Problem 1: Practice with DP (Inventory control)

Consider the problem of controlling the inventory of a product over the time periods \{1, \ldots, T\}. At each time period \(t\), the following sequence of events take place: (i) the inventory position \(x_t\) is observed, (ii) a quantity of \(u_t \geq 0\) is ordered if necessary, and is immediately received and (iii) a random demand of \(D_t\) (of known distribution \(F_t\)) is realized. If there is excess inventory after covering the demand, then a holding cost of \(h\) per unit per time period is incurred. If the demand exceeds the available inventory level, then it is backlogged, whereupon the inventory position becomes negative; for each backlogged unit, we incur a backlogging cost of \(b\) per time period. The purchasing cost for the product is \(c\) per unit, with \(b > c\) (try to reason what would happen if we dropped this assumption). Our objective is to minimize the total expected cost over \(T\) time periods.

Part (a)

Write the problem of optimal ordering via a dynamic programming formulation (dynamics/state/actions/transition/reward and the Bellman equation) to find the optimal policy.

Part (b)

Suppose that the value functions at each time \(t\) are convex functions of the inventory position. Show that there exists a scalar \(b^*_t\) for each time period \(t = 1, \ldots, T\) such that it is optimal to raise the inventory position to \(b^*_t\) after making the replenishment decision at time period \(t\).

Part (c)

Show by induction that the value function at any time period \(t\) is a convex function of the inventory position.

Problem 2: Revenue maximization under dynamic seller bidding

Assume that we have \(C\) units of a particular product that we want to sell over the time periods \{1, \ldots, T\}. At each time period, there is one customer arrival. The customer arriving at time period \(t\) makes a bid for the product; assume that each customer makes an i.i.d bid from a set of prices \{1, \ldots, \(p_n\}\}, where price \(p_j\) is chosen with probability \(\sigma_j\) (with \(\sum_{j=1}^{n} \sigma_j = 1\)). If the customer offers price \(p_j\) and we accept this offer, then we generate a revenue of \(p_j\) and consume one unit of inventory; else, if we reject the offer, the customer departs. The objective is to maximize the total expected revenue over \(T\) time periods.

Part (a)

Write down a dynamic programming formulation to find the optimal policy.
Part (b)

Assume that the value functions are concave functions of the remaining inventory. Show that there exists a scalar $y^*_{jt}$ for each possible price level $j = 1, \ldots, n$ and time period $t = 1, \ldots, T$ such that if the remaining capacity at the beginning of time period $t$ is above $y^*_{jt}$, then it is optimal to accept an offer of price $p_j$ at time period $t$. Otherwise, it is optimal to reject.

Part (c)

Assume that $p_1 \leq p_2 \leq \ldots \leq p_n$. Show that $y^*_{jt} \geq y^*_{j+1,t}$, that is, if the remaining inventory at the beginning of time period $t$ is $x$ and $x > y^*_{jt}$ so that it is optimal to accept an offer of price $p_j$, then we also have $x > y^*_{j+1,t}$, which means that it is also optimal to accept an offer of price $p_{j+1}$.

Part (d)

Show via induction that the value function at each time period is a concave function of the remaining inventory.

Problem 3: Static posted prices and prophet inequalities

We again consider the previous setting, where we have $C$ units of a product which we want to sell in periods $1, 2, \ldots, T$, and wherein one customer arrives in each period. Now however, the customer in period $t$ makes a bid $B_t$ according to some (known) distribution $F_t$. Moreover, instead of using the DP solution, we want to use a single posted price threshold for accepting bids.

Part (a)

Consider the case for $C = 1$: Argue that the optimal revenue is bounded by $\mathbb{E}[\max_{t \in [T]} B_t]$. Moreover, show that:

$$
\mathbb{E}\left[ \max_{t \in [T]} B_t \right] \leq p + \sum_{t=1}^{T} \mathbb{E}[ (B_t - p)^+ ]
$$

*Hint: Add and subtract $p$ in the LHS.*

Part (b)

Now suppose we use a single threshold $p$, and accept the first bid greater than $p$ (keeping the item if no bid exceeds the threshold). Let $\Pi(p)$ denote the revenue earned under this policy, and define $q(p) \triangleq \mathbb{P}[\bigcap_{t=1}^{T} \{B_t < p\}]$ to be the probability that the item goes unsold. Now argue that:

$$
\mathbb{E}[\Pi(p)] \geq p(1 - q(p)) + \sum_{t=1}^{T} \mathbb{E}[(B_t - p)^+] \mathbb{P}[B_k < p \ \forall \ k < t]
$$

$$
\geq p(1 - q(p)) + q(p) \sum_{t=1}^{T} \mathbb{E}[(B_t - p)^+]
$$
Part (c)

Show that by appropriately choosing \( p \), we can ensure that \( E[\Pi(p)] \) is at least \( 1/2 \) the maximum possible reward over all policies.

Part (d)

Extend the above result to the case where the seller has \( C \) units to sell.

Problem 4: Overbooking

Consider the problem of selling \( C \) seats on a flight, where customers can reserve a seat at a price \( p \). The total demand is given by random variable \( D \sim F \). Moreover, each customer with a reservation does not show up independently with probability \( 1 - q \), at which time they are refunded their reservation cost. To counter this loss in revenue, we can overbook by allowing up to \( b \) reservations, where \( b \) can exceed capacity \( C \). However, each customer who is denied admission is refunded an amount \( \theta > p \).

Part (a)

Given booking limit \( b \), let \( S(b) \) denote the (random) number of reservations who show up; Similarly, let \( R(b) \) be the total expected revenue. Argue that \( S(b+1) = S(b) + X \mathbb{1}_{\{D \geq b+1\}} \), where \( X \sim \text{Ber}(q) \).

Part (b)

Show that \( R(b) = p \sum_{j=1}^{\infty} \mathbb{P}[S(b) \geq j] - \theta \sum_{k=c+1}^{\infty} \mathbb{P}[S(b) \geq k] \). Moreover, show that \( \Delta R(b) = R(b+1) - R(b) = q \mathbb{P}[D \geq b + 1] \cdot (p - \theta \cdot \mathbb{P}[\text{Bin}(b, q) \geq C]) \) (where \( \text{Bin}(b, q) \) is the binomial distribution), and use this to characterize the optimal booking level \( b^* \).

Next, we consider overbooking with capacity \( C \) and \( n \) fare classes \( \{(p_i, D_i)\} \) \((p_1 \leq p_2 \leq \ldots \leq p_n)\), which arrive sequentially in the order 1 to \( n \). Each accepted reservation shows up independently with probability \( q \). If a booked reservation for fare class \( j \) does not show up at the departure time, then we give this customer a refund of \( h_j \). For each booked reservation that shows up and is denied boarding, we incur a penalty cost of \( \theta > p_n \).

Part (c)

Let \( x_j^k \) be the number of accepted reservations for fare class \( k \) just before making the decisions for fare class \( j \) (with \( x_j^k = 0 \) for \( k \geq j \)). Using the \( n \)-dimensional vector \( x_j = (x_j^1, x_j^2, \ldots, x_j^n) \) as the state variable just before making the decisions for fare class \( j \), formulate a dynamic program that maximizes the expected profit. Make sure to charge the no-show refunds at departure time.

**Hint:** Let \( e_j \) be the \( j \)-th unit vector; accepting one request for fare class \( j \) results in state change from \( x_j \) to \( x_j + e_j \). What are the boundary conditions in this case?
Part (d)

Suppose instead of refunding no-shows, we discount each customers fare by the expected refund cost. In other words, at the time of accepting a reservation, the fare associated with fare class $j$ is $p_j - (1 - q) h_j$. Let $z_j$ be the total number of accepted reservations that we have on-hand just before making the decisions for fare class $j$. Using the scalar $z_j$ as your state variable, formulate a dynamic program that maximizes the expected profit.

Part (e)

Denote the value function in Part (c) as $V_j(x_j)$ and the value function in Part (d) as $J_j(z_j)$. Use induction to show that $V_j(x_j) = J_j(\sum_{k=1}^{n} x_j^k) - \sum_{k=1}^{n} (1 - \rho) h_k x_j^k$. 