{\infty} (c-b) f_h(x)\,dx$$

$$\Rightarrow \frac{d}{db}\mathbb{E}\left[\min(c-b, D_h)\right] = -(c-b) f_h(c-b) + (c-b) f_h(c-b) - \int_{c-b}^{\infty} f_h(x)\,dx$$

$$= -\bar{F}_h(c-b) \quad \left(\text{or } -\mathbb{P}\left[D_h \geq c-b\right]\right)$$

$$* \quad \frac{d\mathbb{E}[R(b)]}{db} = P_\ell\, \mathbb{P}[D_\ell \geq b] - P_h \int_b^{\infty} \mathbb{P}[D_h \geq c-b]\, f_\ell(y)\,dy$$

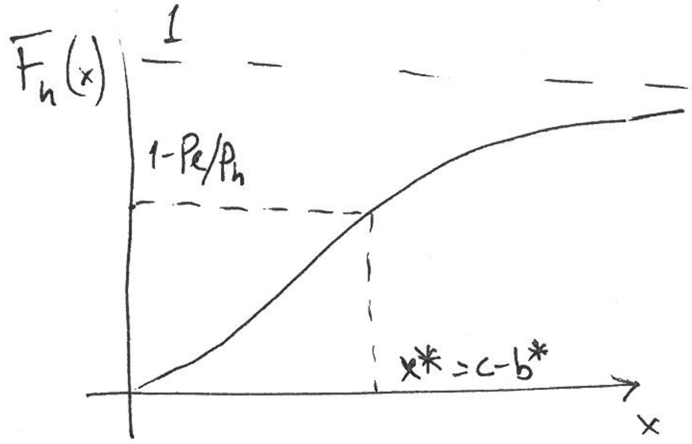
$$= P_\ell\, \mathbb{P}[D_\ell \geq b] - P_h\, \mathbb{P}[D_h \geq c-b]\, \mathbb{P}[D_\ell \geq b]$$

$$= \mathbb{P}[D_\ell \geq b]\left(P_\ell - P_h\, \mathbb{P}[D_h \geq c-b]\right)$$

$$\Rightarrow b^* : \mathbb{P}[D_h \geq c-b^*] = \frac{P_\ell}{P_h}$$

$$\Rightarrow \boxed{c-b^* = \bar{F}_h^{-1}\left(1 - \frac{P_\ell}{P_h}\right) = x^*}$$

Littlewood's Rule



- x^* is indep of F_ℓ
- $P_\ell \nearrow \Rightarrow \left(1 - \frac{P_\ell}{P_h}\right) \searrow$

  $\Rightarrow x^* \searrow$

  If $P_\ell = P_h \Rightarrow x^* = 0$

- Formal proof 2 - If D_ℓ, D_h are discrete

* Let $\Delta R(b) = \mathbb{E}[R(b+1) - R(b)]$. Want first b s.t $\Delta R(b) < 0$

$$\Rightarrow \Delta R(b) = P_\ell \mathbb{E}\left[\min(b+1, D_\ell) - \min(b, D_\ell)\right] + P_h \mathbb{E}\left[\min\{\max(c-(b+1), c-D_\ell), D_h\} - \min\{\max(c-b, c-D_\ell), D_h\}\right]$$

- $$\min(b+1, D_\ell) - \min(b, D_\ell) = \begin{cases} 1 ; & b+1 \le D_\ell \\ 0 ; & \text{ow} \end{cases}$$

- $$\min\{\max(c-(b+1), c-D_\ell), D_h\} - \min\{\max(c-b, c-D_\ell), D_h\} = \begin{cases} -1 ; & \left\{ \begin{array}{l} c-(b+1) \ge c-D_\ell \\ c-b \le D_h \end{array} \right\} \\ 0 ; & \text{ow} \end{cases}$$

$$\Rightarrow \Delta R(b) = P_\ell \, \mathbb{P}[b+1 \le D_\ell] - P_h \, \mathbb{P}[D_\ell \ge b+1, D_h \ge c-b]$$

$$= \mathbb{P}[D_\ell \ge b+1] \left(P_\ell - P_h \, \mathbb{P}[D_h \ge c-b] \right)$$

- The opt protection level $x^* \equiv$ ~~smallest~~ largest x s.t

$$P_\ell - P_h \, \mathbb{P}[x^* \le D_h] < 0$$

$$\Rightarrow \boxed{x^* = \max\left\{ y \in \mathbb{N} \mid \mathbb{P}[D_h \ge y] > \frac{P_\ell}{P_h} \right\}}$$

Assumptions (& the road ahead)

1) 2-fare classes
   - We generalize to multiple-fare classes in the next class

2) Single-leg
   - Generalized via network revenue management

3) Capacity limited
   - Overbooking models
   - Optimal choice of capacity (Cournot competition)

4) Known demand distributions
   - Demand estimation / A/B testing
   - Bandit paradigms

5) Fixed prices
   - dynamic pricing

6) Separation of customer classes (perfect segmentation)
   - Assortment optimization
   - Price discrimination / bundling