Assortment Optimization

(Recall) Choice Model: Given $S \subseteq N = \{1, 2, \ldots, n\}$ (products)

- $\Pi_j(S) = P[\text{Prod } j \text{ purchased from } S]$

- LCA choice models: $\exists \theta_i > 0 \text{ s.t.}$
  
  $\Pi_j(S) = \frac{\theta_j \prod i \in S \theta_i}{\theta_j + \theta(S)}$, $\theta_0 \equiv \text{'attractiveness' of no purchase}$

  $\theta(S) = \sum i \in S \theta_i$

- Mixture of LCA (mixed MNL)
  
  $\Pi_j(S) = \sum_{g \in G} \alpha_g \left( \frac{\theta_j^g}{\theta_j^g + \theta^g(S)} \right)$, $\sum_{g \in G} \alpha_g = 1$

- Markov Chain choice model: $\Lambda = \{\lambda_i^j\}_{i \in N}$
  
  $P = \{P_{ij}\}_{i, j \in N}$

$P[\text{purchase } j \in S]$: $\Pi_j(S) = \lambda_j + \sum_{i \in S} \phi_i(S) p_{ij}$ $\forall j \in S$

$P[\text{consider } j \in S]$: $\phi_j(S) = \lambda_j + \sum_{i \in S} \phi_i(S) p_{ij}$ $\forall j \in S$

$\Pi_0(S) = 1 - \sum_{j \in S} \Pi_j(S)$

The assortment optimization problem

- Exogenous prices (profits) $P_j$ $\forall j \in N$

- $R^* = \max_{S \in N} R(S) = \max_{S \in N} \sum_{j \in S} P_j \Pi_j(S)$
Assortment Opt under LCA / MNL

\[ R(s) = \sum_{j \in S} \frac{p_j v_j}{v_0 + v(s)} \quad , \quad R^* = \max_{S \subseteq N} R(s) \]

\textbf{Thm} - Let \( p_1 \geq p_2 \geq \ldots \geq p_n \)
\( (\text{Nested-by-revenue Sets}) \)
\( E_0 = \emptyset, E_1 = \{1\}, E_2 = \{1, 2\}, \ldots, E_n = N \)

Then \( \exists \ R^* \in E_0, \ldots, n \) s.t. \( E_{R^*} \in \arg\max_{S \subseteq N} R(s) \)

\textbf{Pf} - By definition \( R^* \geq \sum_{j \in S} \frac{p_j v_j}{v_0 + v(s)} \quad \forall S \subseteq N \)

\[ v_0 R^* \geq \sum_{j \in S} v_j (p_j - R^*) \quad \forall S \subseteq N \]

\[ \exists \text{ some } S \subseteq N \text{ s.t. } R(s) = R^* \]

\[ \Rightarrow \arg\max_{S \subseteq N} R(s) = \arg\max_{S \subseteq N} \left\{ \sum_{j \in S} (p_j - R^*) v_j \right\} \]

- Thus we want to find \( S \in \arg\max_{S \subseteq N} \left\{ \sum_{j \in S} (p_j - R^*) v_j \right\} \)

\[ S^* = \{ j \in N \mid p_j \geq R^* \} \]

- Now even if we do not know \( R^* \), it is clear that we only need to consider \( S \subseteq \{ E_0, E_1, \ldots, E_n \} \)
LCA with constraints

- Let \( x^S_j \in \{0,1\}^n \equiv \text{Indicator of set } S \subseteq \mathbb{N} \)
  (i.e., \( x^S_j = 1 \iff j \in S \))

- We now want to solve a constrained assortment opt

\[
\begin{align*}
\max_{x \in \{0,1\}^n} & \quad \sum_{j \in \mathbb{N}} p_j \cdot x_j \\
\text{s.t.} & \quad \sum_{j \in \mathbb{N}} a_{ij} \cdot x_j \leq b_i \quad \forall i \in \mathbb{L} \\
& \quad x_j \in \{0,1\} \quad \forall j \in \mathbb{N}
\end{align*}
\]

- Assumption - \( A = \{a_{ij}\} \) is totally unimodular, \( b_i \in \mathbb{Z} \)
  \((\Rightarrow \text{extreme points of } \{A x \leq b\} \text{ are integral})\)

\[ \sum_{j \in \mathbb{N}} x_j \leq c \]

- If \( N = S_1 \cup S_2 \cup \ldots \cup S_k \)
  \( \sum_{j \in S_i} x_j \in \{b_{s_1}, \ldots, b_{s_k}\} \) \text{partition}

- Joint pricing and assortment opt
  - Products \( N \equiv \{1, \ldots, n^3\} \), prices \( P = \{p_1, p_2, \ldots, p_k\} \)
  - \( \Omega_{ik} \equiv \text{attractiveness of product } i \text{ at price } p_k \)

- Idea - Create virtual products: \( \bar{x}^i_k \equiv \text{product } i \text{ at price } k \)
  - Constraint: \( \text{at most one } \bar{x}^i_k = 1 \text{ for every } i \)
How do we solve constrained MNL pricing?

\[ \text{OPT1: max } \sum_{i \in N} \frac{P_j u_j x_j}{v_0 + v_0 x} \]
\[ \text{s.t. } A x \leq b \]
\[ (L \times N) \rightarrow x_j \in \{0,1\} \]

\[ \text{OPT2: max } \sum_{j \in N} P_j y_j \]
\[ \text{s.t. } \sum_{j \in N} \frac{y_j}{v_0} + \frac{y_0}{v_0} = 1 \]
\[ \sum_{j \in N} \frac{y_j}{v_0} \leq \frac{b_i}{v_0} y_0 \forall i \in I \]
\[ 0 \leq \frac{y_j}{v_0} \leq \frac{y_0}{v_0} \forall j \in N \]

Thm - The above problems have the same optimal objective.

Moreover, given a solution to OPT2, we can construct a solution to OPT1.

**Pf:** First, as in prev result, we have OPT1 is equiv to

\[ \text{OPT3: max } \sum_{j \in N} (P_j - R^*) \frac{y_j}{v_0} x_j \]
\[ \text{s.t. } A x \leq b, \ 0 \leq x_j \leq 1 \]

This follows from LCA + total unimodularity of A form

Thus, we need to show OPT3 \( \equiv \) OPT2
Let \( \{y^*_j\}_{j \in V^*} \) be an optimal soln to OPT2
\( \{x^*_j\}_{j \in V^*} \) be an optimal soln to OPT3

By defn, OPT3(\(x^*_j\)) = \(R^*\)

Now we show \(y^*_j \text{OPT2}(y^*_j) = R^*\)

Let \(\hat{y}_j = \frac{y^*_j x^*_j}{v^*_j + \sum x^*_i x^*_j}\), \(\hat{y}_o = 1 - \sum_{j \in V^*} \hat{y}_j = \frac{v^*_o}{v^*_o + \sum x^*_i x^*_j}\)

then \(\{\hat{y}_j\}\) satisfies constraints of OPT2

\[-\sum_{j \in V^*} \frac{a_{ij}}{v^*_j} \hat{y}_j = \sum_{j \in V^*} \frac{a_{ij} x^*_j}{v^*_o + \sum x^*_i x^*_j} \leq \frac{b_i}{v^*_o + \sum x^*_i x^*_j} = \frac{b_i \hat{y}_o}{v^*_o}\]

\[-\frac{\hat{y}_j}{v^*_j} = \frac{x^*_j}{v^*_o + \sum x^*_i x^*_j} \leq \frac{\hat{y}_o}{v^*_o} \forall j\]

\(\Rightarrow \{\hat{y}_j\}\text{ is feasible for OPT2}\)

\(\Rightarrow y^*_j \geq \text{OPT2}(\{\hat{y}_j\}) = \sum_j p_j \frac{x^*_j y^*_j}{v^*_o + \sum x^*_i x^*_j} = R^*\)

\(\Rightarrow y^*_j \geq \text{OPT2}(y^*_j) \Rightarrow R^*\). Note \(y^*_o > 0\)

Consider \(\hat{x}_j = \frac{y^*_j}{v^*_j}\)

Check that \(\hat{x}_j\) is feasible for OPT3

Then OPT3(\(\hat{x}_j\)) = \(\sum_{j \in V^*} (p_j - R^*) \frac{y^*_j}{v^*_o} \hat{x}_j = \frac{1}{y^*_o} \sum_{j \in V^*} p_j y^*_j - R^*(1 - y^*_o)\)

\(> y^*_j / x^*_j - (y^*_j / y^*_o) = y^*_j / x^*_j = y^*_j \text{ contradiction}\)
Assortment opt\(^n\) for universal approximators

- We now consider assortment optimization for choice models, which serve as universal approximators.

- First consider the mixture of NNL model

\[ \text{Q: Does this have an 'easy' (poly-time) algorithm for assortment opt}^n? \]

\[ \text{A: No!} \]

Consider the problem - 2 classes, \( \alpha^a + \beta^b = 1 \)
- Prices \( p_i \), choice parameters \( \{v_i^a, v_i^b\} \)

\[ \text{Q: Given any } K, \text{ does there exist } S \subseteq N \text{ s.t} \]

\[ (2\text{-class logit}) \sum_{i \in N} \left( \frac{\alpha^a v_i^a p_i}{v_i^a + v_i^a(s)} + \frac{\alpha^b v_i^b p_i}{v_i^b + v_i^b(s)} \right) \geq K? \]

\[ \text{Thm (RSTT 13)} - 2\text{-class logit is NP-complete} \]

(Reduction from set partition)
Example (Nested-by-revenue is not optimal)

\[ a^a = 0.5, \quad a^b = (5, 20, 1) \]
\[ a^b = 0.5, \quad a^b = (\frac{1}{5}, 10, 10) \]

Prices = (8, 4, 3)

Then opt for class \( a \) = \{1\}, \( R^* = 20/3 \)

opt for class \( b \) = \{1, 2\}, \( R^* = 26/7 \)

opt for mixture = \{1, 3\}, \( R^* = 4.48 \)

= What about the Markov Chain choice model (BGG18)

Thm. - Assortment opt under MC model (\( A, P \))

is equivalent to the following LP

\[
\begin{align*}
\min & \quad \sum_{i \in N} g_i \\
\text{s.t} & \quad g_i \geq P_i \quad \forall \ i \in N \\
& \quad g_i \geq \sum_{i \in N} P_{ij} g_j \quad \forall \ i \in N \\
& \quad g_i \geq 0
\end{align*}
\]

. Here \( g_i \) = 'optimal expected revenue starting from \( i \')

. Moreover, \( g_i = P_i \quad \forall \ i \in S^* \), \( g_i = \sum_{ij} P_{ij} g_j \quad \forall \ i \in N \setminus S^* \)