Intermission - The Spiral-Down Effect

- Till now, we assumed perfect segmentation of customers - each customer wants only one product

Eg - In single-resource allocation, we assume demand D_j for fare class j is indep of others

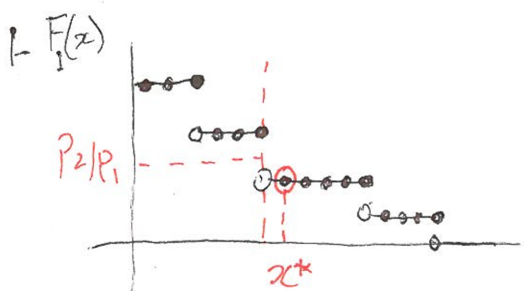
- Moreover, we also assumed we know the distributions $F_j(\cdot)$ from which demand is drawn.
- In practice: We want to learn F_j by observing past sales, which depended on our allocation policies, which depend on past sales ...

Alert - This combination of i) possibly flawed model of customer behavior, and ii) feedback between learning and optimization, can lead to <u>the spiral-down effect</u>

<u>Setting</u> - 2 fare-class, single resource allocation

- C seats, 2 fare-classes $p_1 > p_2$
- Demand for fare class $i \equiv D_i \sim F_i(\cdot)$
  (assume we don't know F_i, but believe $D_1 \perp\!\!\!\perp D_2$)
- Now we know the optimal allocation policy - Set protection level $\boxed{x_1^* = F_1^{-1}(1 - p_2/p_1)}$

(more specifically - $x_1^* = \min_{x \in \{0,1,\ldots\}} \left[p_2 \geq p_1(1 - F_1(x)) \right]$



- This is Littlewood's rule
- <u>Note</u> - We only need $F_1(\cdot)$ to compute x_1^*

- In practice, suppose we have both fare classes open — now some customers may be willing to buy at fare P_1, but choose to buy at fare class P_2 since it is cheaper.
  - We can model this via a <u>customer-choice model</u> - for each customer, we want to define a list of preferred products.
  - Suppose the two fare classes are labelled 1 and 2. We also use 0 for the no purchase option. Now we can have the following ~~ob~~ preference lists
    - 102 - Customers who want only class 1
    - 201 - Customers who want only class 2 } perfect segmentation
    - price-conscious ≡ 210 - Want class 2, but willing to buy class 1
    - quality-conscious ≡ 120 - Want class 1 but willing to buy class 2

- Example of spiral-down effect

  - Suppose there are d customers (deterministic), all of whom have preference list 2 1 0 (ie, they buy class 2 tickets if available, else buy class 1 tickets)

  - Claim: Optimal protection level is $x_1^* = c$ (i.e., only sell class 1 tickets)

  - However, we assume perfect segmentation to compute the protection levels.

  - Define $G_1(y \mid x) \triangleq \mathbb{P}[\text{Demand for fare class 1 is } y \mid \text{protection level} = x]$

    This can be different from $F_1(\cdot)$

- In our example - Suppose our initial protection level was $x_0 \leq c$

(Since there is only one protection level, use x_k for level at time k!)

- We observe demand for class 1 $= [d-(c-x_0)]^+$
- Thus $\hat{G}_1(y \mid x_0) = \begin{cases} 1 & ; \ y \geq [d-(c-x_0)]^+ \\ 0 & ; \ \text{ow} \end{cases}$

(empirical distn)

- After k rounds, suppose we use the empirical distribution as our prediction model

$$\hat{G}_k(y \mid x_0, x_1, \ldots, x_{k-1}) = \frac{1}{k} \sum_{j=0}^{k-1} \mathbb{1}\{y \leq [d-(c-x_j)]^+\}$$

Given this, we set next protection level as

$$x_k = \min_{x \geq 0} \left[\hat{G}_k(y \mid x_0, \ldots, x_{k-1}) \geq 1 - P_2/P_1 \right]$$

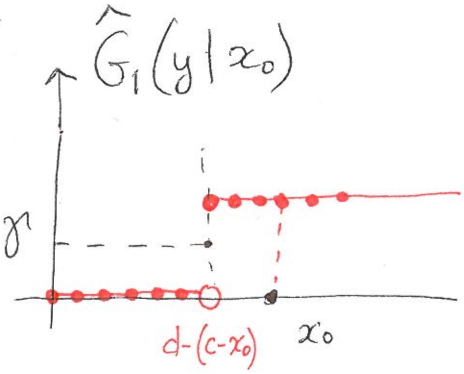
- $\boxed{\text{Claim. If } d < c, \text{ then } x_k \searrow 0}$ Spiral down

In fact, after some finite k^*, we have $x_k = 0$.

Note that this is the worst possible ~~setting~~ Policy!

Proof - Let $\gamma = 1 - p_2/p_1 < 1$

- First consider x_1 by using Littlewood's rule.



$$x_1 = d-(c-x_0) < x_0$$

(since $d<c$, $x_0+d-c < x_0$)

- Now at any ~~stage~~ time k, we have

$$\hat{G}_k(y|x_0,\dots,x_{k-1}) = \frac{1}{k}\sum_{j=0}^{k-1} \mathbb{1}\{y \le [d-(c-x_j)]^+\}$$

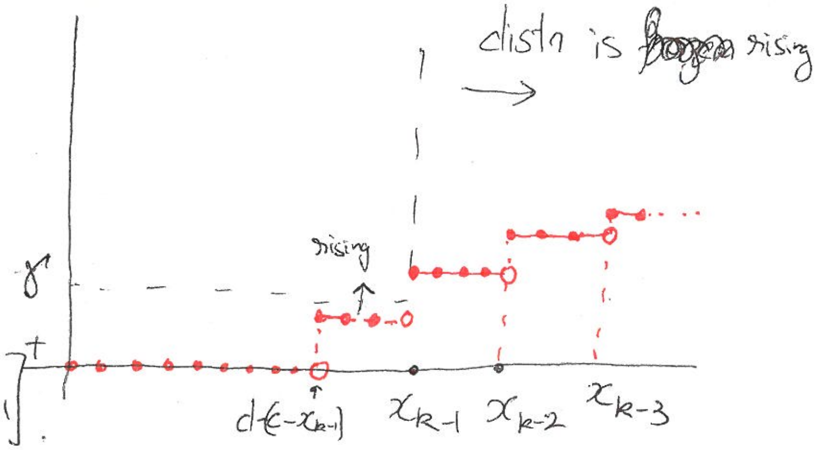
We can now show that $x_k = \min_{y\ge 0}\{\hat{G}_k(y) \ge \gamma\}$

satisfies $x_k \le x_{k-1}$

To see this, note that the number of class-1 fares sold is $[d-(c-x_{k-1})]^+ < x_{k-1}$. Thus, $\hat{G}_k(y) \ge \hat{G}_{k-1}(y)$ for all $y \ge x_{k-1}$ (and this continues to hold for x_{k+1}, $x_{k+2}, \dots$)

- In other words, as long as future sales are less than x_{k-1}, the future empirical distributions $\hat{G}_k, \hat{G}_{k+1}, \ldots$ are ~~frozen~~ increasing beyond the point x_{k-1}

- On the other hand, while the protection level stays frozen at x_{k-1}, the sales are frozen at $[d-c+x_{k-1}]$.



Consequently, $\hat{G}_k(y)$ is 0 for all $y < [d-c+x_{k-1}]$

- However, in between $d-(c-x_{k-1})$ and x_{k-1}, the empirical distribution is rising until it crosses γ! At that point, the protection level decreases to $(d-c+x_{k-1})$

- This continues till the protection level falls to 0!

Eg (from Cooper, Homem-de-Mello, Kleywegt)

$C = 10,\ d = 8,\ P_1 = 500,\ P_2 = 200$ (so $\gamma = 3/5$)

Suppose $x_0 = 10$

k	x_k	Observed Sales of class-1 tickets	Revenue
1	10	8	4000
2	8	6	3400
3	8	6	3400
4	6	4	2800
⋮	⋮	⋮	⋮
8	6	4	2800
9	4	2	2200
⋮	⋮	⋮	⋮
20	4	2	2200
21	2	0	1600
⋮	⋮	⋮	⋮
50	2	0	1600
51	0	0	1600