Low-Complexity Feedback Allocation Algorithms

For Cellular Uplink

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Abstract

It has been well-established both in theory and in practice that the presence of channel state information at the transmitter increases the realizable data rate of a wireless link. In order to enable the transmitter to adapt its transmit strategy to suit the state of the channel, we require the presence of feedback from the receiver. The additional degrees of freedom and hence rate gains made possible with the advent of multiple-antenna technology can be well-exploited by the presence of a feedback channel that enables precoding at the transmitter. That said, feedback bandwidth is a limited resource that needs to be allocated judiciously. This paper, we consider a cellular uplink scenarios where the base-station has limited feedback resources, which it needs to allocate across the users it serves. We propose a general model that captures the effect of feedback allocation on the achievable rates for a user, which allows us to characterize the throughput region for such a system. For unsaturated queueing systems, we show that the optimal feedback allocation policy that stabilizes the queues when possible, involves solving a weighted sum-rate maximization at each scheduling instant. We show that such an online weighted sum-rate maximization policy can also be used for long-term utility maximization, which is applicable to saturated queueing systems.

Having identified the appropriate allocation problem to solve at each scheduling instant, we arrive at the main focus of this paper, which is the issue of computational efficiency. Needless to say, it is critical from the point of view of a network provider to solve the allocation problem as quickly as possible thus motivating the following core algorithmic contributions of this paper. As we will show, a brute-force solution is simply infeasible since the number of possible feedback allocations under our proposed framework is exponentially large. Nevertheless, we develop a dynamic-programming based algorithm that solves the weighted sum-rate maximization with pseudo-polynomial complexity in the number of users and in the total feedback bit budget. We further emphasize the importance of computational complexity by proposing two reduced-complexity approximation algorithms for scenarios where the weighted sum-rate function exhibits more structure. For the case when the weighted sum-rate function is non-decreasing and sub-modular in the feedback allocation, we propose a greedy allocation algorithm and show it is a \(1 - \frac{1}{e}\)-approximation. Finally, we restrict our attention further to single-stream multiple-input-multiple-output beamforming/combining systems with quantized beamformer feedback, an architecture that has been widely-studied in the recent past. We show that a system using Random Vector Quantization codebooks induces a non-decreasing, sub-modular weighted sum-rate function and some additional structure that allows us to further tighten the approximation.

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Index Terms

Limited feedback, multi-user feedback allocation, throughput-optimal, uplink feedback, Random Vector Quantization

I. INTRODUCTION

In many currently-implemented wireless standards, channel state information (CSI) is fed back by the receiver to the transmitter to allow for the latter to adapt its transmit strategy. This includes power and rate adaptation, which is known to increase capacity over the case when there is no CSI at the transmitter (CSIT) and precoder adaptation for a fixed transmission rate in the case of multiple-input-multiple-output (MIMO) systems, which can be used to increase link reliability. Current state-of-the-art opportunistic scheduling algorithms such as multi-user diversity and proportional fairness assume the availability of CSIT through feedback, thus allowing for the transmitters to adapt their respective transmission strategies as a function of their link quality and other network state information. An important example is multi-user diversity downlink scheduling. Here, the user with the best channel is scheduled in each time slot and the base station transmits (ideally) at the Shannon capacity of its link to that user. It is well-known (Sharif and Hassibi [1]) that for this scheduling policy, the sum-rate scales as $\Omega((\log \log K)^{1})$, where $K$ is the number of users. However, as noted by Huang et al. [2] this increase comes with a linear increase in feedback rate. This observation has motivated the development of limited feedback techniques. Past literature on limited feedback, reviewed next, can be broadly classified into techniques for point-to-point links and for multi-user systems, with some overlap between the two.

The impact of limited feedback on the performance of MIMO point-to-point wireless links has been studied extensively. For a comprehensive survey of the current state-of-the-art in limited feedback techniques for point-to-point links, refer to the tutorial paper by Love et al. [3]. Two popular techniques are Grassmannian Quantization and Random Vector Quantization (RVA). The former [4] explores the merits of quantization on the Grassmann manifold. According to the latter technique [5], [6], a codebook is constructed by throwing points uniformly at random on the surface of a complex unit sphere. Bounds have been derived on some suitably-chosen measure of distortion [4]–[12].

A parallel body of work [13]–[15] focuses on developing limited feedback protocols for multi-user systems. Here, past research efforts can be sub-divided into two categories. Work in the first category focuses on traditional single-antenna downlink orthogonal frequency-division multiple-access systems. Chen et al. [13] and Sanayei et al. [15] propose a limited feedback scheme where each user, with associated priority, is restricted to a feedback budget of one bit per tone, i.e., each user transmits a bit that indicates whether its channel is above a certain threshold. Given a set of users with good channels, the base station schedules the user with the highest priority on each tone. The authors compute thresholds that achieve the optimal tradeoff between feedback rate and data rate for this class of data and feedback scheduling policies. While the above work assumes that the feedback window has number of slots equal to the product of the number of users and tones, Agarwal et al. [14] relax this assumption by considering feedback windows of arbitrary size. They propose an opportunistic feedback scheme where a user contends for a feedback slot if their channel strength is greater than a pre-set threshold. Work in the second category focuses on limited feedback for MIMO multiple-access systems. Jindal [16] investigates the impact of finite rate feedback on a downlink space-division multiple-access network where a multiple-antenna base-station serves a number of single-antenna users. This work assumes that the number of users is equal to the number of antennas at the base-station and that the latter uses a zero-forcing precoding transmission policy to simultaneously serve all users.

$$f(n) = O(g(n)) \text{ if } \exists \bar{n} \text{ and } c_{1} > 0 \text{ such that } f(n) \leq c_{1}g(n), \forall n \geq \bar{n}; f(n) = \Omega(g(n)) \text{ if } f(n) = O(g(n)) \text{ and } \exists \bar{n} \text{ and } c_{2} > 0 \text{ such that } f(n) \geq c_{2}g(n), \forall n \geq \bar{n}.$$
Jindal shows that when each mobile uses an RVQ codebook, the feedback budget needs to scale linearly in the signal-to-noise-ratio in order to achieve the full multiplexing gain (equal to the number of users) offered by the channel. Huang et al. [17] study the ergodic sum-rate performance for a space-division multiple-access system that uses the per-user unitary rate control joint scheduling and feedback protocol (defined therein). The authors calculate the sum-rate scaling of this protocol in the number of users and antennas. Unlike time-division-duplexing networks where channel reciprocity cannot be exploited, explicit feedback for the uplink is required for current and future standards (such as Long Term Evolution) that employ frequency-division-duplexing (FDD), thus motivating limited feedback research specifically dealing with the uplink. Dai et al. [18] consider a MIMO uplink where the base-station obtains the channel perfectly for each user and feeds back (broadcasts) an index that maps to a collection of transmit covariance matrices, each for one mobile in the network. The mobile then uses this covariance matrix to design its Gaussian vector codeword. The quantizer design problem that they formulate is as follows: given a constraint on the number of quantization states (or feedback rate equivalently), they seek to find the optimal quantization policy that maximizes the ergodic sum-rate. As this problem is too difficult to solve in its most general form, Dai et al. [18] restrict their attention to a sub-optimal strategy as applicable to a scenario where all users employ single-stream beamforming/combining, thus requiring knowledge of the right singular vector of the channel. The quantization codebook is a composite Grassmannian matrix and given a set of right singular vectors across the users, the base-station chooses an index or quantization point such that the sum-squared chordal distance is minimized. Jorswieck et al. [19] consider a MIMO successive interference cancellation uplink scheme where each mobile uses transmit precoding along with orthogonal-frequency-division-multiplexing. Here, the authors propose a limited feedback protocol that involves reducing the number of precoding matrices, which ideally would be equal to the number of sub-carriers.

While the aforementioned literature considers feedback strategies that are primarily static in nature, dynamic or adaptive feedback bandwidth control (that adapt to the current state of the system) has been recognized by Love et al. [3] as a promising future direction in limited feedback research. Zakhour and Gesbert take a first stride in this direction in a series of papers [20], [21] where they propose an adaptive feedback allocation strategy for a downlink system where the base station serves a subset of users (equal to the number of transmit antennas) using multi-user zero-forcing transmissions. The subset of users is chosen based on limited feedback that it receives during the initial control segment of the time slot. The users are allowed to adapt the quality of their feedback during this control segment, i.e., a user would provide higher-quality feedback if it anticipates being scheduled during this time slot. They essentially seek to design a channel-adaptive feedback scheme that maximizes the expected throughput under an average feedback constraint. As the optimization is difficult to solve, they propose effective sub-optimal solutions to the problem without guarantees on accuracy.

In this paper, we develop dynamic feedback allocation policies for the uplink of a cellular system. Fig. 1 depicts the uplink of an FDD cellular network where the base station serves multiple mobiles or users and has a limited feedback budget represented as a maximum number of feedback bits to communicate a transmit strategy to all users. Feedback allocation is necessary because limited feedback induces errors that predominantly stem from quantization and delay\(^2\). Thus, for the uplink scenario under consideration, if the network objective is rate fairness across users for instance, then a user with a poor channel would demand more accurate CSIT. On the contrary, if the objective is sum-rate maximization, a stronger user might be provided with greater CSIT accuracy. More importantly, as a consequence of the total feedback constraint and independent of the choice of objective, the post-feedback uncertainties in CSIT (and hence throughputs) become coupled

\(^2\)Quantization error is encountered during the process of estimating the channel at the receiver and mapping it to a set of bits or states in order to be sent back to the transmitter. Delay error is due to the fact that the signal passing through the feedback channel is received at the transmitter after some delay depending on the user’s location and the fact that the true channel might have changed over this period.
across the users even in the case when they transmit data on orthogonal channels.

A general transmission policy is a map from the entire content of the feedback packet to a collection of transmission strategies across users. Ideally, this map should be selected dynamically as a function of the system state. However, very little work has gone into this approach, which requires high-dimensional optimization rendering it intractable as recognized by Dai et al. [18]. Thus, we focus on polices that partition the feedback broadcast packet into smaller chunks, each intended for one user. The partition is adapted based on the state of the system. We pursue this intuitively appealing partitioning approach in the interest of analytical tractability and implementability. Henceforth, all claims of optimality are with respect to this space of partitioning policies. A parallel, independent effort by Ouyang and Ying [22] considers OFDMA downlink with a similar partitioning model. In particular, each user reports CSI for at most $F_i$ bands such that $\sum_{i=1}^{N} F_i \leq F$. This work assumes that all wireless links can be modelled as ON-OFF channels. In each time slot, the proposed Longest-Queue-First Feedback Allocation (LQF-FA) policy computes the optimal feedback partition \( \{F_i^*\} \) as one that maximizes the queue-weighted expected throughput. A greedy algorithm is developed that solves the queue-weighted throughput maximization under a mean-field approximation on the channel without guarantees on accuracy.

Our work differs fundamentally from Ouyang and Ying [22] in that we have full observability of the channel and queue state in the uplink and are concerned with how to control the quality of CSIT that we distribute back to the users. This has not been considered before to the best of our knowledge. On the other hand, Ouyang and Ying [22] are interested in acquisition of partial CSI from the users, which is more applicable to the downlink. Furthermore, we deal extensively with the question of computational complexity by proposing a variety of algorithms with analytical guarantees on accuracy.

The main contributions of this paper are the following:

1) We propose a theoretical framework for limited feedback in cellular uplink that models this coupling in throughput performance across users. The scheme proposed by Jorswieck et al. [19] is a special case of our framework, as will be described later in Section II.

2) Optimal – randomized and history-dependent online – multi-user feedback scheduling policies are designed for two long-term network objectives.

   a) Queue stability: This classical network objective [23], [27] is applicable to queueing systems where each user does not have infinitely back-logged data to transmit, henceforth referred to as unsaturated systems.

Fig. 1. FDD Cellular uplink where the base-station has a feedback link to each user.
b) Utility maximization: This second objective applies to systems that have infinitely back-logged data, called saturated systems [28]. Optimal throughput regions are determined in the process.

3) The optimal randomized policy can be obtained by solving a convex optimization problem with linear constraints and with an exponentially large number of variables. An optimal history-dependent online policy, which involves solving a weighted sum-rate maximization problem at each scheduling time slot, is presented as an alternative. The latter policy has the added advantage of not requiring a priori knowledge of the arrival rates in unsaturated systems.

While the above contributions introduce the proposed theoretical framework and the types of scheduling policies of interest for multi-user limited feedback in the uplink, implementability of these policies is an equally-critical design requirement. In light of this, the remaining contributions deal exclusively with the topic of computation complexity, which is the primary focus of this paper. Here, we present a host of practical algorithms to solve the online optimization that explore the tradeoff between computational efficiency, accuracy, and required structure of the weighted sum-rate function. Following the seminal work on link scheduling by Tassiulas and Ephremidis [23] where they show that throughput optimality can be achieved by solving a maximum-weight independent set problem at each scheduling time slot, there has long been immense interest in finding polynomial-time solutions to this problem for special cases. This has resulted in a variety of algorithmic approaches over the last decade (see e.g. [24], [25] and references therein) for specific network structures. Prior to this work and the parallel contributions by Ouyang and Ying [22], the maximum-weight independent set problem has not been considered in the context of feedback allocations, to the best of our knowledge.

4) Notwithstanding the exponential size of the space of all possible feedback allocations, we develop a dynamic programming algorithm that solves the weighted sum-rate maximization with pseudo-polynomial complexity in the number of users and in the total feedback bit budget. This approach is exact and requires no assumptions on the structure of the weighted sum-rate function.

5) We show that in many practical wireless systems, the weighted sum-rate is non-decreasing and sub-modular. Using this observation, we then leverage sub-modular optimization results from combinatorial optimization (e.g. [42]–[44]) and propose a reduced-complexity feedback allocation algorithm with a multiplicative approximation guarantee of \((1 - \frac{1}{e})\).

6) Single-stream multiple-input-multiple-output beamforming and combining is being considered as a potential transmission mode in the Long Term Evolution standard [29]. For such systems, we show that when the popular RVQ codebooks are used, we are able to reduce the complexity even further. We provide additive approximation guarantees for this algorithm.

The rest of this paper is organized as follows. In Section II, we introduce the system model for multi-user feedback scheduling. In Section III, we discuss the two long-term objectives that drive our choice of scheduling policies. We present a convex optimization approach to compute the throughput-optimal randomized feedback allocation policy, introduce an alternate throughput-optimal online feedback policy and provide a result useful later when we obtain approximate but computationally more efficient online feedback allocation schemes. In Section IV, we solve the optimal online feedback optimization problem for both objectives while in Section V, we investigate methods of reducing the complexity of the optimal online optimization problem by exploiting more structure of the objective function. Section VI contains a numerical study of the performance of some of the proposed algorithms. Concluding remarks are made in Section VII.
Notation: $x_{ij}$ denotes element $(i,j)$ of matrix $X$ while $x_i$ denotes element $i$ of vector $x$. Given matrices $X, Y \in \mathbb{R}^{p \times q}$, $X \preceq Y$ means $x_{ij} \leq y_{ij}, \forall i = 1, \ldots, p, j = 1, \ldots, q$. $(\cdot)^T$ and $(\cdot)^\dagger$ are the transpose and Hermitian-transpose operators respectively. The sets $\mathbb{R}_+, \mathbb{N}_0$ and $\mathbb{N}$ represent the non-negative real numbers, non-negative integers and positive integers respectively. Finally, $[x]^+ = \max\{x, 0\}$ and $|| \cdot ||$ is the two-norm operator.

II. System model

Consider the uplink of a slotted-time cellular system with $K$ users scattered across a cell. Each user-base-station channel is modeled as a finite-state discrete-time process where the composite channel across users (in appropriate units) at time $t$, $\mathbf{m}[t]$, takes values in set $\mathcal{M}, |\mathcal{M}| = M$. For example, if we model all the channels as Gilbert-Eliot (or ON-OFF channels), then $\mathcal{M} = \{0,1\}^K$. We assume that the base-station has perfect knowledge of the channel state $\mathbf{m}[t]$ in every time slot. Each user transmits on a separate frequency band thereby removing the need for data scheduling, since the focus of this work is primarily on feedback scheduling. To this effect, we assume that the base station has an error-free control channel that is broadcast in nature, which it uses for feedback purposes. Each feedback packet has a total size $B$ bits and is intended to carry quantized channel state information back to all users. The base station has to allocate $b_k, k = 1, \ldots, K$, bits of each feedback packet to user $k$ such that $\sum_{k=1}^{K} b_k \leq B$. Let $\mathcal{B} = \{\mathbf{b} \in \mathbb{N}_0^K : \sum_{k=1}^{K} b_k \leq B, \mathbf{b} \in \mathbb{N}\}$ represent the set of allowable bit allocation vectors. In each time slot, the base station decides on a bit allocation that it will use to form the feedback packet. An insufficiently large budget $B$ will lead to loss of information in the quantization process. In addition to quantization effects, we assume the presence of delay in the feedback link, which motivates the following general transmission model. In channel state $\mathbf{m} \in \mathcal{M}$, user $k$ chooses its transmission rate $\mu_k(\mathbf{m}_k, b_k) \in \mathbb{R}_+$ based on the bit allocation $b_k$, the quantized CSIT that it receives and its inherent tendency towards tolerating outage or packet drops. Since we assume that maximum tolerable outage probability remains fixed over the entire period that the user is in the system, we do not explicitly include it in the functional definition of rate $\mu_k(\mathbf{m}_k, b_k)$. We assume that the channel process $\{\mathbf{m}[t]\}$ is an ergodic Markov chain and that the feedback link has zero-delay, for simplicity. The limited feedback policy proposed by Jorswieck et al. [19] falls within our framework since their protocol allocates an equal number of bits to each user, i.e. $b_k = \frac{B}{K}$, and given $b_k$ bits, a user transmits at a rate that is determined by the collection of precoding matrices it is assigned by the base-station according to some utility. The above model can also account for delayed feedback with independent and identically distributed (i.i.d.) channels. Here, the user would choose a transmission rate according the distribution of the current channel state conditioned on the delayed channel information received through the feedback link.

III. Long-term Network Objectives

In Sections III.A and III.B, we define the two objectives that we briefly introduced earlier – queue stability and utility maximization – and justify the use of SSS policies, which are randomized policies by definition, to characterize the system rate region for each objective. In the context of feedback, such a characterization has not been made in the past to the best of our knowledge. This characterization immediately allows for the computation of an optimal randomized scheduling policy (under either objective) by solving a convex optimization problem with linear constraints, but one that has an exponential number of variables. Once we establish this initial result, we proceed by proposing an alternate computationally less-demanding online allocation policy that takes into account the history of allocation decisions. We show that in order to achieve either long-term objective, the online allocation problem to be solved is a weighted sum-rate maximization. This allows us to propose an optimal history-dependent feedback allocation algorithm in Section IV, which solves this weighted sum-rate maximization problem at every scheduling instant.
A. Queue stability

Assume that each user $k$, $k = 1, 2, \ldots, K$, has a queue of untransmitted packets with queue-length $q_k[t]$ and associated arrival rate $\lambda_k$. The state of the system at time $t$ is given by $S[t] = \{m[t], q[t]\}$ where $q[t]$ is the vector of queue lengths. A mapping $H$ from the state $S[t]$ to a probability distribution $H(S[t])$ on the set of bit allocations $B$ is called a feedback scheduling or allocation policy. This means that when the system is in state $S[t]$, bit allocation $b$ is picked according to the probability distribution $H(S[t])$.

Let $a_k[t]$ denote the packet arrival process for user $k$. For simplicity, let us assume that $a_k[t]$ is an ergodic Markov chain and that the arrival processes are mutually independent across users. Under these standard assumptions, the queue-state process is Markov and evolves according to

$$q[t] = q[t-1] + a[t] - d[t],$$

where $d_k[t] = \min\{q_k[t], \mu_k(m_k[t], b_k^*[t])\}$; $b^*[t]$ is the allocation decision at time $t$. Queue stability is traditionally defined as the positive recurrence of the queue-state process $q[t]$ under a given scheduling policy.

Let $\mathcal{V}$ be the system rate region, i.e., the set of all long-term stabilizable service rates under all possible feedback allocation policies. While a general policy as introduced above can depend on both queue and channel state, we characterize this set through the use of Static Service Split (SSS) scheduling rules, which are a simplification of it, following the approach pursued by Andrews et al. [27]. We will comment shortly on why it is sufficient to consider SSS feedback allocation policies in order to characterize the system rate region. An SSS rule can be described as follows. In channel state $m$, the scheduler chooses bit allocation $b$ with probability $\phi_{mb}$: an SSS policy is completely characterized by a stochastic matrix $\Phi$. The long-term rate region for this space of policies is written as

$$\mathcal{V} = \left\{ \nu(\Phi) : \sum_{b \in B} \phi_{mb} = 1, \phi_{mb} \in [0, 1], \forall m, b \right\}, \quad (1)$$

where $\nu(\Phi) = \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} \mu(m, b)$ and $\mu(m, b) = [\mu_1(m_1, b_1) \mu_2(m_2, b_2) \ldots \mu_K(m_K, b_K)]^T$; $\nu(\Phi)$ is the long-term average rate under scheduling policy $\Phi$ since $\sum_{b \in B} \phi_{mb} \mu(m, b)$ represents the expected rate while in channel state $m$, which is subsequently averaged over all channel states.

The following theorem states that if some feedback allocation policy (possibly randomized) can stabilize a system, then there exists a SSS policy, as given in (1), that can also stabilize the system. In particular, the theorem says that one can obtain a throughput-optimal feedback allocation strategy by solving a linear program.

**Theorem 1.** If a scheduling rule $H$ exists under which the system is stable, then there exists an SSS scheduling policy $\Phi$ such that the system is stable, i.e., $\lambda < \nu(\Phi)$.

The proof of the theorem follows very similar lines as the proof in the paper by Andrews et al. [27]. Here, the authors prove the above claim under a definition of scheduling policies that maps the state $S[t]$ to a probability distribution on the users indices $\{1, \ldots, K\}$ as opposed to a probability distribution on the set of bit allocations $B$. The core idea of the proof involves a marginalization across the queue states $q[t]$ in order to compute an equivalent SSS probability that picks an allocation or user in a given channel state $m[t]$.

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3A general policy, in principle, is allowed to depend on the entire history of the state (e.g., channel, queues, etc.) of the system but it is well-known [26] that it is sufficient to consider stationary scheduling policies that depend only on the current state at time $t$. 

This theorem, in particular, justifies our use of SSS policies in order to characterize the rate region or stability region, equivalently, of an unsaturated system. The above theorem directly motivates the computation of a stabilizing SSS policy \( \Phi^* \) given arrival rate vector \( \lambda \), through the following linear program

\[
\Phi^* = \arg\min_c \quad \lambda \leq c \nu(\Phi) \\
s.t. \quad \sum_{b \in \mathcal{B}} \phi_{mb} = 1, \ \forall m \in \mathcal{M} \\
\phi_{mb} \in [0, 1], \ \forall m, b
\]

This linear program characterizes the throughput region and also provides the optimal feedback allocation policy. However, there are two key issues. The first issue is that the linear program requires the scheduler to have \textit{a priori} knowledge of the arrival rates. To alleviate the requirement on \textit{a priori} knowledge of arrival rates, Tassiulas and Ephremedis [23] proposed the well-known \textit{max-weight} or \textit{back-pressure} online scheduling algorithm. Observing the natural connection between the independent sets defined by Tassiulas and Ephremedis in [23] and the feedback bit allocations in our model, it follows that if \( \lambda < \nu(\bar{\phi}) \) for some SSS scheduling matrix \( \bar{\phi} \), then the following per-instant scheduling rule

\[
b^*[t] = \arg\max_{b \in \mathcal{B}} q[t]^T \mu(m[t], b)
\]

stabilizes the system.

The second issue is more fundamental and it concerns computational complexity. The linear program (2) has an exponential number of variables since the stochastic matrices \( \Phi \) have dimension \( |\mathcal{M}| \times |\mathcal{B}| = M \times (B+K-1) \). Furthermore, the per-instant scheduling rule of Tassiulas and Ephremides in (3) also requires optimization over the set \( \mathcal{B} \), which may have exponentially many facets. We take up the issue of complexity starting in Section IV.

B. Utility maximization

The following alternate long-term network objective, proposed in [28], is applicable to saturated systems where each user has an infinite amount of data to be served (transmitted). For such systems, the state is given by \( S[t] = m[t] \) and hence, any scheduling rule is automatically a SSS scheduling rule thereby giving us the same characterization of rate region \( V \) as in (1). In such systems, we are concerned with optimizing the vector of long-term service rates \( \nu(\phi) \) such that we maximize some utility function \( H(\nu) \) over the region \( V \) introduced earlier, i.e., we are interested in

\[
\max_{\nu \in V} H(\nu).
\]

The following two classes of long-term utility functions are defined in [28]:

(i) Type I Utility Function - \( H(u) \) is a continuous strictly concave function on \( \mathbb{R}^K_+ \). In addition, \( H(u) \) is continuously differentiable, i.e., the gradient \( \nabla H \) is finite and continuous everywhere in \( \mathbb{R}^K_+ \).

(ii) Type II Utility Function - \( H(u) = \sum_{k=1}^{K} H(u_k) \) where each \( H(u_k) \) is a strictly concave continuously differentiable function, defined for all \( u_k > 0 \) and such that \( H(u_k) \to -\infty \) as \( u_k \to 0 \), e.g. \( H(u) = \sum_{k=1}^{K} \log(u_k) \).

For the aforementioned utility functions, we have a convex optimization problem with linear constraints in (4). As an alternative to solving this problem, which again has a large number of variables, Stolyar [28] shows that using the following

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4The stability region of an unsaturated system is defined as the set of arrival rates \( \Lambda \subset \mathbb{R}^K_+ \) that are stabilizable under any scheduling policy.

5The notion of rate region is slightly different here since we are not stabilizing anything – here it is the region of long term average rates that the system provides.
gradient-weighted sum-rate maximization at each instant solves (4) for $\delta$ sufficiently small
\[ b^*[t] = \arg\max_{b \in B} \nabla H \left( \mu_{\text{emp}}[t] \right)^T \mu(m[t], b) \]  
where
\[ \mu_{\text{emp}}[t] = (1 - \delta)\mu_{\text{emp}}[t] + \delta \mu(m[t], b^*[t]) \]
is the empirical rate vector measured till time $t$. Formally stated, the statement proven in [Theorem 2, [28]] says:

**Theorem 2.** Let $A$ be a bounded subset of $\mathbb{R}_+^K$. Then, for any $\varepsilon > 0$, there exists $T > 0$ (depending on $\varepsilon$ and $A$) such that
\[ \lim_{\delta \to 0} \sup_{\mu_{\text{emp}}[0] \in A, t > T} P \left( \| \mu_{\text{emp}}[t] - \nu^* \| > \varepsilon \right) = 0. \]

As is the case for the stability objective, solving this problem requires optimizing over the set $B$. The computational burden this presents may be non-trivial. We turn to this now.

**IV. Optimal Allocation through Dynamic Programming**

In Section III, we have established that for queue stability in (12) and for Type I/II utility maximization in (5), we are interested in the following online weighted sum-rate maximization problem
\[ \max_{b \in B} w^T \mu(m[t], b), \]  
where $w = [w_1, \ldots, w_K]^T$ is a vector of non-negative weights. The form of the function $\mu(m[t], b)$ would, in general, depend on the underlying system. In fact, for complex modulation/coding schemes the function might only be available as a look-up table. While the optimization problem characterizes optimal performance, solving it exactly may be computationally prohibitive. Thus, the focus of this paper becomes algorithmic. We propose novel solutions to (6) through Theorems 3-9 that explore the natural tradeoffs between accuracy, complexity and the structure of the weighted sum-rate function. We start by showing that using Dynamic Programming, the exact solution can be obtained in pseudo-polynomial time.

**Theorem 3.** The online resource allocation problem (6) can be solved exactly in time $O(KB^2)$.

**Proof:** Order the users arbitrarily. We choose to work with the existing order w.l.o.g. Define
\[ A(i, j) \triangleq w_i \mu_i(m_i, j) \]  
(7)
to be the weighted sum-rate for user $i$ given we allocate $j$ bits to this user and define
\[ R(k, b) \triangleq \max_{\sum_{i=1}^k b_i \leq b, b_i \in \mathbb{N}_0} \sum_{i=1}^k w_i \mu_i(m_i, b_i) \]  
(8)
to be the maximum weighted sum-rate if we have $b$ bits to allocate amongst the first $k$ users with $R(0, b) = 0$. It follows that $R(1, b) = A(1, b) = 0, \ldots, B$.

We can write a recursion
\[ R(k, b) = \max_{j=0, \ldots, b} \{ R(k - 1, b - j) + A(k, j) \}. \]  
(9)

The optimality of the recursion (9) can be established using standard induction arguments. This rule gives rise to a table with a total of $K(b+1)$ elements. In order to compute element $(k, b)$ in the table, using our recursion, we incur a complexity of $O(b+1)$. Hence, the total complexity can be calculated as
\[ \sum_{k=1}^K \sum_{b=0}^B (b+1) = K \sum_{b=0}^B (b+1) = K \left( B + 1 + \frac{B(B+1)}{2} \right) = K \frac{(B+1)(B+2)}{2} = O(KB^2). \]  
(10)
Thus, we have proposed an exact solution using dynamic programming, which has pseudo-polynomial complexity $O(KB^2)$ and which is applicable to any type of weighted sum-rate function.

It is clear that the complexity of this algorithm depends critically on how the bit budget $B$ scales in the number of users $K$. If $B = O(1)$ and is a small constant, then the algorithm provides an implementable linear-complexity solution in the number of users. However, in order to prevent a throughput ceiling, it is necessary for the bit budget to scale with the number of users [32]. In LTE, a physical downlink control channel (PDCCH) carries resource assignments to a user. Each PDCCH can vary in size ranging from 72 bits to 576 bits per user depending on the user’s channel conditions and required robustness [30], [31]. The standard is expected to accommodate an average of 100 users (indoor, high-speed etc.) for services such as VoIP services [33] thus resulting in a complexity of roughly $KB^2 = 100 \times 100 \times 100 = 10^6$ operations for dynamic programming. Here, we are assuming a feedback packet size of 100 bits, a number that will only grow with the advent of technologies such as MIMO-OFDMA coupled with high-data rate applications such as video and gaming. While complexity might not be too large for some applications, others might demand faster running times.

This motivates the development of algorithms with faster running times that might be less accurate. This forms the focus of the remainder of this paper.

V. REDUCED-COMPLEXITY RESOURCE ALLOCATION

In this section, we develop more computationally efficient algorithms that approximately solve (6) for a special class of weighted sum-rate functions. We provide theoretical lower bounds on their performance. The long-term performance of these approximate algorithms in achieving queue stability is characterized by Theorem 4 below.

We say that an algorithm is a multiplicative $\beta$-approximation, $\beta \in (0,1]$, to (3) if it provides a solution $b_{alg}$ such that

$$w^T \mu(m[t], b_{alg}) \geq \beta \max_{b \in B} w^T \mu(m[t], b).$$

We say that an algorithm is an additive $\beta$-approximation to (3) if it provides a solution $b_{alg}$ such that

$$\max_{b \in B} w^T \mu(m[t], b) - w^T \mu(m[t], b_{alg}) \leq \beta w^T 1.$$

The following theorem is a generalization of the original result by Tassiulos and Ephremedis. It essentially states that local approximation is consistent with the long-term objectives we consider.

**Theorem 4.** (i) (Multiplicative) If $\lambda < \beta \nu(\bar{\phi})$, $\beta \in (0,1]$ for some SSS scheduling matrix $\bar{\phi}$, then a $\beta$-approximation to the following per-instant scheduling rule

$$b^*[t] = \arg\max_{b \in B} q[t]^T \mu(m[t], b)$$

(11)

stabilizes the system.

(ii) (Additive) If $\lambda + \beta < \nu(\bar{\phi})$ where $\beta = \beta 1$, $\beta > 0$ for some SSS scheduling matrix $\bar{\phi}$, then the approximate bit allocation policy $\bar{b}[t]$ satisfying

$$q[t]^T \mu(m[t], \bar{b}[t]) \geq q[t]^T [\mu(m[t], b^*[t]) - \beta]$$

(12)

stabilizes the system. Here, $b^*[t] = \max_{b \in B} q[t]^T \mu(m[t], b)$.

An algorithm has pseudo-polynomial complexity if its running time is a polynomial in the size of the input in unary. The size of the input to (6) in unary at most $KB A_{max} + B = O(KB)$ where $A_{max} = \max_{(i,j)} A(i,j)$. 
The theorem essentially states that for unsaturated systems: (i) If we calculate a multiplicative $\beta$-approximate solution, $\beta \in (0, 1]$ to (3) in every time slot, one can achieve a $\beta$-fraction of the stability region $\mathcal{V}$, and (ii) if we calculate a solution that is within $\beta q[t]^T 1$, $\beta > 0$ of (3) in every time slot, one can achieve all rates within the region $(\mathcal{V} - \beta 1)^+$, where $(x)^+ = \max\{0, x\}$. This is the set formed by subtracting $\beta 1$ from each vector in $\mathcal{V}$. Of course, it is understood that if $\beta$ is large leading to vectors with negative elements, these elements are made zero since we cannot have negative rates. This result paves the way for the design of computationally efficient algorithms for the long-term objectives, by constructing approximations to (6).

In Section A, we consider weighted sum-rate functions that are non-decreasing and sub-modular in the bit allocation. In short, sub-modularity refers to diminishing returns with respect to the allocation of resources. This is a property that is exhibited quite frequently by wireless systems in general since transmission rates typically behave logarithmically. Sub-modularity enables us to propose a greedy bit allocation algorithm that has complexity $O((B + K) \log_2 K)$ with multiplicative approximation factor $(1 - \frac{1}{e})$. In the example above, this reduces the running time from $10^6$ operations to roughly $10^3$ operations. Our main contributions are contained in Lemma 2 and Theorem 6.

In Section B, we focus on a class of weighted sum-rate functions that arise in uplink scenarios where all nodes (including the base-station) are equipped with multiple antennas and the adopted transceiver scheme is single-stream beamforming and combining with quantized beamformer feedback. Single-stream beamforming and combining multiple-input-multiple-output (MIMO) systems have been extensively studied in the past [4]–[6], [9], [12], [39], [40]. This is an attractive method for achieving reliable data transmission through significant diversity and array gain making them part of standards such as W-CDMA [41] and LTE [30]. We show that for this choice of physical layer signalling protocol, the weighted sum-rate maximization problem in (6) is sub-modular for certain types of beamformer quantizers. More importantly, this sub-class of non-decreasing, sub-modular functions allows for the development of an approximation algorithm with a further-reduced complexity of $O(K \log_2 K)$ thus reducing the running time even further from 1000 operations to roughly 600 operations in the example above. We prove an additive approximation factor for this algorithm that is a function of the beamformer quantizer parameters.

### A. Reduced-complexity resource allocation through sub-modularity

In this section we show that under some mild assumptions, bit allocation has sub-modular structure. Roughly speaking, this means that a users’ performance exhibits diminishing returns with respect to the number of feedback bits received. This allows us to leverage results from sub-modular function optimization. We begin this section with a quick primer on sub-modular optimization (summarized from [42]–[44]) that will be useful for our purposes. In keeping with the literature, the approach pursued in this section will be graph theoretic in contrast to the rest of this paper. A sub-modular function is defined as follows:

**Definition (Sub-modular Function):** Let $E$ be a finite set and $2^E$ represent all its subsets. Then, $F : 2^E \to \mathbb{R}_+$ is a non-decreasing, normalized, sub-modular function if:

- $F(\emptyset) = 0$ (normalized)
- $F(A) \leq F(B)$ if $A \subseteq B \subseteq E$ (non-decreasing)
- $F(A \cup \{e\}) - F(A) \geq F(B \cup \{e\}) - F(B)$, $\forall A \subseteq B \subseteq E$ and $e \in E \setminus B$ (sub-modular)

The following property of sub-modular functions is useful for reasons that are obvious.
Lemma 1. If $F_k$, $k = 1, \ldots, K$, are sub-modular on set $E$, then $\sum_{k=1}^{K} w_k F_k(A)$, $A \subseteq E$ is a sub-modular function for $w_k \geq 0, \forall k$.

Proof: The proof follows from direct application of the definition of sub-modularity. Let $F_k(A)$, $k = 1, \ldots, K$, $A \subseteq E$, be sub-modular functions on set $E$. Then, for all $A \subseteq B \subseteq E$ and $e \in E \setminus B$, we have the property

$$\sum_{k=1}^{K} w_k F_k(A \cup \{e\}) - \sum_{k=1}^{K} w_k F_k(A) = \sum_{k=1}^{K} w_k (F_k(A \cup \{e\}) - F_k(A)) \geq \sum_{k=1}^{K} w_k (F_k(B \cup \{e\}) - F_k(B)).$$

(13)

Having provided the definition of sub-modularity along with a useful property, we now introduce the kinds of constraint sets that are typically considered in the context of sub-modular optimization.

Definition (Independence System): A set system $(E, \mathcal{I})$ where $E$ is a finite set and $\mathcal{I}$ is a collection of subsets of $E$ is called an independence system if it satisfies the following properties:

- $0 \in \mathcal{I}$
- $A \subseteq B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$

Definition (Matroid): An independence system is called a matroid if it satisfies the following additional property; if $A, B \in \mathcal{I}$ and $|A| < |B|$, then there exists $e \in B \setminus A$ such that $A \cup \{e\} \in \mathcal{I}$.

We are interested in a special class of matroids called uniform matroids, defined as follows.

Definition (Uniform Matroid): $\mathcal{I}$ is a uniform matroid if $\mathcal{I} = \{F \subseteq E : |F| \leq k\}$ for $k \in \mathbb{N}$.

The optimization problem that has been considered in the context of sub-modular functions and independence systems is

$$F^* = \maximize_{A \in \mathcal{I}, A \subseteq E} F(A)$$

(14)

Since many NP-hard problems can be reduced to a sub-modular function maximization over an independence system, significant research has focused on developing efficient approximation algorithms. In particular, the performance of the greedy algorithm in solving special cases of (14) has been extensively studied. Nemhauser et al. [45] considered problem (14) over uniform matroids and showed that the greedy algorithm provides a $(1 - \frac{1}{e})$ approximation factor for this special case.

At each step, this algorithm augments the existing subset solution with an additional element from the set $E$ such that the new subset solution belongs to the independence system. The additional element is selected to maximize the incremental utility. Given sets $S, T \subseteq E$, we define

$$\rho_T(S) = F(S \cup T) - F(S)$$

(15)

and write the greedy algorithm (borrowing some notation from [42]) when $\mathcal{I}$ is a uniform matroid, parametrized by size $k$, as follows.
Algorithm (Greedy algorithm for maximizing non-decreasing, normalized, sub-modular functions over uniform matroids):

- **Step 1:** Set $i = 1$ and $S_{g,0} = \emptyset$.
- **Step 2:** Select element $e_i \in E$ such that
  \[
  e_i = \maximize_{e \in E \setminus S_{g,i-1}} \rho_e(S_{g,i-1})
  \]  
  \[
  \text{s.t} \quad e \in E \setminus S_{g,i-1}
  \]  
- **Step 3:** Set $S_{g,i} = S_{g,i-1} \cup \{e_i\}$.
- **Step 4:** Stop if $i = k$; else set $i = i + 1$ and go to Step 2.

In the above, $S_{g,i}$ represents the set constructed by the greedy algorithm after $i$ iterations. The approximation factor result is formally stated in Theorem 5 below. We re-state the proof of Theorem 5 as it provides insight into the operation of the greedy algorithm.

**Theorem 5.** Let $F : 2^E \rightarrow \mathbb{R}^+$ be a normalized, non-decreasing, sub-modular function on set $E$, $\mathcal{I}$ be a uniform matroid, and $F_{\text{greedy}}$ be the solution provided by the greedy algorithm. Then $\frac{F_{\text{greedy}}}{\rho_{\text{opt}}} \geq (1 - \frac{1}{e})$.

**Proof:** See Appendix B. \qed

Please refer to Goundan et al. [42], Calinescu et al. [43] and Vondrak [44] for a summary of related results on sub-modular function optimization over other families of constraint sets.

**Sub-modularity in feedback allocation:** We now show that the optimal bit allocation problem in (6) is indeed a sub-modular function maximization over a uniform matroid. Let $G = (U, V, E)$ be a bipartite graph where $U$ contains $K$ user nodes and $V$ contains $B$ bit nodes, both ordered arbitrarily, i.e. $|U| = K$ and $|V| = B$. Let $E$ contain the set of all edges $E = \{e_{kb} : i = 1, \ldots, K$ and $j = 1, \ldots, B\}$. Given $A \subseteq E$, we define $|A|_i \overset{\Delta}{=} |\{e_{kb} \in A : k = i\}|$ to represent the number of bits allocated to user $i$, i.e., $|A|_i = b_i$. The independence we are interested in is $\mathcal{I} = \{A \subseteq E : |A| \leq B\}$ where $B$ is the total bit budget. By definition, $\mathcal{I}$ is a uniform matroid and furthermore, $\mathcal{I}$ is the set of all valid allocations since if $A \in \mathcal{I}$, then $\sum_{k=1}^{K} b_k = \sum_{k=1}^{K} |A|_k \leq B$ and if $A \notin \mathcal{I}$, then $\sum_{k=1}^{K} b_k = \sum_{k=1}^{K} |A|_k = |A| > B$. Now the weighted sum-rate maximization problem in (6) when the channel is in state $\mathbf{m}[t]$ in time slot $t$ as

\[
\maximize_{\mathbf{b} \in \mathcal{B}} \quad \sum_{k=1}^{K} w_k \mu_k(m_k[t], b_k) \equiv \maximize_{\mathbf{b} \in \mathcal{B}} \quad \sum_{k=1}^{K} w_k \mu_k(m_k[t], b_k) - \mu_k(m_k[t], 0) \quad \text{s.t} \quad b_k = |A|_k, \sum_k |A|_k \leq B, \ A \subseteq E \\
= \maximize_{A \in \mathcal{I}} \quad \sum_{k=1}^{K} w_k \mu_k(m_k[t], |A|_k) - \mu_k(m_k[t], 0).
\]

The following result becomes immediate.

**Lemma 2.** If the function $\mu_k(m_k, b_k)$ is non-decreasing and sub-modular in the bit allocation $b_k = |A|_k$, $A \subseteq E$ for all users $k = 1, \ldots, K$, and channel states $\mathbf{m} \in \mathcal{M}$, then $\sum_{k=1}^{K} w_k \mu_k(m_k[t], |A|_k) - \mu_k(m_k[t], 0)$ is a normalized, non-decreasing, sub-modular function on set $E$ for all channel states $\mathbf{m} \in \mathcal{M}$.

**Proof:** The result follows from Lemma 1. \qed

Hence, the result in Theorem 5 is applicable and the greedy algorithm can be used to solve the optimal bit allocation problem in (17) with approximation factor $(1 - \frac{1}{e})$. The greedy algorithm for the specific case of our bit allocation problem
in time slot $t$ can be written as follows where

$$u_k(b_k) = u_k(m_k, b_k + 1) - u_k(m_k, b_k)$$  \hspace{1cm} (18)$$

is the increase in rate or marginal utility if user $k$ is given one extra bit.

**Algorithm (Greedy algorithm for feedback bit allocation):**

- **Step 1:** Set $b = 1$ and $b_k = 0, \forall k$, which is essentially a bit counter for each user.
- **Step 2:** Compute $u_k(b_k), \forall k$.
- **Step 3:** Sort this list of marginal utilities.
- **Step 4:** Assign a bit to user $k^*$ who is on top of this list, update $b_{k^*} = b_{k^*} + 1$ and re-compute $u_{k^*}(b_{k^*})$
- **Step 5:** If $b < B$, set $b = b + 1$, and go to Step 3; else exit.

We end this section by investigating the complexity of the above algorithm in the following theorem.

**Theorem 6.** The greedy algorithm has complexity $O((B + K)\log_2 K)$ when applied to the optimal bit allocation problem in (6).

**Proof:** Step 2 of this algorithm incurs complexity $O(K\log_2 K)$ for the first iteration $b = 1$. Subsequently, every re-sort in Step 3 costs $O(\log_2 K)$ with a maximum of $B$ such re Sorts. Thus, the total complexity is $O((B + K)\log_2 K)$. \hfill $\Box$

**B. Reduced-complexity resource allocation for MIMO systems**

By assuming that the rate $\mu_k(m_k, b)$ is a non-decreasing sub-modular function in the bit allocation $b$ in every channel state $m_k$, we use the greedy algorithm in Section A to approximately solve the online feedback allocation problem in (6) with complexity $O((B + K)\log_2 K)$. In this section, we show that when single-stream beamforming and combining with quantized beamformer feedback is used as the physical layer transmission scheme, the weighted-sum-rate maximization problem in (6) is non-decreasing and sub-modular for a broad class of quantizers. Thus, we are able to develop an approximation algorithm with a further-reduced complexity of $O(K\log_2 K)$ with an additive guarantee that depends on the parameters of the quantizer. The technique involves relaxing the integral constraint on the bits, solving the weighted sum-rate maximization using fractional bits under an assumed form on the expected post-quantization signal-to-noise ratio (SNR) or quantized SNR in short, followed by rounding to obtain the integer solution. Thus, aside the usual impact on precision that are typically omitted from running time calculations, the running time of our algorithm no longer depends on the feedback budget $B$.

We begin this section by investigating the effects of limited feedback on the aforementioned class of MIMO systems.

1) **Single-stream MIMO with limited feedback:** The classical $N_t \times N_r$ single-stream beamforming and combining MIMO link for a typical user (shown in Fig. 2) can be described using the following received signal model,

$$y = \sqrt{\nu}z^\dagger Hg s + z^\dagger n,$$  \hspace{1cm} (19)
where

\[ s \sim \text{Complex Gaussian transmit codeword with } E[|s|^2] = P \]
\[ n \in \mathbb{C}^{N_r} \sim \mathcal{CN}(0, N_0I) \text{ is additive white Gaussian noise} \]
\[ g \in \mathbb{C}^{N_t} : \text{ Transmit beamformer with } ||g||^2 = 1 \text{ to satisfy the transmit power constraint} \]
\[ z \in \mathbb{C}^{N_r} : \text{ Receive combiner} \]
\[ H \in \mathbb{C}^{N_r \times N_t} : \text{ Complex-valued MIMO channel} \]
\[ \alpha \in \mathbb{R}^+ : \text{ Large-scale fading gain} \]

The model in (19) is a comprehensive description of the wireless channel in that it explicitly accounts for the composite effects of small-scale (SS) fading and large-scale (LS) fading. We use \( \alpha \) to represent the path-loss or shadowing effects of the channel, henceforth referred to as LS effects, while the matrix \( H \) denotes SS fading. Composite models have been used in past literature (see [36] and references therein). The SNR for this system can be written as

\[ \text{SNR} = \frac{|z^H H g|^2 P_\alpha}{|z|^2 N_0}. \tag{20} \]

For simplicity, we assume that all users have the same number of antennas \( N_t \) although all results presented in the remainder of this section can be extended to scenarios where this is not true. It is well-known that the SNR in (20) can be maximized by setting \( g^* = v \) and \( z^* = H g^* \) where \( v \) is the right singular vector corresponding to the maximum singular value \( \sigma \) of the channel matrix \( H \). By introducing user indices, the maximum SNR for user \( k \) can be written as

\[ \text{SNR}_{k,PF} = \frac{\alpha_k P_k |H v_k|^2}{N_o} = \frac{\alpha_k P_k \sigma_k^2}{N_o}. \tag{21} \]

The choice of notation reflects the fact that the user requires Perfect Feedback of the right singular vector \( v_k \) from the base-station in order to achieve this maximum SNR. However, feedback in realistic systems is imperfect due to limited feedback budgets, the primary motivation for this work. Through the remainder of this section, we restrict our attention to quantization error: error that is introduced when the base-station quantizes the optimal precoder \( v_k \) using \( b_k \) bits in preparation for feedback, the feedback link is assumed to be delay- and error-free. We assume that user \( k \) uses a quantized beamformer \( \hat{v}_k \) for which we can write the SNR with Imperfect Feedback as

\[ \text{SNR}_{k,IF} = \frac{\alpha_k P_k |H \hat{v}_k|^2}{N_o}. \tag{22} \]

2) Time-scales and structure of rate vector \( \mu(m,b) \): In this section, we describe the structure of rate vector \( \mu(m,b) \) that arises out of employing the single-stream MIMO physical layer scheme described earlier.

We consider changing feedback allocations once every LS fading coherence time, which typically spans multiple SS fading coherence times, say \( D \) of them, as shown in Fig. 3. In other words, we provide feedback about the faster time-scale...
can be written as 
\[ \sigma \]

system. From (22), the SNR with imperfect feedback is a random variable whose distribution depends on the joint distribution 
\[ \gamma \]

al. [37]) when transmitting at the maximum possible rate chosen transmission rate in accordance with Shannon’s capacity formula.

In this framework, outages arise due to delay constraints that dictate that a packet must be decoded within a SS coherence time. This means that a particular SS fading realization within the larger coherence time might not be able to support the chosen transmission rate in accordance with Shannon’s capacity formula.

To compute \[ \gamma_k(\alpha_k, b_k) \], we need to quantify the outage probability of the single-stream beamforming/combining MIMO system. From (22), the SNR with imperfect feedback is a random variable whose distribution depends on the joint distribution of \[ \sigma_k^2 \] and \[ \nu_k \] along with the quantization policy. Thus, the outage probability for user \( k \) that transmits at rate \( \gamma_k(\alpha_k, b_k) \) can be written as

\[
\mathbb{P} \left( \frac{\alpha_k P_k ||H_k\nu_k||^2}{N_0} \leq 2^{\gamma_k(\alpha_k, b_k)} - 1 \right)
\]

(23)

Fig. 3. Composite effects of small-scale fading and large-scale fading in a wireless channel with \( D = 4 \).

(small-scale fading) and the quality of feedback is varied at a slower time-scale (large-scale fading). Such a design choice has two benefits: First, it might require too much overhead to compute and communicate optimal allocations on the SS fading time-scale, which typically spans a few milliseconds. Second, this allows each user to estimate their LS coefficient \( \alpha_k \) without the need for feedback from the base-station by exploiting reciprocity on the downlink. This is possible since path-loss and/or shadowing are dependent solely on the distance between the user and the base-station. The increasing availability of GPS-enabled devices also offers the user an alternate means to compute their path-loss.

Capturing the two separate time-scales, we define the channel state as

\[ m[t] = \{\alpha[t], [H_k](t-1)D+1, \ldots, H_k[tD], k = 1, \ldots, K\} \]

for the single-stream MIMO system we are considering. We assume that \( \{\alpha[t]\} \) is a finite-state process that is either (i) i.i.d. across time or (ii) an ergodic Markov chain, taking values from the set \( P \) with a unique stationary distribution \( \{\pi_\alpha\}_{\alpha \in P} \). On the faster time-scale, we assume that \( \{[H_k](t-1)D+1, \ldots, H_k[tD], k = 1, \ldots, K\} \) is again a finite-state process that is either i.i.d. across time or ergodic Markov taking values from the set \( H \). Traditionally, each element of the channel matrix \( H_k \) is modeled as a complex Gaussian random variable. However, we can consider a finite-state process by discretizing this random variable and creating set \( H \) by sampling the support of its probability density function sufficiently finely. As is the case in past literature (see [36] and references therein), large-scale fading is assumed to be independent of the small-scale fading. Finally, the small-scale fading channels are assumed to be identically distributed across users.

In each state \( m \in M = P \times H \), given bit allocation \( b \), we assume that user \( k \) transmits at a rate \( \mu_k(\alpha_k, b_k) \) that is independent of the realization \( \{[H_k](t-1)D+1, \ldots, H_k[tD], k = 1, \ldots, K\} \). Given a fixed \( \alpha_k \) and bit allocation \( b_k \) through the course of a large coherence time, we define \( \mu_k(\alpha_k, b_k) \) to be the goodput (a notion that is discussed by Lau et al. [37]) when transmitting at the maximum possible rate \( \gamma_k^*(\alpha_k, b_k) \) while allowing for an outage probability of at most \( \epsilon_k \), i.e.,

\[
\mu_k(\alpha_k, b_k) \triangleq \gamma_k^*(\alpha_k, b_k)(1 - \epsilon_k).
\]

In this framework, outages arise due to delay constraints that dictate that a packet must be decoded within a SS coherence time. This means that a particular SS fading realization within the larger coherence time might not be able to support the chosen transmission rate in accordance with Shannon’s capacity formula.

Markovian and i.i.d. models for user mobility in a cell (and hence path-loss) have been utilized by El Gamal et al. [34] and Toumpis et al. [35] respectively in studying how mobility impacts the performance of a wireless network.
We can use the Markov inequality to bound (23):

\[ P \left( \frac{\alpha_k P_k \|H_k v_k\|^2}{N_0} \leq 2 \gamma_k(\alpha_k, b_k) - 1 \right) = P \left( \frac{1}{\alpha_k P_k \|H_k v_k\|^2} \geq \frac{2 \gamma_k(\alpha_k, b_k) - 1}{N_0} \right) \leq \frac{1}{\alpha_k P_k E \left\{ \|H_k v_k\|^2 \right\}} \left( 2 \gamma_k(\alpha_k, b_k) - 1 \right) \]  

(24)

by Markov’s inequality.

From Jensen’s inequality, we know that \( E \left\{ \frac{1}{\|H_k v_k\|^2} \right\} \geq \frac{1}{E \{\|H_k v_k\|^2\}} \). In order to proceed, we note that one can find a function \( e(\cdot) \) such that

\[ E \left\{ \frac{1}{\|H_k v_k\|^2} \right\} = e(b_k) \frac{1}{E \{\|H_k v_k\|^2\}}. \]

(25)

While it is true that \( e(\cdot) \) is dependent on the quantization codebook/policy and the channel distribution as well, we do not explicitly write down this dependence since we are interested only in optimizing bit allocations. This function can be computed numerically at the beginning of the communication session and furthermore, we can find a bound \( e_{\text{max}} = \max_k e(b_k) \) such that

\[ E \left\{ \frac{1}{\|H_k v_k\|^2} \right\} \leq e_{\text{max}} \frac{1}{E \{\|H_k v_k\|^2\}}, \forall k. \]

(26)

For our analysis, we use the popular Random Vector Quantization (RVQ) technique [5], [6]. According to this approach, a codebook \( C_k(b) \) for user \( k \), corresponding to a bit allocation of \( b \) bits, is constructed by throwing \( 2^b \) points uniformly at random on the surface of a complex unit sphere. These codebooks offer the combined advantages of analytical tractability along with implementability [9]. On the other hand, Grassmannian codebooks [4], [10], which are optimal maximum-SNR fixed codebooks for single-stream transmission over a Rayleigh fading channel are unfortunately not available for all combinations of feedback bits and transmit antennas [9]. Recent results [9], [11], [12] quantify the loss in SNR due to quantization when using RVQ codebooks. In these works, the authors show that the expected SNR with feedback quantization using \( b \) bits for a single-stream beamforming/combining MIMO system can be described accurately by a function of the form

\[ E_{\mathcal{C}(0,1)} \{ \|H v\|^2 \} = E[\sigma^2] \left( 1 - c_1(N_t, N_r) 2^{-c_2(N_t, N_r)b} \right) \text{ where } c_1(N_t, N_r) \in (0, 1], \ c_2(N_t, N_r) > 0. \]

(27)

Some user indices have been dropped in the above expression since all users transmit through i.i.d. Rayleigh MIMO channels and employ the same codebook, i.e., \( C_k(b) = C(b) \), \( \forall k \). Now since (27) is true on an average over all realizations of the codebook \( C(b) \), it follows that there exists at least one codebook \( C^*(b) \) with quantized SNR

\[ E_{C^*(b)} \{ \|H v\|^2 \} \geq E[\sigma^2] \left( 1 - c_1(N_t, N_r) 2^{-c_2(N_t, N_r)b} \right). \]

(28)

We can collect codebooks across all \( b = 0, \ldots, B \), to form a super codebook \( C^* = \bigcup_{b=0}^{B} C^*(b) \). Through the remainder of our analysis, we assume that the system uses such a codebook \( C^* \) and do not include an explicit dependence on \( C^* \) in our notation henceforth.

Substituting (26) and (28) in (24), we get

\[ \begin{align*}
  P \left( \frac{\alpha_k P_k \|H_k v_k\|^2}{N_0} \leq 2 \gamma_k(\alpha_k, b_k) - 1 \right) &\leq \frac{N_0}{\alpha_k P_k} E \left\{ \frac{1}{\|H_k v_k\|^2} \right\} \left( 2 \gamma_k(\alpha_k, b_k) - 1 \right) \\
  &\leq \frac{N_0}{\alpha_k P_k} E \frac{e_{\text{max}}}{\sigma^2} \left( 1 - c_1(N_t, N_r) 2^{-c_2(N_t, N_r)b} \right) \left( 2 \gamma_k(\alpha_k, b_k) - 1 \right) \\
  &\leq \frac{N_0}{\alpha_k P_k} E \frac{e_{\text{max}}}{\sigma^2} \left( 1 - c_1(N_t, N_r) 2^{-c_2(N_t, N_r)b} \right) \left( 2 \gamma_k(\alpha_k, b_k) - 1 \right).
\end{align*} \]

Therefore, by using

\[ \begin{align*}
  \left\{ \frac{N_0}{\alpha_k P_k} E \frac{e_{\text{max}}}{\sigma^2} \left( 1 - c_1(N_t, N_r) 2^{-c_2(N_t, N_r)b} \right) \left( 2 \gamma_k(\alpha_k, b_k) - 1 \right) \right\} \text{ as our outage event, we are being conservative. We enforce the maximum outage probability constraint of } \varepsilon_k \text{ and explicitly compute } \gamma_k^*(\alpha_k, b_k) \text{ as}
\end{align*} \]

\[ \begin{align*}
  \varepsilon_k &\geq \frac{N_0}{\alpha_k P_k} E \frac{e_{\text{max}}}{\sigma^2} \left( 1 - c_1(N_t, N_r) 2^{-c_2(N_t, N_r)b} \right) \left( 2 \gamma_k(\alpha_k, b_k) - 1 \right) \\
  \Rightarrow \gamma_k^*(\alpha_k, b_k) &= \log_2 \left( 1 + \frac{\alpha_k P_k e_{\text{max}}}{\sigma^2} \left( 1 - c_1(N_t, N_r) 2^{-c_2(N_t, N_r)b} \right) \left( 2 \gamma_k(\alpha_k, b_k) - 1 \right) \right),
\end{align*} \]
Thus, we have computed the goodput when transmitting at $\gamma_k^*(\alpha_k, b_k)$ while incurring outage probability of at most $\epsilon_k$ as
\[
\mu_k(\alpha_k, b_k) \triangleq \log_2 \left( 1 + a_k \Delta(b_k) \right) (1 - \epsilon_k).
\]
where $a_k = \frac{P_k \alpha_k^2}{\epsilon_k + N_k}$ and $\Delta(b_k) = \mathbb{E}[\sigma^2] \left( 1 - c_1(N_t, N_r) 2^{-c_2(N_t, N_r) b_k} \right)$, $c_1(N_t, N_r) \in (0, 1]$, $c_2(N_t, N_r) > 0$ is the quantizer SNR function. Recall that $\gamma_k^*(\alpha_k, b_k)$ represents the maximum possible transmission rate that obeys the outage constraints.

From (1), the rate region for a system that employs the single-stream MIMO physical layer structure described thus far can be expressed in terms of
\[
\nu(\Phi) = \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} \mu(m, b)
\]
\[
= \sum_{\alpha \in \mathcal{P}} \pi_\alpha \sum_{b \in B} \phi_{\alpha b} \mu(\alpha, b)
\]
\[
= \sum_{\alpha \in \mathcal{P}} \pi_\alpha \sum_{b \in B} \phi_{\alpha b} \left( \log_2 (1 + a_1 \Delta(b_1)) (1 - \epsilon_1), \ldots, \log_2 (1 + a_K N_o \Delta(b_K)) (1 - \epsilon_K) \right)^T
\]
and the optimization in (6) takes the specific form
\[
\maximize_{b \in B} \sum_{k=1}^{K} w_k \log_2 \left( 1 + a_k \Delta(b_k) \right) (1 - \epsilon_k).
\]
We absorb the success probability $(1 - \epsilon_k)$ into weight $w_k$ henceforth.

While the above analysis calls for the use of a specific super codebook $C^*$, in Section VI, we consider a $N_t = N_r = 2$ MIMO system with a randomly-generated super codebooks. We estimate the constants $c_1(N_t, N_r)$ and $c_2(N_t, N_r)$ thereby forming a lower bound (28) on quantized SNR for many codebook realizations. We also compute the function $e(b)$ (and hence $e_{\max}$) to demonstrate the feasibility of this approach.

3) Relaxation and approximation guarantees: In Theorems 7-9 below, we develop an approximation algorithm to solve (29) in closed-form while incurring a complexity of $O(K \log_2 K)$\(^8\). We provide an additive approximation guarantee of $\log_2 \left( 1 + \max \left\{ \frac{1 - c_1(N_t, N_r) 2^{-c_2(N_t, N_r)} \cdot 1 - c_1(N_t, N_r)}{1 - c_1(N_t, N_r) 2^{-c_2(N_t, N_r)} \cdot 1 - c_1(N_t, N_r)} \right\} \right) \approx 2$ bits per second for a $N_t = N_r = 2$ MIMO system.

**Theorem 7.** Consider the following continuous relaxation of (29):
\[
b^*_k[i] = \arg\max_{b_k \leq B, b_k \in \mathbb{R}_+} \maximize_{b \in B} \sum_{k=1}^{K} w_k \log_2 \left( 1 + \frac{P_k \alpha_k}{N_o} \Delta(b_k) \right) (1 - \epsilon_k).
\]
The solution to this relaxation is
\[
b^*_k = (N_t - 1) \left[ \log_2 \left( \left( 1 + \frac{1}{(N_t - 1) \eta^*} \right) \left( \frac{a_k^* (\mathbb{E}[\sigma^2] - N_t)}{a_k^* \mathbb{E}[\sigma^2] + 1} \right) \right) \right]^{-1}
\]
where $\eta^*$ is chosen such that $\sum_k b^*_k = B$.

*Proof:* See Appendix C.

Now, we argue that the weighted sum-rate function in (29) is non-decreasing and sub-modular on set $E = \{ e_{kb} : i = 1, \ldots, K \text{ and } b = 1, \ldots, B \}$.

**Lemma 3.** The weighted sum-rate function in (29) where $b_k = |A|_k$, $A \subseteq E$, $E = \{ e_{kb} : i = 1, \ldots, K \text{ and } b = 1, \ldots, B \}$ is non-decreasing and sub-modular on this set $E$.

\(^8\)We recognize that there is an additional storage cost of $O(\log B)$. 
Proof: By setting \( b_k = |A_k| \), \( A \subseteq E \) and defining a function \( F : b_k \rightarrow \mathbb{R} \) on the bit allocation for the \( k \)-th user, we observe that it is sufficient to show that \( F(b_k) \) is (i) non-decreasing in \( b_k \in \mathbb{N}_0 \) and that (ii) \( F(b_k + n) - F(b_k) \geq F(\bar{b}_k + n) - F(\bar{b}_k) \), \( b_k \leq \bar{b}_k \), \( \bar{b}_k \in \mathbb{N}_0 \), \( n \in \mathbb{N} \) in order to prove the claim.

Consider the relaxed function \( f : b_k \rightarrow \mathbb{R}, b_k \in \mathbb{R}_+ \) (of course \( f(b_k) = F(b_k) \) for \( b_k \in \mathbb{N}_0 \)) and assume that this function is non-decreasing, concave, and twice differentiable. Then, the conditions (i) is trivially satisfied while condition (ii) is satisfied due to following argument. Since \( f(b_k) \) is concave and twice differentiable, we know that \( f'(b_k) \geq f'(\bar{b}_k) \) for \( b_k \leq \bar{b}_k \). Thus, for any \( y \in \mathbb{R}_+ \), we can write

\[
\frac{d}{db_k} [f(b_k + y) - f(b_k)] = f'(b_k + y) - f'(b_k) \leq 0,
\]

which implies that condition (ii) is satisfied. Since the continuous relaxation of (29) is non-decreasing, concave, and twice differentiable, the result follows.

\[\square\]

**Theorem 8.** Computing the above solution in (30) incurs a complexity of \( \mathcal{O}(K \log_2 K) \).

Proof: See Appendix C.

Comparing the results in Theorems 6 and 8, we see that by assuming less about the exact form of the communication system, we are incurring an added complexity cost of \( \mathcal{O}(B \log_2 K) \), while providing a system-independent multiplicative approximation guarantee of \( (1 - \frac{1}{c}) \).

Once we solve for \( b_k^* \), we apply a floor operation in order to enforce the integer constraints, i.e., we set

\[
b_{k,INT}^* = \begin{cases} 
|b_k^*|, & b_k^* \geq 1 \\
0, & b_k^* < 1 
\end{cases}
\]

This leads us to the task of quantifying loss due to integrality, which we address in Theorem 9 below.

**Theorem 9.** The bit allocation obtained by relaxing integer constraints followed by flooring gives an additive approximation factor of \( \log_2 \left( 1 + \max \left\{ \frac{1}{1 - c_1(N_t, N_r)}^2, \frac{1}{1 - c_2(N_t, N_r)} \right\} \left( \sum_{k=1}^K w_k \right) \right) \).

Proof: See Appendix C.

By applying Theorem 4 with \( w = q \), we can conclude that the proposed relaxation-based algorithm will result in a throughput loss of at most \( \log_2 \left( 1 + \max \left\{ \frac{1}{1 - c_1(N_t, N_r)}^2, \frac{1}{1 - c_2(N_t, N_r)} \right\} \right) \) bits per second for unsaturated systems. Furthermore, the result in Theorem 9 tells us that for single-stream beamforming/combining MIMO systems, the performance of relaxation-based algorithm approaches the optimal as \( c_1(N_t, N_r) \) and \( c_2(N_t, N_r) \) approach zero. This agrees with intuition because as \( c_1(N_t, N_r) \) becomes small, the loss due to quantization decreases. Similarly, as \( c_2(N_t, N_r) \) becomes small, we are dealing with codebooks that exhibit a slow rate of decay. This would mean that the flooring operation to obtain integral bits would not impact the SNR too much.

**VI. PERFORMANCE OF RELAXATION-BASED ALGORITHM**

In this section, we evaluate the accuracy of the convex-relaxation-based algorithm by plotting the distribution of the approximation factor \( \log_2 \left( 1 + \max \left\{ \frac{1}{1 - c_1(N_t, N_r)}^2, \frac{1}{1 - c_2(N_t, N_r)} \right\} \right) \) over many RVQ codebook realizations. The goal of these experiments is to demonstrate that the quantized SNR functional form proposed in (28) is accurate for RVQ codebooks.
We generate an RVQ codebook, compute $c_1(N_t, N_r)$ and $c_2(N_t, N_r)$ for each codebook and the resulting approximation factor. We repeat this process for 1000 codebooks and plot the distribution of the approximation factor in Fig. 4. The distribution in Fig. 4 shows us that the convex relaxation technique offers us a guarantee of roughly 2 bits per second. Note that the computation of $c_1(N_t, N_r)$ and $c_2(N_t, N_r)$ for each codebook is not optimized meaning that the above guarantee is conservative.

Finally, we compute $e(b)$ in Fig. 5 for one such RVQ codebook in order to demonstrate the implementability of this approach. From Fig. 5, it is clear that $e_{\text{max}} \approx 1.5$ for this codebook.

![Fig. 4. Distribution of log$_2\left(1 + \max\left\{\frac{1}{1 - c_1(N_t, N_r)2^{-2(N_t+ N_r)}, 1 - c_1(N_t, N_r)}\right\}\right)$ over 1000 codebook realizations.](image)

![Fig. 5. The function $e(b)$ in Fig. 5 for a $2 \times 2$ MIMO system over a Rayleigh fading channel with a randomly chosen codebook and $B = 10$.](image)

VII. CONCLUDING REMARKS

We summarize the algorithmic contributions presented in Sections IV and V in Table I. We observe from the table that these algorithms explore the tradeoffs between accuracy, computational efficiency and the structure of the weighted sum-rate function.

An interesting question and future direction pertaining to the section on single-stream MIMO systems is whether such an analysis can be extended to cover other commonly-deployed MIMO architectures. Finally, the design of joint data scheduling and feedback allocation policies is another direction for future research.
In summary, we propose optimal feedback allocation policies for cellular uplink systems where the base station has a limited feedback budget. The optimality is in the sense of queue stability for unsaturated queueing regimes and long-term utility maximization for saturated queueing regimes. We show that a randomized optimal allocation policy can be computed by solving a convex optimization problem with linear constraints and with an exponentially large number of optimization variables. An optimal online allocation policy, one that involves solving a weighted sum-rate maximization problem at every scheduling instant, is presented as an alternative. This problem is solved using dynamic programming incurring pseudo-polynomial complexity in the number of users and the total bit budget. When the weighted sum-rate is a non-decreasing sub-modular function, we leverage the theory of sub-modular function maximization to propose a greedy algorithm with polynomial complexity in the number of users and the total bit budget. When the weighted sum-rate is a non-decreasing and sub-modular function, we recognize that the weighted sum-rate function is non-decreasing and sub-modular for RVQ codebooks. More importantly, it takes a special form that allows us to develop an approximation algorithm based on convex relaxations that can be solved in closed-form, incurring further-reduced complexity than the greedy algorithm. We connect the performance of the proposed approximate online algorithms to the long-term stability region of the system.

APPENDIX A

PROOF OF THEOREM 4

The proof is essentially the same as in [23] with some minor modifications. It uses Foster’s theorem to prove positive recurrence of the queue state process $q[t]$. We define a quadratic potential function $V(x) = \frac{1}{2} \sum x_i^2$. The standard Lyapunov drift function can be computed and bounded as

\[
d(q) = \mathbb{E} [V(q[t+1]) - V(q[t]) | q[t] = q] \\
= \frac{1}{2} \sum_{k=1}^{K} \mathbb{E} [q_k^2(t+1) - q_k^2(t) | q_k[t] = q_k] \\
= \frac{1}{2} \sum_{k=1}^{K} \mathbb{E} [(q_k[t] + a_k[t+1] - d_k[t+1])^2 - q_k^2(t) | q_k[t] = q_k] \\
\leq \frac{1}{2} \sum_{k=1}^{K} \mathbb{E} [a_k^2(t+1)] + \mathbb{E} [d_k^2(t+1)] + 2\mathbb{E} [q_k[t] (a_k[t+1] - d_k[t+1]) | q_k[t] = q_k].
\]

(31)

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Required structure on weighted sum-rate</th>
<th>Complexity</th>
<th>Multiplicative approximation factor</th>
<th>Additive approximation factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamic Programming</td>
<td>None</td>
<td>$\mathcal{O}(KB^2)$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Greedy</td>
<td>Non-decreasing sub-modular weighted sum-rate</td>
<td>$\mathcal{O}((B+K)\log_2 K)$</td>
<td>$(1 - \frac{1}{c})$</td>
<td>$-c$</td>
</tr>
<tr>
<td>Convex Relaxation</td>
<td>Non-decreasing sub-modular weighted sum-rate for specific single-stream beamforming/combining MIMO systems</td>
<td>$\mathcal{O}(K\log_2 K)$</td>
<td>$-c$</td>
<td>$\log_2 (1 + M)$ where $M = \max \left{ \frac{1}{1 - c_1 (N_t, N_r) 2^2 (N_t, N_r)}, \frac{1}{1 - c_1 (N_t, N_r)} \right}$</td>
</tr>
</tbody>
</table>
As is standard in the literature, we assume that the arrival and departure processes have bounded second moments, i.e.,

\[ \mathbb{E} \left[ a_k^2 [t + 1] \right] + \mathbb{E} \left[ d_k^2 [t + 1] \right] \leq c_k, \text{ for some } c_k > 0, \forall k. \]

For Part (i), we can continue to bound the drift as

\[
\begin{align*}
d(q) & \leq \frac{1}{2} \sum_{k=1}^{K} c_k + 2E \left[ |a_k[t] (a_k[t + 1] - d_k[t + 1])| \right] \quad |q_k[t] = q_k| \\
& = \frac{1}{2} \sum_{k=1}^{K} c_k + 2E \left[ |a_k[t + 1] - E[d_k[t + 1]|q_k[t] = q_k]| \right] \\
& = \frac{1}{2} \sum_{k=1}^{K} c_k + 2q_k (E[a_k[t + 1] - E[d_k[t + 1]|q_k[t] = q_k]) \\
& = \left( \frac{1}{2} \sum_{k=1}^{K} c_k \right) + q^T (\lambda - \beta \nu(\phi) + \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} \mu(m, b) - E[d[t + 1]|q[t] = q]) \\
& = \left( \frac{1}{2} \sum_{k=1}^{K} c_k \right) + q^T (\lambda - \beta \nu(\phi)) + \beta \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} (q^T \mu(m, b)) - E[q^T d[t + 1]|q[t] = q].
\end{align*}
\]

(32)

Since \( q^T (\lambda - \beta \nu(\phi)) < 0 \) by the conditions of the theorem, we need to show that

\[
\beta \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} (q^T \mu(m, b)) - E[q^T d[t + 1]|q[t] = q] \leq 0,
\]

(33)

in order to prove positive recurrence or stability according to Foster’s Theorem. To this end, we define \( \chi_{mb}[t + 1] \in \{0, 1\} \), to be random variables that represent the scheduling decision at time \( t + 1 \); \( \chi_{mb}[t + 1] = 1 \) if bit allocation \( b \) is selected at time \( t + 1 \) and \( \chi_{mb}[t + 1] = 0 \) otherwise. Since only one bit allocation can be selected at each time, we have the constraint \( \sum_{b \in B} \chi_{mb}[t + 1] = 1 \). We re-write (33) using these newly introduced scheduling variables as

\[
\begin{align*}
& \beta \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} (q^T \mu(m, b)) - E[q^T d[t + 1]|q[t] = q] \\
& = \beta \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} (q^T \mu(m, b)) - \sum_{m \in M} \pi_m E \left[ \sum_{b \in B} \chi_{mb}[t + 1]q^T d[t + 1]|q[t] = q, m[t] = m \right] \\
& = \beta \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} (q^T \mu(m, b)) - \sum_{m \in M} \pi_m \sum_{b \in B} \chi_{mb}[t + 1]q^T \sum_{\mathbf{m} \in \mathcal{M}} \sum_{\mathbf{b} \in \mathcal{B}} \phi_{mb} (q^T \mu(m, b)) - E[q^T d[t + 1]|q[t] = q] \\
& = \beta \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} (q^T \mu(m, b)) - \beta \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} (q^T \mu(m, b)) \text{ for some } \mathbf{b} \in \mathcal{B} \\
& \leq \beta \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} (q^T \mu(m, b)) - \beta \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} (q^T \mu(m, b)).
\end{align*}
\]

(34)

The last inequality follows since our scheduling rule dictates that we choose allocation \( \mathbf{b} \) such that \( q^T \mu(m, b) \geq \beta \max_{b \in B} q^T \mu(m, b) \). Finally, it is straightforward to see that

\[
\max_{\phi_{mb} \in [0, 1], \forall \mathbf{m} \in \mathcal{M}, b \in \mathcal{B}} \left[ \beta \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} q^T \mu(m, b) \right] = \beta \sum_{m \in M} \pi_m \left[ \max_{b \in B} q^T \mu(m, b) \right]
\]

(35)

Thus, the drift \( d(q) \) is strictly negative for \( q_k \) sufficiently large which proves stability.

For Part (ii), as done in the earlier proof, we bound the drift as

\[
\begin{align*}
d(q) & \leq \frac{1}{2} \sum_{k=1}^{K} c_k + 2E \left[ |a_k[t] (a_k[t + 1] - d_k[t + 1])| \right] \quad |q_k[t] = q_k| \\
& = \left( \frac{1}{2} \sum_{k=1}^{K} c_k \right) + q^T (\lambda - \nu(\phi)) + \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} (q^T \mu(m, b)) - E[q^T d[t + 1]|q[t] = q].
\end{align*}
\]

(36)

We now show that

\[
\beta \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} (q^T \mu(m, b)) - E[q^T d[t + 1]|q[t] = q] \leq q^T \beta.
\]

(37)
as
\[
\sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb}(q^T \mu(m, b)) - E [q^T d[t + 1] | q[t] = q] = \\
\sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} \left[ q^T \mu(m, b) \right] - \sum_{m \in M} \pi_m E \left[ \sum_{b \in B} \chi_{mb} [t + 1] q^T d[t + 1] | q[t] = q, m[t] = m \right]
\]

\[
= \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} \left[ q^T \mu(m, b) \right] - \sum_{m \in M} \pi_m \sum_{b \in B} \chi_{mb} [t + 1] q^T E [\mu(m[t + 1], b) | q[t] = q, m[t + 1] = m]
\]
since given \( q[t] = q \) and \( m[t + 1] = m \), the scheduling variables \( \chi_{mb}[t + 1], \forall m, b \), are no longer random

\[
= \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} \left[ q^T \mu(m, b) \right] - \sum_{m \in M} \pi_m \sum_{b \in B} q^T \mu(m, b)
\]

\[
\leq \sum_{m \in M} \pi_m \sum_{b \in B} \phi_{mb} \left[ q^T \mu(m, b) \right] - \sum_{m \in M} \pi_m \sum_{b \in B} q^T \mu(m, b)
\]

since \( b \) satisfies \( q^T \mu(m, b) \geq q^T \mu(m, b^*) - q^T \beta \)

\[
\leq q^T \beta \text{ by the argument in (35)}
\]

The proof is complete from the fact that \( q^T (\lambda + \beta - \nu(\bar{\phi})) < 0 \) by the conditions of the theorem.

**APPENDIX B**

**PROOF OF THEOREM 5**

Firstly, we present an alternate characterization of sub-modular functions from Nemhauser et al. [45] that is useful for the proof. For any two disjoint subsets \( S \) and \( T = \{t_1, \ldots, t_N\}, S, T \subseteq E \), we can write

\[
F(S \cup T) = \left[ \sum_{i=2}^N F(S \cup \{t_1, \ldots, t_i\}) - F(S \cup \{t_1, \ldots, t_{i-1}\}) + (F(S \cup \{t_1\}) - F(S)) \right] + F(S)
\]

through telescoping. By the sub-modularity of \( F \), we have

\[
F(S \cup T) \leq \left[ \sum_{i=1}^N F(S \cup t_i) - F(S) \right] + F(S)
\]  

\[
= F(S) + \sum_{i=1}^N \rho_i(S)
\]

and furthermore, for \( S \subseteq T \), this simplifies to

\[
F(T) \leq F(S) + \sum_{t \in T \setminus S} \rho_t(S).
\]  

Now, let \( S^* \) and \( S_0 \) be the optimal solution and the solution generated by the greedy algorithm respectively; \( \rho_i \) represents the incremental value that is obtained during the \( i \)-th iteration of the greedy algorithm. Then, by setting \( S = S_{g,0} = \emptyset \) in (41) and noting that \( |S^*| \leq k \) since it is a uniform matroid, we calculate

\[
F^* \leq \sum_{e \in T} F(\{e\}) \leq k \rho_1 = k \max_{e \in E} F(\{e\}).
\]  

Recalling that \( F(S_{g,0}) = 0 \) due to normalization and applying (41) to set \( S_{g,j} \) generated by the greedy algorithm after \( j \) iterations, we have

\[
F^* \leq F(S_{g,j}) + \sum_{t \in T \setminus S_{g,j}} \rho_t(S_{g,j})
\]  

\[
= \sum_{i=1}^j (F(S_{g,i}) - F(S_{g,i-1})) + \sum_{t \in T \setminus S_{g,i}} \rho_t(S_{g,i})
\]  

\[
\leq \sum_{i=1}^j \rho_i + k \rho_{j+1}.
\]

By dividing both sides by \( k \), re-arranging and adding \( \sum_{i=1}^j \rho_i \) to both sides, we get

\[
\sum_{i=1}^{j+1} \rho_i \geq \frac{1}{k} F^* + \frac{k-1}{k} \sum_{i=1}^j \rho_i.
\]

The following result is proved through induction in order to solve the recursion.

\[
\sum_{i=1}^j \rho_i \geq \left( \frac{k^j - (k-1)^j}{k^j} \right) F^*.
\]  


For $j = 1$, we get $\rho_1 \geq \frac{E}{k}$, which is true since, for $S^* = \{s_1^*, s_2^*, \ldots, s_K^*\}$, we have

$$F^* = F(S^*) = \sum_{i=1}^{K} \left[ F \left( \{s_1^*, s_2^*, \ldots, s_i^*\} \right) - F \left( \{s_1^*, s_2^*, \ldots, s_{(i-1)}^*\} \right) \right] + F \left( \{s_i^*\} \right)$$

$$\leq (k - 1) F \left( \{s_1^*\} \right) + F \left( \{s_i^*\} \right)$$

$$= kF \left( \{s_1^*\} \right).$$

Assuming the statement holds true for $(j - 1)$, and substituting it in (44), we get

$$\sum_{i=1}^{j} \rho_i \geq \frac{j}{k} F^* + \frac{k-1}{k} \left( \frac{k^{j-1} - (k-1)^{j-1}}{k^{j-1} - (k-1)^{j-1}} \right) F^*$$

$$= \frac{j}{k} F^* + (k - 1) \left( \frac{k^{j-1} - (k-1)^{j-1}}{k^{j-1} - (k-1)^{j-1}} \right) F^*$$

$$= \frac{k^{j-1} + (k-1)(k^{j-1} - (k-1)^{j-1})}{k^j} F^*$$

which proves the claim. Now, by setting $j = k$, we calculate

$$F_y = \sum_{i=1}^{k} \rho_i \geq \left( \frac{k^k - (k-1)^k}{k^k} \right) F^*,$$

or in other words,

$$F_y \geq \left( \frac{k^k - (k-1)^k}{k^k} \right) = 1 - \left( 1 - \frac{1}{k} \right)^k.$$

The result follows since $\lim_{k \to \infty} \left( 1 - \frac{1}{k} \right)^k = \frac{1}{e}$ and the fact that $(1 - \frac{1}{k})^k$ is increasing in $k$.

**Appendix C**

**Proof of Theorems 7-9**

**A. Proof of Theorem 7**

With $\Delta(b_k) = E[\sigma^2] \left( 1 - c_1(N_t, N_r)2^{-c_2(N_t, N_r)b_k} \right)$, the objective function (omitting dependence on $t$) becomes

$$\text{minimize}_{b \in B} - \sum_{k=1}^{K} w_k \log_2 \left( 1 + a_k E[\sigma^2] \left( 1 - c_1(N_t, N_r)2^{-c_2(N_t, N_r)b_k} \right) \right).$$

(50)

The objective function is clearly convex since $2^{-c_2(N_t, N_r)b_k}$ is convex. By studying (50) closely, we can also say that $b_k^*$ is such that $\sum_{k=1}^{K} b_k^* = B$ since if this not true, we can increase the bit allocation for at least one user thereby decreasing the objective function. Since $B > 0$, $b_k = 0, \forall k$ is in the interior of our constraint set $B$ which implies that Slater’s constraint qualification condition holds. Consequently, the Karush-Kuhn-Tucker (KKT) conditions become sufficient in nature. The Lagrangian cost function can be written as

$$\mathcal{L}(b_k, \lambda_k, \eta) = - \sum_{k=1}^{K} w_k \log_2 \left( 1 + a_k E[\sigma^2] \left( 1 - c_1(N_t, N_r)2^{-c_2(N_t, N_r)b_k} \right) \right) - \lambda_k b_k + \eta \left( \sum_{k} b_k - B \right)$$

(51)

for which the KKT conditions are

$$b_k^* \geq 0, \quad \lambda_k^* \geq 0, \quad \eta^* = \frac{a_k E[\sigma^2] c_1(N_t, N_r) c_2(N_t, N_r) (\log 2)}{(1 + a_k E[\sigma^2])^2 (N_t, N_r) b_k - E[\sigma^2] c_1(N_t, N_r)} + \lambda_k^*,$$

(52)

Since $\frac{a_k E[\sigma^2] c_1(N_t, N_r) c_2(N_t, N_r) (\log 2)}{(1 + a_k E[\sigma^2])^2 (N_t, N_r) b_k - E[\sigma^2] c_1(N_t, N_r)}$ is a decreasing function in $b_k$, if $\eta^* \leq \frac{a_k E[\sigma^2] c_1(N_t, N_r) c_2(N_t, N_r) (\log 2)}{(1 + E[\sigma^2]) (a_k c_1(N_t, N_r))}$, then $\lambda_k^* = 0$ and

$$b_k^* = \left[ \frac{\eta^*}{c_2(N_t, N_r) \log_2 \left( \frac{E[\sigma^2] c_1(N_t, N_r)}{(1 + a_k E[\sigma^2])} \left( \frac{a_k c_2(N_t, N_r) (\log 2)}{\eta^*} + 1 \right) \right)} \right]^+$$

(53)

is a valid solution to (50). If $\eta^* > \frac{a_k E[\sigma^2] c_1(N_t, N_r) c_2(N_t, N_r) (\log 2)}{(1 + E[\sigma^2]) (a_k c_1(N_t, N_r))}$, $\lambda_k^* = \eta^* - \frac{a_k E[\sigma^2] c_1(N_t, N_r) c_2(N_t, N_r) (\log 2)}{(1 + E[\sigma^2]) (a_k c_1(N_t, N_r))}$ and $b_k^* = 0$.

Hence, we can write the solution as

$$b_k^* = \left[ \frac{\eta^*}{\log_2 \left( \frac{E[\sigma^2] c_1(N_t, N_r)}{(1 + a_k E[\sigma^2])} \left( \frac{a_k c_2(N_t, N_r) (\log 2)}{\eta^*} + 1 \right) \right)} \right]^+$$

(54)

where $\eta^*$ is chosen such that $\sum_{k} b_k^* = B$. 
In order to compute (30), we first need to sort \( \left\{ \frac{a_k E[\sigma^2] c_1(N_1, N_r) c_2(N_1, N_r) (\log 2)}{(1+E[\sigma^2]) (a_m-c_1(N_1, N_r))} \right\} \) in ascending order, which has complexity \( O(K \log_2 K) \). Call this sorted set \( \left\{ \frac{a_m E[\sigma^2] c_1(N_1, N_r) c_2(N_1, N_r) (\log 2)}{(1+E[\sigma^2]) (a_m-c_1(N_1, N_r))} \right\} \). Once sorted, we need to set \( \eta^* = \frac{a_m E[\sigma^2] c_1(N_1, N_r) c_2(N_1, N_r) (\log 2)}{(1+E[\sigma^2]) (a_m-c_1(N_1, N_r))} \) for each \( m \) and test feasibility. Testing feasibility incurs \( O(K) \), as it is a \( K \)-term addition and scanning through each \( a_m E[\sigma^2] c_1(N_1, N_r) c_2(N_1, N_r) (\log 2) \) incurs \( O(\log_2 K) \) through the use of binary search. As we increase \( \eta^* \), more \( b_m^* \) terms are set to zero. Once we locate \( m_1 \) and \( m_2 \) such that \( \eta^* = \frac{a_m E[\sigma^2] c_1(N_1, N_r) c_2(N_1, N_r) (\log 2)}{(1+E[\sigma^2]) (a_m-c_1(N_1, N_r))} \) is infeasible while \( \eta^* = \frac{a_m E[\sigma^2] c_1(N_1, N_r) c_2(N_1, N_r) (\log 2)}{(1+E[\sigma^2]) (a_m-c_1(N_1, N_r))} \) is feasible, we can compute \( \eta^* \) in closed-form since it satisfies \( \sum_{m \geq m_2} b_m^* = B \). Hence, the total complexity is \( O(K \log_2 K + \log(\log_2 K)) = O(K \log_2 K) \).

\[ \text{REFERENCES} \]


