Single-Parameter Environments

We now generalize the single-item auction to a more general setting of single-parameter environments.

- \( n \) bidders, each with private valuations
- For bidder \( i \), private value \( v_i \) is value 'per-unit of stuff' that it receives
- Mechanism decides an allocation \( (x_1, x_2, \ldots, x_n) \), where \( x_i = \) 'amount of stuff' given to bidder \( i \)
- \( X \) is the set of feasible allocations \( (X \subseteq \mathbb{R}^n) \)

Sealed bid mechanism

- Collected bids \( b = (b_1, b_2, \ldots, b_n) \)

(Allocation Rule) Choose allocation \( (x_1, \ldots, x_n) \in X \) as fn of bids

(Payment Rule) Choose payments \( p(b) \) as fn of bids
Given allocation and payment rules \((x, p)\), the utility of bidder \(i\) is (under bids \(b\))
\[
u_i(b) = v_i \cdot x_i(b) - p_i(b)
\]
We focus on payments satisfying \(p_i(b) \in [0, b_i \cdot x_i(b)]\).

Using this notation, we can define IC, IR:

**IC** - For every bidder \(i\) and vector \(b_{-i}\), the allocation and payment rules \((x(b), p(b))\) obey
\[
u_i(b_i, b_{-i}) \leq u_i(v_i, b_{-i}) \quad \forall b_i
\]

**IR** - For every bidder \(i\), we have
\[
u_i(v_i, b_{-i}) \geq 0
\]

Possible objectives of interest:
1) Revenue: \(R = \sum_i p_i(b)\)
2) Welfare: \(W = \sum_i x_i(b) v_i\)
Examples of single-parameter environments

i) Single-item auction: \( x_i \in \{0,1,3 \}, X = \text{set of vectors in } \{0,1,3\}^n \text{ such that } \sum_{i=1}^{n} x_i \leq 1 \)

ii) \( k \) identical items: Assuming each person wants at most one item, we have \( X \in \{0,1,3\}^n \) s.t. \( \sum_{i=1}^{n} x_i \leq k \)

iii) Knapsack auction: Suppose we want to sell ad-time on a TV show. Each bidder \( i \) has an ad with private value \( v_i \), public run-time \( w_i \). If we have ad-time of size at most \( W \), then \( x_i \in \{0,1,3\} \) and \( X = \{x \in \{0,1,3\}^n \mid \sum_{i=1}^{n} x_i w_i \leq W \} \)

iv) Sponsored search: \( k \) slots \( 1,2,..,k \), where slot \( j \) has a click-through-rate (CTR) of \( \alpha_j \) (i.e., \( P[\text{Visitor clicks on slot } j \text{ ad}] \propto \alpha_j \)) assume \( \alpha_1 \geq \alpha_2 \geq \alpha_k \)

- Each bidder has private value \( v_i \), public quality \( \beta_i \); \( P[\text{Visitor clicks on ad for bidder } i \text{ in slot } j] = \beta_i \alpha_j \).

Visitor clicks then bidder gets value \( v_i \).

\( X \) = bipartite matching between bidders, slots
Myerson's Lemma

In order to design DSIC mechanisms, we want to first characterize what such mechanisms look like.

- **Implementable Allocation Rule** - An allocation rule $x(b)$ is implementable if there is a payment rule $p(b)$ such that $(x, p)$ is DSIC.

  **Eg.** For single-item auctions, $x_i = \{ i \mid \exists j \geq i: b_j \geq b_i \}$ (i.e., allocate to maximum bidder) is implementable (via the second price auction).

- **Monotone Allocation Rule** - An allocation rule is monotone if for every bidder $i$, bids $b_i$, the allocation $x_i(z, b_{-i})$ is non-decreasing in $z$.

  **Eg.** Allocate to highest bidder is monotone. Allocate to second highest bidder is not. (check)
Theorem (Myerson '81) For a single-parameter setting

a) An allocation rule \( x(\cdot) \) is implementable iff it satisfies

b) If \( x(\cdot) \) is monotone, then there is a unique payment rule \( P(\cdot) \) such that \((x, P)\) is DSIC (and \( P \) has an explicit formula)

Note - the second statement needs a condition that \( p_i(b) = 0 \) if \( b_i = 0 \)

Proof - Suppose \((x, P)\) is DSIC. Then for every bidder \( i \), every value \( v_i \), and every bid vector \( b-i \), we have for every \( z \)

\[
\forall i: x_i(v_i, b-i) - P_i(v_i, b-i) \geq \max_{z} \left( x_i(z, b-i) - P_i(z, b-i) \right)
\]

(\*)

For shorthand, let \( x_i(z) = x_i(z, b-i) \), \( P_i(z) = P_i(z, b-i) \)

- (Swapping-trick) \( x_i(z) \) for any \( z_1, z_2 \), the equation \( \forall i: z_1 \leq z_2 \) holds for \( v_i = z_1, z = z_2 \) and \( v_i = z_2, z = z_1 \)
For \( z_1 = z_1 \), we have

\[
Z_1 x_i(z_1) - P_i(z_1) \geq Z_1 x_i(z_2) - P_i(z_2)
\]

For \( z_2 = z_2 \), we have

\[
Z_2 x_i(z_2) - P_i(z_2) \geq Z_2 x_i(z_1) - P_i(z_1)
\]

Re-arranging, we get (assume \( 0 \leq z_2 < z_1 \))

\[
Z_2 (x_i(z_1) - x_i(z_2)) \leq P_i(z_1) - P_i(z_2) \leq Z_1 (x_i(z_1) - x_i(z_2))
\]

**Proven by monotonicity, \( x_i(t) = x_i(z_1) \)**

Since \( z_1 > z_2 \), we must have

\[
2c_i(z_1) - x_i(z_2) \geq 0 \implies x_i(z) \text{ is monotone (non-decreasing)}
\]

Now we want to find a pricing rule. For this, we assume henceforth that \( x \) is monotone, and piecewise constant; we then generalize to when \( x \) is continuous.
First consider the equation
\[ Z_2 \left[ x_i(z_1) - x_i(z_2) \right] \leq P_i(z_1) - P_i(z_2) \leq Z_1 \left[ x_i(z_1) - x_i(z_2) \right] \]

If we take, \( z_1 \geq z_2 \), then both the left and right sides become 0 if \( x_i(z_2) = x_i(z_2^+) \) (i.e., no jump at \( z_2 \)). If however there is a jump at \( z_2 \), of size \( h \), then both sides tend to \( z_2 - h \). Therefore
\[ (\text{jump in } P_i \text{ at } z) = z \cdot (\text{jump in } x_i \text{ at } z) \]

Finally, if \( P_i(0) = 0 \), then we have
\[ P_i(z, b_{-i}) = \sum_{j=1}^{l} z_j \cdot (\text{jump in } x_i(\cdot, b_{-i}) \text{ at } z_j) \]

where \( Z_1, Z_2, \ldots, Z_l \) are breakpoints of \( x_i(\cdot, b_{-i}) \) in \([0, Z]\)

If instead \( x_i(\cdot) \) is continuous, we take \( Z_1 \cup Z_2 \) together
\[ \frac{dP_i(z)}{dz} = z \cdot \frac{dx_i(z)}{dz} \]

Therefore
\[ P_i(b^i, b_{-i}) = \int_{0}^{Z} z \cdot x_i(z) \cdot dz \]
By integrating by parts, we can get an easier form:
\[ P_i(b_i, b_j) = \int_0^{\infty} \int_{b_i}^{b_j} \left[ \frac{d x_i(z)}{dz} \right] dz \]

\[ = \int_0^{\infty} x_i(z) \bigg|_{b_i}^{b_j} - b_i \int_0^{\infty} x_i(z) dz \]

Pictorially, this can be depicted as follows (Assume \( b_i \) is fixed)

\[
\begin{align*}
\text{Note: Monotone } x_i(z) & \\
\square : b_i x_i(b_i) & \\
\text{Area} : b_i \int_0^{b_i} x_i(z) dz & \\
\text{ : } P(b_i) & \\
\end{align*}
\]

Thus the unique payment (assuming \( P(0) = 0 \)) is always the area of the rectangle \( b_i x_i(b_i) \) which is above the curve \( x_i(b, z) \). Now, using this fact, we can easily prove that this price is DSC, IR.
We can show DSIC via pictures. Recall \( p_i(b_i) = b_i x_i(b_i) - \int x_i(z) \, dz \).

Case 1 - \( b_i = v_i \)

Case 2 - \( b_i < v_i \)

Case 3 - \( b_i > v_i \)
The above proof also works when $x_i(v)$ is discrete (i.e., has discontinuous jumps).

Figure 2: Proof by picture that the payment rule in (6), coupled with the given monotone and piecewise constant allocation rule, yields a DSIC mechanism. The three columns consider the cases of truthful bidding, overbidding, and underbidding, respectively. The three rows show the surplus $v \cdot x(b)$, the payment $p(b)$, and the utility $v \cdot x(b) - p(b)$, respectively. In (h), the solid region represents positive utility and the lined region represents negative utility.

(Courtesy: Tim Roughgarden, Jason Hartline)