\[ \hat{V}_k(S_k) = p_k S_k + \mathbb{E}\left[ \max_{Y_k \in \{(S_k-D_k)^+, \ldots, S_k\}} \{ -p_k Y_k + \hat{V}_{k+1}(S_{k+1}) \} \right] \]

where \( \hat{V}_k(S_k) = \mathbb{E}[\hat{V}_k(S_k | D_k)] \)

Consider \( Y = \max_{x \in [A,B]} \{ f(x) \} \)

\[ f(x) \]

\[ a_1 \quad b_1 \quad a_2 \quad b_2 \]

- For general \( f \), this is some complicated random variable

Suppose \( f \) is **concave**. Let \( x^* = \arg\max f(x) \)

Now \( Y \) is easy to define

\[ Y(A,B) = \begin{cases} 
B & ; B < x^* \\
x^* & ; A \leq x^* \leq B \\
A & ; A > x^* 
\end{cases} \]
Now consider $h_k(y) = -P_k y + \hat{V}_{k+1}(y)$

Then $\hat{V}_k(S_k) = P_k S_k + \mathbb{E} \left[ \max_{y \in \{(S_k-D_k)^+,...,S_k\}} h_k(y) \right]$.

Suppose $h_k(\cdot)$ is concave, $x_{k-1}^* = \arg\max_{y \in [0,c]} h_k(y)$,

Then $y_k^* = \begin{cases} S_k & S_k < x_{k-1}^* \\ x_{k-1}^* & (S_k-D_k)^+ \leq x_{k-1}^* \leq S_k \\ (S_k-D_k)^+ & (S_k-D_k)^+ > x_{k-1}^* \end{cases}$

This is a **protection level policy** (with protection level $x_{k-1}^*$)

i.e., at each stage $k$, for all states $S_k$ and demand $D_k$, we do the following

1) If $S_k < x_{k-1}^*$ (i.e., already below protection level),
   then do not admit anyone

2) If $(S_k-D_k)^+ \leq x_{k-1}^* \leq S_k$ (i.e., excess demand), then admit up to protection level $x_{k-1}^*$

3) If $(S_k-D_k)^+ > x_{k-1}^*$ (i.e., excess supply), then admit all customers $D_k$ up to capacity limits
Is $h_k(x)$ concave?

- $\hat{V}_{s_1}(s_1|D_1) = \max_{(s_1-D_1)^+ \leq y \leq s_1} \left[ P_1 s_1 - P_1(s_1-D_1)^+ \right]$
  
  * $\hat{V}_{s_1}(s_1|D_1)$ concave in $S_n$
  
  * $\hat{V}_{s_1}(s_1) = E[\hat{V}_{s_1}(s_1|D_1)]$
  
  is concave in $S_1$
  
  (linear combination of concave funs)

Moreover $h_2(y) = -P_2 y + \hat{V}_1(y) \Rightarrow$ concave!

- Now we prove $h_k(y)$ are concave by induction

  - Assume $\hat{V}_{k-1}(y)$ is concave in $y$

  - $h_k(y) = -P_k y + \hat{V}_{k-1}(y) \Rightarrow$ concave

  - $\hat{V}_k(s|D_k) = \max_{(s-D_k)^+ \leq y \leq s} \left[ h_k(y) \right] + P_k s_k$

  + Is this convex?
Let \( x_{k-1}^* = \arg \max_{y \in [0, \infty)} [h_k(y)] \).

(Note: \( x_1^* = F_{\theta_1}^{-1} (1 - \frac{P_2}{P_1}) \) - Littlewood's Rule!)

(check this by writing out \( h_2(y) \), and comparing to first class)

We know \( h_k(y) \) is convex, so we can maximize it over \((s - D_k)^+ \leq y \leq s\).

For a fixed value of \( D_k \):

\[ y_k^* = \arg \max_{y \in [(s-D_k)^+, s]} [h_k(y)] \]

\[ = \begin{cases} 
  s & ; s \leq x_{k-1}^* \\
  x_{k-1}^* & ; s - D_k < x_{k-1}^* \leq s \\
  s - D_k & ; s - D_k > x_{k-1}^* 
\end{cases} \]

Thus, \( \max_{y \in [(s-D_k)^+, s]} [h_k(y)] \) is concave in \( s \).

\( \Rightarrow \hat{V}_k(s|D_k) \) is concave \( \Rightarrow \hat{V}_k(s) \) is concave.
The above argument also works when \( D_j \) are discrete.

**Main Idea** - Consider the linear interpolation of the discrete functions \( V_k(s) \).
- This is concave for \( V_1(s)(D) \).
- The rest of the argument is identical.

In particular, the argument gives:

\[ \Delta \hat{V}_k(s) = \hat{V}_k(s+1) - \hat{V}_k(s) \leq \Delta \hat{V}_k(s-1) \]

- **Diminishing returns to revenue from increasing capacity**

Finally, observe that the optimal policy given \( D_k \) is to accept as many customers as we can till:

\[ \# \text{ of remaining seats} = X^*_k \]

This can be implemented without knowing \( D_k \)!!