Concavity and random LPs

- In the process of bounding the revenue in the single-resource allocation problem, we obtained the following LP

\[ V_n^{ub}(c | \{D_1, D_2, \ldots, D_n\}) = \max \sum_{i=1}^{n} p_i x_i \]

\[ \text{s.t. } \sum_{i=1}^{n} x_i \leq c \]

\[ x_i \leq D_i \quad \forall i \]

\[ x_i \geq 0 \quad \forall i \]

This is sometimes called the randomized-LP bound.

- We also defined the fluid-LP bound

\[ V_n^{fl}(c) = \max \sum_{i=1}^{\hat{n}} p_i x_i \]

\[ \text{s.t. } x_i \leq \mu_i \quad (\mu_i = \mathbb{E}[D_i]) \]

\[ \sum_{i=1}^{\hat{n}} x_i \leq c \]

\[ x_i \geq 0 \quad \forall i \]
Finally we showed \[ V_n^{pl}(c) \geq E[V_n^{ub}(c \{D_i\})] \]

We will now see how to derive such results directly using convexity and Jensen's inequality.

Main tools from convexity

- **Defn** - A function \( f(x) \) is convex if \( \forall x, y \) and \( \forall t \in [0, 1] \), the function obeys
  \[ f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \]

- **Notes** - i) Function \( f(x) \) is concave if \( -f(x) \) is convex (alternately, replace \( \leq \) with \( \geq \) in \( \circledast \) to get concavity)
  
  ii) The definition works even if \( x \) is a vector

  iii) Graphical way to remember:

\[ f(tx + (1-t)y) \]

- Suppose \( t = \frac{1}{2} \)
- \( tx + (1-t)y \) is the pt midway bel" \( x, y \)
Operations which preserve convexity (see HW2)

1) Scaling - \( f(x) \) convex, \( a \geq 0 \), then
\[
f(ax + b) \text{ is convex}
\]

2) Linear Combination - \( f_1(x), f_2(x) \) convex, \( a_1, a_2 > 0 \)
\[
a_1f_1(x) + a_2f_2(x) \text{ is convex}
\]

3) Maximization - \( f_1(x), f_2(x) \) convex, then
\[
\max (f_1(x), f_2(x)) \text{ is convex}
\]
(there are many others, but these suffice for us)

For concavity, it is preserved under scaling, linear combination and minimization.

A linear function \( f(x) = ax + b \) is both convex and concave.

Jensen's Inequality - If \( f(x) \) is convex, then
\[
E[f(x)] \geq f(E[x])
\]
(\( E \) denotes expectation)
Now we show how to use those for LPs.

- **Primal Argument**

- We want to show that the expected value of the randomized LP is bounded by the fluid LP.

\[
\mathbb{E} \left[ \max \sum_{i=1}^{\hat{c}} p_i x_i \right] \leq \max \sum_{i=1}^c \hat{p}_i x_i
\]

\[
\text{st. } \sum_{i=1}^{\hat{c}} x_i \leq \hat{c} \\
\hat{x}_i \leq D_i \\
x_i \geq 0
\]

\[
\text{st. } \sum_{i=1} c \hat{x}_i \leq c \\
x_i \leq \mathbb{E}[D_i] + \delta_i \\
x_i \geq 0
\]

- To see that this can be shown via Jensen’s, we need to first understand that the optimization problem above is a function \$G\$ which maps ‘capacities’ \$(D_1, D_2, \ldots, D_n)\$ to an output. Visualize this as follows:

\[
(d_1, d_2, \ldots, d_n) \rightarrow \max \sum_{i=1}^{\hat{c}} p_i x_i
\]

\[
\text{st. } \sum_{i=1}^{\hat{c}} x_i \leq \hat{c} \\
\hat{x}_i \leq D_i \\
x_i \geq 0
\]

\[
\rightarrow \ y = G(d_1, d_2, \ldots, d_n)
\]
Thus, if we show \( g(d_1, d_2, \ldots, d_n) \) is concave in \((d_1, d_2, \ldots, d_n)\), we can use Jensen's Inequality to get **

- Suppose we have 2 inputs 
  \[
  d = (d_1, d_2, \ldots, d_n) \\
  s = (s_1, s_2, \ldots, s_n)
  \]

  let 
  \[
  x_d^* = \arg \max \sum_{i=1}^{n} p_i x_i : s.t. \sum_{i=1}^{n} x_i \leq c \\
  0 \leq x_i \leq d_i + \epsilon
  \]
  \[
  x_s^* = \arg \max \sum_{i=1}^{n} p_i x_i : s.t. \sum_{i=1}^{n} x_i \leq c \\
  0 \leq x_i \leq s_i
  \]

  (and \( g(d) \), \( g(s) \) the corresponding objective value)

- For some \( t \in [0,1] \), consider \( \Delta = td + (1-t)s \)
  (i.e., for each \( i \), \( \Delta_i = td_i + (1-t)s_i \))

  To show concavity, we need to show
  \[
  g(\Delta) \geq t g(d) + (1-t)g(s)
  \]
To see this, we first observe that the point \( x_{\Delta} = t x_d^* + (1-t) x_g^* \) is feasible for demand vector \( \Delta \), i.e.

\[
\sum_{i=1}^{n} x_{\Delta,i} = t \sum_{i=1}^{n} x_{d,i}^* + (1-t) \sum_{i=1}^{n} x_{g,i}^* \leq tc + (1-t)c = c
\]

\[
x_{\Delta,i} = t x_{d,i}^* + (1-t) x_{g,i}^* \leq td_i + (1-t)s_i = \Delta_i
\]

\[
\Rightarrow \sum_{i=1}^{n} p_i x_{\Delta,i} \leq \left[ \max_{x} \sum_{i=1}^{n} p_i x_i \right]
\]

\[
\begin{align*}
&\text{s.t. } \sum_{i=1}^{n} x_i \leq c, \\
&\quad 0 \leq x_i \leq \Delta_i \quad \forall i
\end{align*}
\]

\[= g(\Delta)\]

However, \[\sum_{i=1}^{n} p_i x_{\Delta,i} = t \sum_{i=1}^{n} p_i x_{d,i}^* + (1-t) \sum_{i=1}^{n} p_i x_{g,i}^* = t g(d) + (1-t) g(s)\]

Thus \( g(\Delta) \geq t g(d) + (1-t) g(s) \Rightarrow g \) is concave.
Dual argument

We will now see an alternate (easier?) way to see the same result via duality.

- We need one fact for this: Earlier, we said that if $f_1, f_2$ are convex, then $\max(f_1(x), f_2(x))$ is convex. Similarly, if $g_1, g_2$ are concave fns, then $\min(g_1(x), g_2(x))$ is concave. (Try to check this!)

- Now, by LP duality, we have

$$
G(d_1, \ldots, d_n) = \begin{bmatrix}
\max & \sum_{i=1}^{\hat{n}} p_i x_i \\
\text{s.t.} & \sum_{i=1}^{\hat{n}} x_i \leq C \\
& 0 \leq x_i \leq d_i \forall i \end{bmatrix}
$$

\[ \text{PRIMAL} \]

\[ = \begin{bmatrix}
\min & Cz + \sum_{i=1}^{\hat{n}} d_i \beta_i \\
\text{s.t.} & \beta_i + z \geq p_i \forall i \\
& \beta_i \geq 0, \ z \geq 0
\end{bmatrix}
\]
Now let \( \mathcal{X} \) be the set of extreme points of the dual constraint set, i.e.

\[
\mathcal{X} = \{(p, z) \mid (p, z) \text{ is an extreme pt of } \mathcal{P}_i \geq 0, z \geq 0\}
\]

Then

\[
g(d_1, \ldots, d_n) = \min_{(p, z) \in \mathcal{X}} \left\{ c Z + \frac{\sum_{i=1}^{n} d_i \beta_i}{\lambda} \right\}
\]

is linear (hence concave) in \((d_1, d_2, \ldots, d_n)\)

\[
\Rightarrow g(d_1, \ldots, d_n) \text{ is concave in } d_1, \ldots, d_n
\]

Thus,

\[
E \left[ g(D_1, D_2, \ldots, D_n) \right] \leq g(E[D_1], \ldots, E[D_n])
\]

\[
\begin{bmatrix}
\max \sum_{i=1}^{n} x_i \beta_i \\
\text{s.t. } \sum_{i=1}^{n} x_i \leq c \\
0 \leq x_i \leq D_i
\end{bmatrix}
\]

\[
\begin{bmatrix}
\max \sum_{i=1}^{n} x_i \beta_i \\
\text{s.t. } \sum_{i=1}^{n} x_i \leq c \\
0 \leq x_i \leq E[D_i]
\end{bmatrix}
\]