We used Leibniz integral rule in this lecture.

10.1 Overview of the last lecture

In last lecture, we talked about Linkage Principle: the more closely the winning bidder’s payment is linked to his signal, the higher is the expected revenue.

10.2 Overview of this lecture

In this lecture, we will introduce a new topic: price theory of 2-sided platforms.

10.3 Price theory of 2-sided platforms

10.3.1 Examples of 2-sided platforms

Here are some real-life examples of 2-sided platforms:

1. Newspaper: Reader v.s. Advertiser. Newspapers get paid by advertisers by putting up advertisements, but may lose readers if they have too many advertisements.

2. Credit Card Companies: Customers v.s. Merchants. Customers generally prefer credit cards and are likely to purchase more when using credit cards, but credit card companies charge a per transaction fee from merchants and merchants might increase the price for credit card customers or set lower bounds for credit card purchases.

10.3.2 The basic model of 2-sided platforms

As observed in [1], the outcomes of a transaction in a 2-sided marketplace depend not only on value or cost but also on the payment structure.

The model proposed by Weyl in [2] is structured as follows:

- Two Sides: L (left) and R (right).
- Number of people on both sides are normalized to unit mass.
- Users of side $S$ derives 2 benefits from the platform. (Both values are scaled and can be negative.)
  
  · Membership benefit $B^S$,
  
  · Interaction benefit $b^S$

In this model, the utility functions are:

$$U^L = B^L + b^L N^R - P^L(N^R)$$ (10.1)

$$U^R = B^R + b^R N^L - P^R(N^L)$$ (10.2)

Note that under these assumptions, the user values are exogenous and quasi-linear. We also assume that there are only homogeneous cross-network interactions and that there are no price discrimination within a side.

### 10.3.3 Analysis of the model

Assume that the distribution of the benefits are $F^L(B^L, b^L)$ and $F^R(B^R, b^R)$, which lie on some convex set in $\mathbb{R}^2$ and have density functions $f^L$ and $f^R$.

By the IR constraint, users join the platform if and only if $U^S \geq 0$, which means

$$B^L \geq P^L(N^R) - b^L N^R \quad \text{and} \quad B^R \geq P^R(N^L) - b^R N^L$$

Then,

$$N^L(N^R, P^L) = \int_{-\infty}^{\infty} \int_{P^L(N^R) - b^L N^R}^{\infty} f^L(b^L, B^L) dB^L db^L$$ (10.3)

$$N^R(N^L, P^R) = \int_{-\infty}^{\infty} \int_{P^R(N^L) - b^R N^L}^{\infty} f^R(b^R, B^R) dB^R db^R$$ (10.4)

As shown in Figure 10.1, the shaded area represents $N^L$. We call users on the dotted line 'marginal users' since they have 0 utility and call users in the shaded area 'loyal users'.

The system can have multiple equilibriums. For example, let $b^L = b^R = 1$, $B^L = B^R = 0$ and $P^L = P^R = \frac{1}{2}$. Then the system reaches equilibrium when $(N^L, N^R) = (0, 0)$ or $(1, 1)$.

Now we assume that the platform wants to maximize welfare or revenue. Observe that whenever we increase $P^L$, the dotted line in Figure 10.1 shifts up and $N^L$ decreases. Hence,

$$\frac{\partial}{\partial P^L} N^L(N^R, P^L) < 0$$

Then, $\tilde{P}^L(N^R \mid N^L)$ is well defined as the inverse of $N^L(N^R, P^L)$. Similarly, we have $\tilde{P}^R(N^L \mid N^R)$. 

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Suppose we want to achieve \((N^L, N^R) = (\hat{N}^L, \hat{N}^R)\), then we may define the insulating tariffs as

\[ P^L(N^R) = \tilde{P}^L(N^R | \hat{N}^L) \]

and

\[ P^R(N^L) = \tilde{P}^R(N^L | \hat{N}^R) \]

The value to side L is

\[ V^L(N^L, N^R) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (b^L N^R + B^L) f^L(b^L, B^L) dB^L db^L \tag{10.5} \]

Similarly, we can define \(V^R(N^L, N^R)\).

The social welfare is

\[ W(N^L, N^R) = V^L(N^L, N^R) + V^R(N^L, N^R) - c^L N^L - c^R N^R - c N^L N^R. \tag{10.6} \]

where \(c^L, c^R\) and \(c\) are platform costs.

We want to choose \((N^L, N^R)\) to maximize \(W(N^L, N^R)\).

\[ \frac{\partial W}{\partial N^L} = \frac{\partial V^L}{\partial N^L} + \frac{\partial V^R}{\partial N^L} - c^L - c N^R \]

Using Leibniz integral rule and Tonelli’s theorem, we can differentiate (10.3) with respect to \(N^L\),

\[ 1 = \left( - \int_{-\infty}^{\infty} f^L(b^L, P^L - b^L N^R) db^L \right) \cdot \left( \frac{\partial P^L}{\partial N^L} \right) \]

Then,

\[ \frac{\partial P^L}{\partial N^L} = - \frac{1}{\int_{-\infty}^{\infty} f^L(b^L, P^L - b^L N^R) db^L} \]
Differentiate (10.3) with respect to $N^R$, we get

$$0 = \int_{-\infty}^{\infty} f^L(b^L, P^L - b^L N^R) \left( \frac{\partial P^L}{\partial N^R} - b^L \right) db^L$$

Thus,

$$\frac{\partial P_L}{\partial N^R} = \frac{\int_{-\infty}^{\infty} b^L f^L(b^L, P^L - b^L N^R)db^L}{\int_{-\infty}^{\infty} f^L(b^L, P^L - b^L N^R)db^L}$$

Let $\bar{b}^R(N^L, N^R)$ represent the average interaction value of marginal users. Then,

$$\frac{\partial P_L}{\partial N^R} = \bar{b}^R(N^L, N^R)$$

Hence, we have

$$P_L(N^L, N^R) = c^L + cN^R - N^R \bar{b}(N^L, N^R)$$
Bibliography
