Multiple fare-class capacity allocation

- \( n \) fare classes, capacity \( C \)
- \( \text{prices} \quad P_1 \leq P_2 \leq \ldots \leq P_n \)
  
  
  \( \text{(demand-distrib)} \quad (D_{ij}, F_i) \quad (D_{ij}, F_i^2) \quad (D_{ij}, F_i^n) \)

- Assumptions
  
  i) \( D_i \) are independent
  ii) \( D_i \) realized before \( D_{i+1} \) (sequential) arrival

- Controls - protection levels (nested)
  
  \( x_j \) = protection level for fare classes \( j+1, j+2, \ldots, j \n
- Let \( S_j \) = capacity available when fare class \( j \) arrives \( (S_1 = C) \)

then

\[
V_j (s_j) = \max_{x_j \in \{0,1, \ldots, s_j\}} \mathbb{E}_{x_j} \left[ \prod_j \min \{ D_j, s_j - x_j \} + V_{j+1} (s_{j+1}) \right]
\]

where \( s_{j+1} = \max \{ s_j - D_j, x_j \} \)
- **Question**: Why protection levels?

- **Alternative**: Let's solve an 'easier' problem
  - Assumption 1: We know $D_j$ before allocating class $j$ seats
  - Assumption 2: Can choose exact number of seats for $j$

$$V_j(s_j) = \mathbb{E}_{F_j} \left[ \max_{x \in \{0, ..., s_j\}} \{ p_j \cdot \min(x, D_j) + V_{j+1}(s_{j+1}) \} \right]$$

$$\max(s_j - x, s_j - D_j) = s_j - \min(x, D_j)$$

- **Easier state variable**: Let $y_j = s_j - \min(x, D_j)$

$$V_j(s_j) = \mathbb{E}_{F_j} \left[ \max_{y_j \in \{(s_j - D_j), ..., s_j\}} p_j(s_j - y_j) + V_{j+1}(y_j) \right]$$

$$\Rightarrow V_j(s_j) = p_j s_j + \mathbb{E}_{F_j} \left[ \max_{y_j \in \mathcal{Y}(s_j, D_j)} (-p_j y_j + V_{j+1}(y_j)) \right]$$

What can we say about the solution?

- Assumption 3: Consider continuous $D_j$
- Maximizing a fixed fn over a random set (interval) 

h(y) 

$Y(s, D_i = d_1)$ $Y(s, D_i = d_2)$ $y$

- In general, this is some complicated random variable

What if $h(y)$ is concave

Let interval be $[L, U]$

$soln = \arg \max_{y \in \mathbb{R}}$ 

$y^*[L; U] = \begin{cases} 
L & L > y^* \\
{y^*} & L \leq y^* \leq U \\
U & y^* > U 
\end{cases}$

(Do we need concave? What about $\mathbb{R}$?)

(Ref: Quasicconcavity) (What if $y \in$ discrete set?)

In our case -

$\mathbb{E}_F \left[ \max_{y \in \{s_j D_j^t, \ldots, s_j\}} \left\{ -P_j y_i + V_{j+1} (y_i) \right\} \right]$ 

(Asm 4) Suppose $V_{j+1} (\cdot)$ is concave \( \Rightarrow \) $h^*_j = P_j y_i + V_{j+1} (y_i)$ is concave in $y_i$

* Define $\mathbb{x}^* = \arg \max_{y \in \mathbb{R}} \{ h^*_j (y) \} = \mathbb{x}^*_j$

Then $opt$ $y^*_j = \begin{cases} 
(s_j - D_j)^t (s_i - D_i)^t \mathbb{x}^*_j & \mathbb{x}^*_j \in \mathbb{R} \\
\mathbb{x}^*_j & s_j - D_j \leq \mathbb{x}^*_j \leq s_i \\
s_j & s_i < \mathbb{x}^*_j 
\end{cases}$
Thus, given

\( A1 \) - 'Oracle access' to \( D_j \)

\( A2 \) - Control \( \equiv \) Exact # of seats at fare \( p_j \)

\( A3 \) - Continuous \( D_j \)

\( A4 \) - Concave \( \bar{V}_j (\cdot) \)

Then optimal capacity allocation \( \Rightarrow \) protection level \( \{x^*_j\} \)

\( \frac{1}{2} \text{ where } x^*_j = \text{argmax}_{x \in [0,c]} V_j (x) = \text{argmax}_{x \in [0,c]} (-p_j x + \bar{V}_j (x)) \)

- we don't need to store the value functions! Only \( x^*_j \)

* Let's remove the assumptions. \( A4 \), then \( A1 \) and \( A2 \)

* Concavity of \( \bar{V}_j (\cdot) \)

- Induction on \( n, n-1, \ldots, 1 \)

\[
\bar{V}_n (x_n | D_n) = \max \left[ -p_n (x_n - D_n)^+ + p_n x_n \right]
\]

\( (x_n - D_n)^+ \leq y \leq x_n \) \( \text{ or } p_n \cdot \min \{x_n, D_n\} \)

\( \bar{V}_n (x_n) - D_n \)

- concave in \( x \) \( \forall D_n \)

\[
\bar{V}_n (x_n) = \mathbb{E}_{F_n} \left[ \bar{V}_n (x_n | D_n) \right]
\]

- Concave in \( x_n \)

(linear combination of concave functions)
Assume $V_{j+1}(x_{j+1})$ is concave in $x_{j+1}$.

Let $F_j(y) = \max \left[ P_j y + V_{j+1}(y) \right]$, and

$V_j(x_j|D_j) = \max \left[ F_j(y) \right] + P_j x_j \quad \forall (x_j-D_j)^+ \leq y \leq x_j$

Sufficient to show this is concave

Consider $W(x,a) = \max \left( \min g(x) \right)$

Let $x^* = \arg\max g(x)$ \quad \text{concave}

Then $y^*(x) = \begin{cases} x & x \leq x^* \\ x^* & x - a \leq x^* \leq x \\ x - a & x^* \leq x - a \end{cases}$

$\Rightarrow W(x,a) = g(y^*(x))$ is concave

Notes: this works for discrete demands (A3)

Protection levels are optimal even if we control exact allocations (A2)

What about A1? Claim: It does not matter...