ORIE 4154 - Pricing and Market Design

Module 1: Capacity-based Revenue Management
(Two-stage capacity allocation, and Littlewood’s rule)

Instructor: Sid Banerjee, ORIE
Here’s the Story…

- Until 1978, US airline industry was heavily regulated
  - US Airline Deregulation Act can in 1978
  - Price controls lifted
  - Free entry and exit from markets
- This led to the rise of new low-cost carriers
  - They provide bare-bone service, passengers paying for meals and all luggage handling, non-union employees
  - Their service structure allow them to offer low fares
  - One such carrier was People Express, started in 1981

Courtesy: Huseyin Topaloglu
What Happened to Major Airlines

- Major airlines were heavily affected, especially by the loss of discretionary leisure travelers
- Dilemma faced by American Airlines
  - If it matched People Express’ fares, it can retain customers but not cover cost
  - If it does not, then it would lose customers

Courtesy: Huseyin Topaloglu
American Airlines’ Solution

• Bob Crandall, VP of Marketing at AA then, recognized the following key facts
  – Many AA flights departed with empty seats
  – Marginal cost of using these seats was very small
  – AA could use these “surplus seats” to compete on cost

Courtesy: Huseyin Topaloglu
But how?

- Create new restricted, discounted fares called “Super Saver” and “Ultimate Super Saver” fares
  - Must book at least 2 weeks prior to departure and stay at destination over a Saturday night
  - Passengers not meeting this restriction are charged a higher fare
  - Restrict number of discount seats sold on each flight to save seats for full-fare passengers book late
  - DINAMO – Dynamic Inventory Allocation and Maintenance Optimizer
- People Express allowed every seat to be sold at a low fare!

Courtesy: Huseyin Topaloglu
Results of the New Strategy

- AMR shares initially plunged on announcement of “Ultimate Super Saver” fares Jan. 1985
  - Analysts thought it was the start of a price war
  - “American cannot operate profitably at these fares”
- DINAMO proved to be surprisingly effective
  - AA total revenues rose
  - Competitors suffered: e.g. People Express
    - 1984 $60M profit (all-time high)
    - 1985 $160M loss
    - 1986 Bankruptcy, sold to Continental

Courtesy: Huseyin Topaloglu
Problem: Single-resource two-stage capacity allocation

Want to maximize revenue from selling multiple copies of a single resource (e.g., $C$ seats on a single flight)
Problem: Single-resource two-stage capacity allocation

• Buyer behavior:
  - (Dynamics) Buyers arrive sequentially to the market
  - (Choice) Each buyer wants either a discount-fare (i.e., low price) ticket or a full-fare (i.e., high-price) ticket
Problem: Single-resource two-stage capacity allocation

- **Buyer behavior:**
  - (Dynamics) Buyers arrive sequentially to the market
  - (Choice) Each buyer wants either a *discount-fare* (i.e., low price) ticket or a *full-fare* (i.e., high-price) ticket

- **Seller constraints:**
  - (Capacity) Has \( C \) identical units (seats) to sell
  - (Prices) Prices fixed to \( p_h \) (full-fare) and \( p_l \) (discount fare), with \( p_h > p_l \)
  - (Control) Can choose how many discount-fare and full-fare tickets to sell
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• Information structure:
  - (Dynamics) All discount-fare customers arrive before full fare customers
  - (Demand Distributions) Demand for full-fare tickets is $D_h \sim F_h$, discount fare tickets is $D_l \sim F_j$
Aside: Customer Segmentation (Price Discrimination)

In our first lecture, we considered a single customer with (unknown) value $V$ for an item, and we charge a single price $p$

- **Note:** The best achievable revenue is $V$
- **Customer segmentation:** use pricing to get revenue closer to $V$
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There are three high-level ways of achieving this:

- **First-degree (Complete discrimination)**: ‘learn’ each buyers’ value, and charge $p = V$ (e.g., negotiations/haggling)
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- **Third-degree (Direct segmentation)**: use some ‘feature’ to segment buyers into classes, and charge different price to each class (e.g., student discounts)
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**Second-degree (Indirect segmentation)**: Rely on some proxy to offer a ‘choice’ of products (e.g., bulk discounts, coupons)
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Profits and information requirements increase going up the list
Single-resource two-stage capacity allocation

Timeline of optimization problem

\[ R(b, D^\ell, D^h) = p^\ell \min\{b, D^\ell\} + p^h \min\{D^h, \max\{C - b, C - D^\ell\}\} \]

Aim: Choose \( b^* = \arg\max_{b \in [0, C]} E[R(b, D^\ell, D^h)] \)
Single-resource two-stage capacity allocation

Timeline of optimization problem

\( C \) units of capacity available
Single-resource two-stage capacity allocation

Timeline of optimization problem

C units of capacity available

$D_1$ customers arrive desiring discount-fares

Inputs: prices $p_\ell, p_h$, demand distributions $F_\ell, F_h$

Control variable: Booking limit $b$ for discount-fare seats

Revenue (as a function of $b, D_\ell$ and $D_h$):

$$R(b, D_\ell, D_h) = p_\ell \min\{b, D_\ell\} + p_h \min\{D_h, \max\{C - b, C - D_\ell\}\}$$

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Single-resource two-stage capacity allocation

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Single-resource two-stage capacity allocation

Timeline of optimization problem

- \( C \) units of capacity available
- Accept up to \( b \) customers at discount-fare \( p_l \)
- \( D_l \) customers arrive desiring discount-fares
- \( D_h \) customers arrive desiring full-fares

Revenue (as a function of \( b \), \( D_l \) and \( D_h \)):

\[
R(b, D_l, D_h) = p_l \min\{b, D_l\} + p_h \min\{D_h, \max\{C - b, C - D_l\}\}
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Aim: Choose \( b^* = \arg\max_{b \in [0, C]} E[R(b, D_l, D_h)] \)
Single-resource two-stage capacity allocation

Timeline of optimization problem

- **C** units of capacity available
- **D_l** customers arrive desiring discount-fares
- Accept up to **b** customers at discount-fare **p_l**
- **D_h** customers arrive desiring full-fares
- Fill remaining seats at the full-fare **p_h**

Revenue (as a function of **b**, **D_l**, and **D_h**):

\[
R(b, D_l, D_h) = p_l \min\{b, D_l\} + p_h \min\{D_h, \max\{C - b, C - D_l\}\}
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Aim: Choose **b^* = \arg\max_{b \in [0, C]} E[R(b, D_l, D_h)]**
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Inputs: prices $p_\ell, p_h$, demand distributions $F_\ell, F_h$

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Revenue (as a function of $b, D_\ell$ and $D_h$):

$$R(b, D_\ell, D_h) = ?$$
Single-resource two-stage capacity allocation

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**Inputs:** prices $p_\ell, p_h$, demand distributions $F_\ell, F_h$

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Single-resource two-stage capacity allocation
Heuristic derivation of Littlewood’s rule

The problem

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Aim: Choose \( b^* = \arg\max_{b \in [0, C]} \mathbb{E}[R(b, D_\ell, D_h)] \)
  - Equivalently, can choose opt protection level \( y^* = C - b^* \)

Marginal-revenue heuristic: Suppose we have \( y \) units left, and a discount-fare customer arrives
Single-resource two-stage capacity allocation

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Single-resource two-stage capacity allocation

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Marginal-revenue heuristic: Suppose we have \( y \) units left, and a discount-fare customer arrives

- If we accept, then \( \Delta R = p_{\ell} \)
- If we reject, then \( \Delta R = p_h \mathbb{P}[D_h \geq y] \)

Thus, optimal protection level \( y^* = \max_{y \in \mathbb{N}} \left\{ \mathbb{P}[D_h \geq y] > \frac{p_{\ell}}{p_h} \right\} \)
Single-resource two-stage capacity allocation

Formal derivation 1: Continuous RV

\[ R^* = \max_{b \in [0,C]} \mathbb{E} \left[ p_\ell \min\{b, D_\ell\} + p_h \min\{D_h, \max\{C - b, C - D_\ell\}\} \right] \]

- Assume \( D_h, D_\ell \) are continuous, \( b \) can be fractional
Single-resource two-stage capacity allocation

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\[ \mathbb{E}[R(b, D_\ell, D_h)] = p_\ell \mathbb{E}[\min\{b, D_\ell\}] + p_h \mathbb{E}[\min\{D_h, \max\{C - b, C - D_\ell\}\}] \]

(By linearity of expectation)
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\mathbb{E}[R(b, D_\ell, D_h)] = p_\ell \mathbb{E} [\min\{b, D_\ell\}] + p_h \mathbb{E} [\min\{D_h, \max\{C - b, C - D_\ell\}\}] \\
\quad (By \ linearity\ of\ expectation)
\]

\[
= p_\ell \cdot \left( \int_{-\infty}^{b} x \cdot f_\ell(x)dx + \int_{b}^{\infty} b \cdot f_\ell(x)dx \right) + \ldots
\]
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(By linearity of expectation)

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\[ p_h \cdot \left( \int_{-\infty}^{b} \mathbb{E}[\min\{C - x, D_h\}] \cdot f_\ell(x)dx + \ldots \right) \]

\[ \int_{b}^{\infty} \mathbb{E}[\min\{C - b, D_h\}] \cdot f_\ell(x)dx \]
Single-resource two-stage capacity allocation

Formal derivation 1: Continuous RV

Thus we want to choose $b$ to maximize:

$$r(b) = \mathbb{E}[R(b, D_{\ell}, D_h)] = p_{\ell} \cdot (L_1(b) + L_2(b)) + p_h \cdot (H_1(b) + H_2(b))$$

Where

$$L_1(b) = \int_{-\infty}^{b} x \cdot f_{\ell}(x) \, dx$$

$$L_2(b) = \int_{b}^{\infty} b \cdot f_{\ell}(x) \, dx$$

$$H_1(b) = \int_{-\infty}^{b} \mathbb{E}[\min\{C - x, D_h\}] \cdot f_{\ell}(x) \, dx$$

$$H_2(b) = \int_{b}^{\infty} \mathbb{E}[\min\{C - b, D_h\}] \cdot f_{\ell}(x) \, dx$$

We now need to check the first-order condition $\frac{dr(b)}{db} = 0$
Aside: Leibniz rule of integration

Let \( f(x,t) \) be such the partial derivative w.r.t. \( t \) exists and is continuous. Then:

\[
\frac{d}{dx} \left[ \int_{A(x)}^{B(x)} f(x,t) \, dt \right] = \int_{A(x)}^{B(x)} \frac{\partial f(x,t)}{\partial x} \, dt + \ldots \\

f(x,B(x)) \frac{dB(x)}{dx} - f(x,A(x)) \frac{dA(x)}{dx}
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Aside: Leibniz rule of integration

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\[
f(x, B(x)) \frac{dB(x)}{dx} - f(x, A(x)) \frac{dA(x)}{dx}
\]

As an example, consider \( L_2(b) = \int_b^\infty b \cdot f_\ell(x) dx \):

\[
\frac{dL_2(b)}{db} = \int_b^\infty \frac{\partial b f_\ell(x)}{\partial b} dx - b f_\ell(x) \bigg|_{x=b} \cdot \frac{db}{db} = \int_b^\infty f_\ell(x) dx - b f_\ell(b)
\]
Single-resource two-stage capacity allocation

Formal derivation 1: Continuous RV

\[
\frac{dr(b)}{db} = p_\ell \cdot \left( \frac{dL_1(b)}{db} + \frac{dL_2(b)}{db} \right) + p_h \cdot \left( \frac{dH_1(b)}{db} + \frac{dH_2(b)}{db} \right)
\]

where we have

\[
L_1(b) = \int_{-\infty}^{b} x \cdot f_\ell(x) dx, \quad L_2(b) = \int_{b}^{\infty} b \cdot f_\ell(x) dx
\]

\[
H_1(b) = \int_{-\infty}^{b} \mathbb{E}[\min\{C - x, D_h\}] \cdot f_\ell(x) dx
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H_2(b) = \int_{b}^{\infty} \mathbb{E}[\min\{C - b, D_h\}] \cdot f_\ell(x) dx
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\]

\[
H_2(b) = \int_{b}^{\infty} \mathbb{E}[\min\{C - b, D_h\}] \cdot f_{\ell}(x) dx
\]

- Differentiating we have (check these for yourself):

\[
\frac{dL_1(b)}{db} = bf_{\ell}(b), \quad \frac{dL_2(b)}{db} = \mathbb{P}[D_{\ell} \geq b] - bf_{\ell}(b)
\]
Single-resource two-stage capacity allocation

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where we have

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\frac{dL_1(b)}{db} = b f_\ell(b), \quad \frac{dL_2(b)}{db} = \mathbb{P}[D_\ell \geq b] - b f_\ell(b)
\]
\[
\frac{dH_1(b)}{db} = \mathbb{E}[\min\{C - b, D_h\}] f_\ell(b)
\]
\[
\frac{dH_2(b)}{db} = -\mathbb{E}[\min\{C - b, D_h\}] f_\ell(b) + \frac{d\mathbb{E}[\min\{C - b, D_h\}]}{db} \int_{b}^{\infty} f_\ell(x) dx
\]
Single-resource two-stage capacity allocation

Formal derivation 1: Continuous RV

- Combining all terms, we have

\[
\frac{dr(b)}{db} = p_\ell \mathbb{P}[D_\ell \geq b] + p_h \left( \frac{d\mathbb{E}[\min\{C - b, D_h\}]}{db} \right) \mathbb{P}[D_\ell \geq b]
\]

Setting \( \frac{dr(b)}{db} = 0 \), we get

\[
\frac{d\mathbb{E}[\min\{C - b, D_h\}]}{db} + \frac{p_\ell}{p_h} = 0.
\]
Single-resource two-stage capacity allocation
Formal derivation 1: Continuous RV

- Combining all terms, we have

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\frac{dr(b)}{db} = p_\ell \mathbb{P}[D_\ell \geq b] + p_h \left( \frac{d\mathbb{E}[\min\{C-b,D_h\}]}{db} \right) \mathbb{P}[D_\ell \geq b]
\]

Setting \( \frac{dr(b)}{db} = 0 \), we get \( \frac{d\mathbb{E}[\min\{C-b,D_h\}]}{db} + \frac{p_\ell}{p_h} = 0 \).

- Finally, we can again use the Leibniz rule to simplify the LHS

\[
\frac{d\mathbb{E}[\min\{C-b,D_h\}]}{db} = \frac{d}{db} \left( \int_{-\infty}^{C-b} xf_h(x)dx + \int_{C-b}^{\infty} (C-b)f_h(x)dx \right)
\]
\[
= -(C-b)f_h(C-b) + (C-b)f_h(C-b) - \mathbb{P}[D_h \geq C-b]
\]

Thus, the optimal \( b^* \) satisfies:

\[ \mathbb{P}[D_h \geq C-b^*] = p_\ell p_h, \quad \text{and hence:} \]

\[ C-b^* = y^* = F_{-1}(1-p_\ell p_h) \]
Single-resource two-stage capacity allocation

Formal derivation 1: Continuous RV

• Combining all terms, we have

\[
\frac{dr(b)}{db} = p_\ell P[D_\ell \geq b] + p_h \left( \frac{dE[\min\{C - b, D_h\}]}{db} \right) P[D_\ell \geq b]
\]

Setting \( \frac{dr(b)}{db} = 0 \), we get \( \frac{dE[\min\{C - b, D_h\}]}{db} + \frac{p_\ell}{p_h} = 0 \).

• Finally, we can again use the Leibniz rule to simplify the LHS

\[
\frac{dE[\min\{C - b, D_h\}]}{db} = \frac{d}{db} \left( \int_{-\infty}^{C-b} x f_h(x) dx + \int_{C-b}^{\infty} (C - b) f_h(x) dx \right)
\]

\[
= -(C - b) f_h(C - b) + (C - b) f_h(C - b) - P[D_h \geq C - b]
\]

Thus, the optimal \( b^* \) satisfies: \( P[D_h \geq c - b^*] = \frac{p_\ell}{p_h} \), and hence:

\[
C - b^* = y^* = F_h^{-1} \left( 1 - \frac{p_\ell}{p_h} \right)
\]