Network Revenue Management

- Capacity control across multiple resources linked by demands for sets of resources

  - Eg. Hub and spoke networks (airlines)
  - Linear networks (hotels, rental cars)

  - No natural ordering of fare-classes ⇒ Use time for DP formulation

Problem setting

- \( m \) resources, \( n \) products (or ODFs = Origin - Dest - Fare)
- \( \mathbf{c} = (c_1, c_2, \ldots, c_m) \in \mathbb{Z}^m \)
- \( t = \text{time to go} \in [0, T] \)
- Demand for product \( i \sim PP(\lambda_i(t))\)
- Product \( i \) ∈ requirements \( \mathbf{a}_i \in \{0, 1\}^n \), price \( p_i \)

- State \( (t, \mathbf{z}) \)

  - Bellman Eqn. \( V(t,0) = V(0, \mathbf{z}) = 0 \)

\[
\frac{\partial V(t, \mathbf{z})}{\partial t} = \sum_{i=1}^{n} \lambda_i(t) \max_{u_i \in \{0,1\}} \left[ p_i u_i - \Delta T \cdot V(t+\Delta T, \mathbf{z}) \right] + V(t, \mathbf{z}) - V(t, \mathbf{z} - \mathbf{a}_i) > 0
\]
\[ R(\theta) = \max_{u \in \{0,1\}^n} \sum_{i=1}^{n} \beta_i(t) [P_i u_i - \theta_i]^+ \]

Then \( u^* \equiv \{ u_i = 1 \text{ if } P_i > \theta_i \} \Rightarrow R(\theta) = \sum_{i=1}^{n} \beta_i(t) [P_i - \theta_i]^+ \)

\[ \frac{\partial V(t,x)}{\partial t} = \sum_{i=1}^{n} \beta_i(t) \left[ R(\Delta V(t,x)) \right] \]

and \( u^*(t,x) = \{ u_i = 1 \text{ if } x - a_i > 0 \text{ and } P_i > \Delta_i V(t,x) \} \)

**Difficulty:**

i) \( V(t,x) \) may be difficult to compute

ii) \( u^*(t,x) \) needs too much storage / is complicated

We want simpler policies with good properties

**Idea - bid-price controls / probabilistic admission control**

Suppose \( \Delta V(t,x) = V(t,x) - V(t,x - a_i) \)

\[ \approx \sum_{j=1}^{n} \frac{\partial V(t,x)}{\partial x_j} a_{ij} \]

\( \triangleq M_j(t) \) - bid-price for resource \( j \) at time \( t \)

(This seems likely when \( a_i \ll x \))

Now condition for admission \( \equiv P_i > \sum_{j=1}^{n} M_j(t) a_{ij} (\text{and } x - a_i > 0) \)
First, we rescale time to get a discrete time equivalent to the Bellman eqn

- Given \( a \geq 1 \), we set \( T \rightarrow aT \), \( \lambda_i(t) \rightarrow \lambda_i(at) \)
and consider discrete times \( \{Ta, aT-1, \ldots, 0\} \)
(henceforth, we use \( T \) and \( \lambda_i(t) \) to refer to scaled values)
- \( a \) is chosen s.t. \( \lambda_i(at)/a \leq 1 \quad \forall \ t \)

Now we have

\[
V(t, x) - V(0, x) = \mathcal{R} \left( \Delta V(t-1, x) \right)
\]

(where again \( \mathcal{R}(\Theta) = \sum_{i=1}^{n} \lambda_i(t) \left[ p_i - \Theta_i \right]^+ \) \( V(0) = V(0, x) = 0 \))

and opt control \( u^*_i(t, x) = \mathbb{1} \{ a_i \leq x \text{ AND } p_i \geq \Delta_i \Delta V(t, x) \} \)

To implement bid-prices, we need a linear approximations of \( V(t, x) \)

(Fluid) upper bound on \( V(t, x) \)

- Oracle based bound: Suppose we know realization of demand
- Let \( D_j \sim \text{Poi} \left( \beta_j \int_0^T \lambda(s) \, ds \right) \) be total arrivals to class \( j \)
  (i.e., total demand for product \( j \)). Define \( \Lambda_j = \int_0^T \lambda_j(s) \, ds \)

\[
V^u(T, c, D) = \max_{i=1}^n \sum_{i=1}^n \pi_i \gamma_i \quad \text{s.t.} \quad \sum_{i=1}^n a_{i, j} y_i \leq \xi_j, \forall j, v_i \in [M]
\]

\[
0 \leq y_i \leq D_j, \forall j, v_i \in [N]
\]
Claim - $V^0(T, c|D)$ is concave in $D$

Proof - If $y_*$ and $y_*$ are solutions to $V^0(T, c|D)$ and $V^0(T, c|D')$, then $\alpha y + (1-\alpha) y'$ is feasible for $D + D'(1-\alpha)$

$\Rightarrow$ $V^0(T, c|D + D'(1-\alpha)) \geq \alpha V^0(T, c|D) + (1-\alpha) V^0(T, c|D')$

Thus, if we replace $D_j$ with $E[D_j] = \Lambda_j$ in $\circ$, we get $V^{\text{fluid}}(T, c)$, which by Jensen's satisfies $V^{\text{fluid}}(T, c) \geq E[V^0(T, c|D)]$

Primal - Dual forms for fluid problem $V^{\text{fluid}}(T, c)$

**Primal**

$$\begin{align*}
\text{max} & \quad \sum_{i=1}^{n} \Lambda_i y_i \\
\text{s.t.} & \quad \sum_{i=1}^{m} a_{i}(j) y_i \leq C_j \quad \forall j \in [M] \\
& \quad 0 \leq y_i \leq \Lambda_i \quad \forall i \in [N]
\end{align*}$$

**Dual**

$$\begin{align*}
\text{min} & \quad \sum_{i=1}^{n} \Lambda_i \beta_i + \sum_{j=1}^{m} C_j z_j \\
\text{s.t.} & \quad \sum_{j=1}^{m} a_{i}(j) z_j + \beta_i \geq P_i \quad \forall i \in [N] \\
& \quad z_j \geq 0, \quad \beta_i \geq 0
\end{align*}$$
By complementary slackness
\[ z_j^* > 0 \Rightarrow \sum_{i=1}^{n} a_{i}(j) y_i = c_j \]
\[ \beta_i^* > 0 \Rightarrow y_i = \Lambda_i \]

\( z_j^* \) = 'marginal cost' of unit of resource \( j \)
\( \beta_i^* \) = 'marginal cost' of additional customer for product \( i \)

\[ \beta_i^* = (P_i - \sum_{j=1}^{m} a_{i}(j) z_j^*)^+ \quad \forall i \]

\[ \sum_{i=1}^{n} \Lambda_i \beta_i^* = \sum_{i=1}^{n} \Lambda_i (P_i - \sum_{j=1}^{m} a_{i}(j) z_j^*)^+ = R(A^Tz^*) \]

\[ V_{\text{fkw}}(T, c) = \min_{z > 0} \left\{ R(A^Tz) + c^Tz \right\} \]

(Equivalently - we are approximating \( \Delta V(T, c) \approx \sum_{i=1}^{m} a_{i}(i) z_i^* \))

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Now given \( y_i^* \) ( primal opt vars) or \( z_i^* \) (dual opt vars), we have 2 simple heuristics

i) Bid price = Admit i if \( \sum_{j=1}^{m} a_{i}(j) z_j^* \leq P_i \) and \( x - q_i \geq 0 \)

ii) Probabilistic admission control = Admit i w/ \( y_i^* / \Lambda_i \)

(PAC)

The notion of a bid price is more general. Given any linear approx of \( \Delta V(t, x) = \sum_{i \in M} a_i(\cdot) \), \( M \) are bid prices

(Similarly for PAC policies)
Are bid-prices optimal? No

\[ P_1 = P_2 = 250 \]
\[ P_3 = 450 \]

Suppose \[ \Lambda(2) = (0.3, 0.3, 0.4) \]
\[ \Lambda(1) = (0, 0, 0.8) \quad \text{(and no arrival wp 0.2)} \]

Claim. \( \text{OPT} = \text{Accept only customer for product 3} \)
\[ R^* = (0.4 + 0.6 \times 0.8) \times 450 = 396 \]

However, to implement this via bid prices \( \mu_i \), we need
\[ \mu_1 + \mu_2 \leq 450, \quad \mu_1 \geq 250, \quad \mu_2 \geq 250 \rightarrow \text{impossible} \]

Can we show bid-prices perform well? Yes!

Idea - Consider the problem under a 'large market' scaling
\[ c \rightarrow \Theta c, \quad \Lambda_i(t) \rightarrow \Theta \Lambda_i(t) \forall i, t \text{ for some } \Theta > 1 \]

- Note \( \{ Z^*_j \} \) remains the same

\[ V^\theta_{\text{fluid}}(T, c) = \max \Theta \left( \sum_{i=1}^{n} \Lambda_i \beta_i + \sum_{i=1}^{n} c_i z_i \right) \]
\[ \text{s.t. } \sum_{j \in \mathcal{J}(i)} z_j + \beta_i \geq \lambda_i \forall i, \quad z_j, \beta_i \geq 0 \]

- Let \( V^\theta(T, c) \) be the value fn under \( \Theta \) scaling

We know \[ V^\theta(T, c) \leq E[V^\theta_{\text{fluid}}(T, c)] \leq V^\theta_{\text{fluid}}(T, c) \]
We first need an additional Lemma

**Lemma** - For any random variable $X$ with $E[X] = \mu$, $Var(X) = \sigma^2$, we have

$$E[(X - c)^+] \leq \frac{1}{2} \left( \sqrt{\sigma^2 + (c - \mu)^2} - (c - \mu) \right)$$

**Pf** Consider $f(x) = (X - c)^+$

Moreover, for any $\alpha > 0$, define

$$g(x) = \frac{(x - (c + \alpha))^2}{4\alpha}$$

Then (from figure) we have $g(x) \geq f(x)$ for $x \\geq c$

$$\Rightarrow E[f(x)] \leq E[g(x)] = \frac{1}{4\alpha} E\left[x^2 - 2x(c + \alpha) + (c + \alpha)^2\right]$$

$$\Rightarrow E[(X - c)^+] \leq \min_{\alpha \geq 0} \left( \frac{\alpha^2 + \sigma^2 - 2\mu(c + \alpha) + (c + \alpha)^2}{4\alpha} \right)$$

Setting $\alpha = \sqrt{\sigma^2 + (c - \mu)^2}$, we get

$$E[(X - c)^+] \leq \sqrt{\sigma^2 + (c - \mu)^2} - (c - \mu)$$

Note: $\mathbb{E}[(X - c)^+] \leq 0.5 \sigma + 0.5(1 - c - \mu) \iff \sigma = 0.5 \sigma$ if $c = \mu$. 
Thm (Talluri & Van Ryzin '98) - Let $B^\theta$ be the total expected revenue under the bid price heuristic using bid prices $\{Zj^*\}$ from the fluid LP. Then

$$\frac{B^\theta}{V^\theta(T,c)} \geq 1 - O\left(\frac{1}{\sqrt{\theta}}\right)$$

(Strictly speaking - this requires a small modification to the policy - see below)

Pf: We consider a small modification of the basic bid-price heuristic, as follows (based on Reinhardt-Wang '07).

Recall for $V^\theta_{\text{fluid}}(T,c)$, the primal soln is $\{y^*_i : i \in [m]\}$, and dual soln is $\{Zj^* : j \in [m]\}$. Now consider the following policy:

- If $P_i > \sum_{j=1}^m a_{i,j} Z_j^*$ (and $x - a_i \geq 0$): Admit $i$
- If $P_i = \sum_{j=1}^m a_{i,j} Z_j^*$ (and $x - a_i \geq 0$): Admit $i$ wp $\frac{y^*_i}{\lambda_i}$
- Else reject $i$

Now we show that under this policy, revenue $\bar{B}^\theta$ satisfies

$$\bar{B}^\theta \geq (1 - O\left(\frac{1}{\sqrt{\theta}}\right)) V^\theta(T,c)$$
Now we have the following:

1) \( \forall \Theta, \quad V^\Theta, \text{fluid} (T, c) = \Theta V^\text{fluid} (T, c) \)
   \( V^\Theta (T, c) \leq V^\Theta, \text{fluid} (T, c) \)

\[ \Rightarrow \frac{\bar{B}^\Theta}{V^\Theta (T, c)} \geq \frac{\bar{B}^\Theta}{\Theta V^\text{fluid} (T, c)} \quad (\text{Note: by Jansen's}) \]

We can write \( y_i^* = \Lambda_i \left( \frac{y_i^*}{\Lambda_i} \right) \quad \forall i \)

\( \Rightarrow \) The fluid LP is

\[ V^{\text{fluid}} (T, c) = \sum_{i \in [n]} \Lambda_i \left( \frac{y_i^*}{\Lambda_i} \right) P_i \]

and also

\[ \sum_{i \in [n]} \Lambda_i \left( \frac{y_i^*}{\Lambda_i} \right) a_i (j) \leq c_j \quad \forall j \quad (\text{by feasibility}) \]

2) Now consider an alternate admission policy, where we admit all arriving customers in class \( i \) w.p. \( \frac{y_i^*}{\Lambda_i} \).

Ignoring capacity constraints; however, we incur a cost of

\( P_{\text{max}} = \max_{i \in [n]} \{ P_i \} \) for each additional unit of capacity used on any leg.
Let $L^θ = \text{revenue under this new policy}$

\[ L^θ = \sum_{i \in [n]} \theta \Lambda_i \left( \frac{y_i^*}{\Lambda_i} \right) p_i - E \left[ \sum_{j=1}^m \rho_{\text{max}} \left( \sum_{i \in [n]} \left( \frac{y_i^*}{\Lambda_i} \right) a_i(j) - \theta_{\text{cover}} \right) \right] \]

Overbooking cost

\[ = 0 \left( \sum_{i \in [n]} \theta \Lambda_i \left( \frac{y_i^*}{\Lambda_i} \right) p_i \right) - \theta_{\text{max}} \theta \theta_{\text{cover}} \]

\[ = 0 \left( V^{\text{fluid}} \right) \left( 1 - \frac{\theta_{\text{cover}}}{\theta_{\text{V^{\text{fluid}}}}} \right) \]

ii) $L^θ \leq \overline{B}^θ \text{ sample-pathwise}$

This is because admitting a customer when capacity is unavailable costs more than revenue earned.

Combining the 4 inequalities, we get

\[ \frac{\overline{B}^θ}{V^{\theta(\text{tie})}} \geq 1 - \frac{\theta_{\text{cover}}}{\theta_{V^{\text{fluid}}}} \]

where

\[ V^{\theta_{\text{fluid}}} = \Theta \left( \sum_{i \in [n]} y_i^* p_i \right) \]

and

\[ \text{Cover} = \rho_{\text{max}} \sum_{j \in [m]} E \left[ \left( \sum_{i \in [n]} D^θ \left( \frac{y_i^*}{\Lambda_i} \right) a_i(j) - \theta_{\text{cover}} \right) \right] \]
Finally, \[ \mathbb{E} \left[ \sum_{i \in [n]} D_i^\theta \left( \frac{y_i^*}{\lambda_i} \right) a_i(j) \right] = \sum_{i \in [n]} \theta y_i^* a_i(j) \]

\[ \Rightarrow \theta c_j \geq \theta \sum_{i \in [n]} y_i^* a_i(j) \]

\[ \Rightarrow \text{We can use } \mathbb{E} \left[ (x-c)^+ \right] \leq 0.5 \sigma \]

\[ \Rightarrow \sqrt{\text{Var} \left( \sum_{i \in [n]} D_i^\theta \left( \frac{y_i^*}{\lambda_i} \right) a_i(j) \right)} \leq \sqrt{\theta \left( \sum_{i \in [n]} a_i(j) y_i^* \right)} \]

\[ \Rightarrow \frac{B^\theta}{V^{\theta(t,c)}} \geq 1 - \frac{P_{\max} \sqrt{\theta} \sqrt{\sum_{i \in [n]} a_i(j) y_i^*}}{2 \left( \sum_{i \in [n]} y_i^* p_i \right) \theta} \]

\[ = 1 - \Theta^0 \left( \frac{1}{\sqrt{\theta}} \right) \]