

Impact of Information in a Simple Multiagent Collaborative Task

Jason R. Marden, Behrouz Touri, Ragavendran Gopalakrishnan, and J. Patrick O'Brien.

Abstract—In this paper we study the impact of information in a simple multiagent collaborative task – graph coloring. Inspired by the experimental study in [4], we study distributed algorithms for graph coloring where individual nodes are periodically given the opportunity to adjust their color in response to information regarding the color choice of neighboring nodes. When granted such an opportunity, each node chooses an admissible color (if available) that is more prevalent than its current color in its neighborhood. Focusing on the family of ring graphs, our findings demonstrate that there is an inherent trade-off between efficiency and convergence rates for such an algorithm. While increasing the information to the nodes improves the efficiency of the emergent coloring profile, it also degrades the underlying convergence rates. The degradation in convergence rates provides one possible explanation for the findings in [4], which demonstrate that providing additional information to the nodes, which were controlled by human participants, can actually lead to losses in the efficiency of the emergent coloring profile. These losses could be a byproduct of the human participants not having the desire or time to stay engaged long enough in the revision process.

I. INTRODUCTION

Information plays a fundamental role in the coordination of multiagent systems. The information available to the individual agents, either attained through communication or sensing, invariably influence their local decisions which in turn influence the emergent global behavior. These informational restrictions are often viewed as hard constraints that the underlying control design must satisfy; however, by allowing agents to communicate with one another, information transitions from being a hard constraint to a design choice. The focus of this paper is to shed some light on how information, or the lack thereof, impacts performance guarantees in multiagent systems.

The field of networked control systems has long sought to attain a general understanding of how the architecture associated with distributed decision-making systems impacts achievable performance guarantees. Here, we use the term architecture to encompass both the structure of the decision making rule and the structure of the underlying informational dependence. Recent studies have focused on characterizing the implications of architecture on achievable performance guarantees when viewing the underlying architecture from the perspective of information structures [7], sparsity patterns [5], desired structures on the agents' control policies [6], [14], among others. Irrespective of the specific problem setting, the underlying message associated with these results is clear – architectural constraints can dramatically impact performance guarantees in multiagent systems.

Our understanding of the relationship between architecture and performance is becoming increasingly unclear as the complexity associated with multiagent systems expands with the addition of human decision-making entities, adversarial components, and system uncertainty. For example, consider the classical sensor-coverage problem where the goal is to design local agent control policies that ensure desirable collective behavior irrespective of the state of the mission space [9]. While this example has been extensively studied in the literature [1]–[3], [9], it is currently unresolved as to what are the agent control policies that optimize efficiency guarantees for a given measure of locality.

A recent result in [8] has sought to shed light on this issue by identifying how a measure of locality in the agents' control laws directly translate to a bound on the achievable efficiency guarantees in spatially distributed multiagent systems. With regards to the above sensor coverage problem, the vast majority of the literature interprets local as agents having access to information pertaining to their local mission space as defined by their Voronoi partition [1]–[3], [9]. The result in [8] demonstrate that if agents only have access to this information, then it is *impossible* to guarantee that the emergent collective behavior of a system comprised of n agents performs any better than a system comprise of just a single agent. Hence, we inherit the complexity associated with a multiagent system without any provable performance gains in comparison to a single agent system. Nonetheless, [8] demonstrates that providing agents with additional information can lead to control designs with much improved performance guarantees.

This prompts the following question: does providing agents with additional information always lead to improvements in the performance guarantees associated with a multiagent system? A recent experimental study in [4] explores this question in the context of humans participating in a simple collaborative task – graph coloring. The goal of a graph coloring problem is to associate each node in a graph with a color such that (i) two neighboring nodes do not use the same color and (ii) this color profile uses the least number of colors possible. This experimental study focused on identifying the quality of the emergent coloring profile when human participants played the role of nodes in the network, repeatedly adjusting the color of their nodes in response to varying degrees of information regarding the coloring choice of other nodes in the network. Focusing on three degrees of information (Low, Medium, and High), the findings showed that the efficiency of the emergent coloring profile increased significantly as the degree of information went from Low to Medium. However, the efficiency of the emergent coloring profile degraded significantly as the degree of information went from Medium to High. Hence,

This research was supported by AFOSR grant #FA9550-12-1-0359, ONR grant #N00014-12-1-0643, and NSF Grant #ECCS-1351866.

J. R. Marden, B. Touri, and J.P. Obrien are with the Department of Electrical, Computer, and Energy Engineering, University of Colorado, Boulder. R. Gopalakrishnan is with the Xerox Research Centre India.

providing the agents, i.e., human participants, with too much information was detrimental from a system-wide perspective.

The focus of this paper is to understand the conflicting stories between the two results highlighted above. In particular, is providing the agents with too much information detrimental from a system-wide perspective? To shed light onto this question, we focus on a distributed graph coloring problem inspired by [4], [12] where individual nodes independently select their color using information regarding the color choice of a set of neighboring nodes. To model the human participants in [4], we associate each node with a utility function that is increasing in the frequency of the node’s color over its observable neighborhood. Here, we focus on characterizing how the level of information impacts both the efficiency of the stable coloring profile (i.e., coloring profile that is a Nash equilibrium) and the underlying convergence rates. Informally, our results are as follows:

- In Theorem 3.2, we show that increasing the information to the agents *improves* the efficiency of the stable coloring profiles.

- In Theorem 3.3, we show that increasing the information to the agents *degrades* the convergence rates associated with a distributed adjustment process where agents seek to improve their utility.

These findings demonstrate that increasing the level of information to the agents provides improvements according to certain performance metrics (e.g., efficiency stable coloring profiles) but losses according to other performance metrics (e.g., convergence rates). This offers a potential explanation for the findings in [4] as it is possible that the humans did not have the desire (or time) to engage long enough to see improvements in the efficiency of the emergent coloring profile. Whether this apparent tradeoff between efficiency and convergence rates is fundamental to multiagent systems remains an open and interesting question.

II. MODEL AND PERFORMANCE METRICS

Graph coloring problems serve as a simple platform to formally study the impact of information in multiagent collaboration. The goal of a graph coloring problem is to assign each node in a graph $G = (V, E)$ a color $c_i \in \mathcal{C}$ such that (i) two neighboring nodes do not share the same color and (ii) the coloring assignment uses the least number of colors possible. Existing research has focused on both centralized and distributed mechanisms for attaining near optimal coloring assignments, e.g., [12], [13]. Here, we focus on the distributed mechanisms where our goal to is to gain an understanding of how the information available to the agents impacts achievable performance guarantees.

To that end, we consider a game theoretic model for a graph coloring problem where each node $i \in V = \{1, 2, \dots, n\}$ independently selects its color $c_i \in \mathcal{C}$ using information pertaining to the color choice of “local” nodes. The information available to each node (or agent) is modeled as the color choice of all nodes within k -hops, denoted by $\mathcal{N}_i^{(k)} \subseteq V$, where $k \in \{1, \dots, n\}$ is referred to as the *locality*

coefficient. A natural choice for the utility function of node $i \in V$ is

$$U_i(c_1, \dots, c_n) = \sum_{j \in \mathcal{N}_i^{(k)}} I\{c_i = c_j\}, \quad (1)$$

where $I\{\cdot\}$ is the indicator function and attention is restricted to *proper* coloring profiles, i.e., coloring profiles of the form (c_1, \dots, c_n) where $c_i \neq c_j$ for any distinct nodes $i, j \in V$ where $(i, j) \in E$. Such utility functions can serve as a simple platform to study distributed algorithms for graph coloring problems where nodes gravitate to admissible colors that are more prevalent in their observable neighborhoods. For simplicity, we will denote a graph coloring game of the above form merely by the tuple $\{G, k\}$.¹

The goal of this paper is to understand how the information available to the agents directly translates to achievable performance guarantees with regards to the (i) efficiency of the emergent behavior and (ii) the underlying rates of convergence. The following game theoretic concepts will be instrument in the forthcoming arguments.

- **Potential games and pure Nash equilibria:** The graph coloring game $\{G, k\}$ is a potential game, introduced in [10], if there exists a potential function $\phi : \mathcal{C}^n \rightarrow \mathbb{R}$ such that for any coloring profile $c \in \mathcal{C}^n$, agent $i \in V$, and alternative color choice $c'_i \in \mathcal{C}$, we have

$$\phi(c'_i, c_{-i}) - \phi(c_i, c_{-i}) = U_i(c'_i, c_{-i}) - U_i(c_i, c_{-i}), \quad (2)$$

where we often denote a coloring profile c as (c_i, c_{-i}) where $c_{-i} = (c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n)$ represents the coloring choice of all agents $j \neq i$. The existence of a potential game ensures the existence of a pure Nash equilibrium², which is a coloring profile $c^{\text{ne}} \in \mathcal{C}^n$ such that

$$U_i(c_i^{\text{ne}}, c_{-i}^{\text{ne}}) = \max_{c_i \in \mathcal{C}} U_i(c_i, c_{-i}^{\text{ne}}), \quad \forall i \in V. \quad (3)$$

- **Better reply paths:** A better reply path is a sequence of coloring profiles $c^1, c^2, \dots, c^m \in \mathcal{C}^n$ such that for each successive coloring profiles $c^\ell, c^{\ell+1}$ there is exactly one agent $i \in V$ such that $c_i^\ell \neq c_i^{\ell+1}$, i.e., $c^{\ell+1} = (c_i^{\ell+1}, c_{-i}^\ell)$, and for that agent $U_i(c_i^{\ell+1}) > U_i(c_i^\ell)$. In other words, one node adjusts his color at a time, and each time a node adjusts his color he increases his own utility. In potential games, a better reply path is acyclic and terminates at an equilibrium. Accordingly, the upper and lower bounds regarding the length of a better reply path serves as a simple proxy for the convergence rates associated with a better reply process [15]. Informally, a better reply process is a learning algorithm that follows a better reply path with high probability.

- **Efficiency of pure Nash equilibria:** If a distributed algorithm guarantees convergence to a Nash equilibrium, then the efficiency guarantees associated with this distributed

¹The agents’ utility functions could be naturally extended to all coloring profiles by assigning a value of $-\infty$ to any coloring profile that is not proper. Furthermore, the proposed utility functions could be used to model the behavior of the human subjects in the experimental study on collaborative graph coloring in [4].

²We will henceforth refer to a pure Nash equilibrium as just an equilibrium.

algorithm are aligned with the efficiency of the worst performing Nash equilibrium. To that end, consider a global objective $W : \mathcal{C}^n \rightarrow \mathbb{R}$, where $W(c)$ represents the number of colors used in an admissible coloring profile c . We focus on evaluating the performance associated with the worst-performing equilibrium. Such a measure is commonly referred to as the price of anarchy [11], and takes on the form

$$\text{PoA}(G, k) = \frac{\max_{c^{\text{ne}}} W(c^{\text{ne}})}{W(c^{\text{opt}})} \geq 1 \quad (4)$$

where c^{opt} represents the proper coloring profile that uses the least number of colors. Here, the max operator in the numerator is due to the fact that the equilibria need not be unique for a given coloring game $\{G, k\}$.

III. MAIN CONTRIBUTIONS

In this section we present a formal analysis of the graph coloring game depicted in Section II. We begin by demonstrating that the graph coloring game is a potential game irrespective of the locality coefficient or the structure of the underlying graph provided that it is undirected. We then proceed to formally analyze both the price of anarchy and the length associated with the family of better reply paths for the class of ring graphs.

A. Potential game structure

The first result in this manuscript demonstrates that the graph coloring game depicted in Section II is a potential game irrespective of the locality coefficient provided that the underlying graph is undirected. The following theorem makes this claim precise.

Theorem 3.1: Consider any graph coloring game $\{G, k\}$ where the graph G is undirected. For any locality coefficient $k \geq 1$, the coloring game is a potential game where the potential function is given by

$$\phi(c) = \frac{1}{2} \sum_{i \in V} U_i(c). \quad (5)$$

We omit the proof of Theorem 3.1 for brevity. Theorem 3.1 demonstrates that the system behavior will eventually reach an equilibrium when individual agents gravitate to commonly used colors in their neighborhood, i.e., when agents follow a better reply process. While this result seems natural and intuitive, it is important to point out that each agent is responding to a limited view of the network that may be unique to that particular agent. While these informational limitations do not change the property that any better reply process will converge to an equilibrium, they do have quite severe implications on the efficiency of the emergent behavior and the underlying convergence rates as we will show in the ensuing sections.

B. A price of anarchy analysis for ring graphs

The remainder of this paper will focus on characterizing properties regarding the price of anarchy and the length of better reply paths when focusing purely on the simple class of ring graphs. By ring graphs, we mean graphs of the form $G = (V, E)$ where the vertex set is $V = \{1, \dots, n\}$ and the edge set is $E = \{(1, 2), (2, 3), \dots, (n-1, n), (n, 1)\}$.

Accordingly, we will now represent the above graph coloring game on a ring graph merely by the tuple $\{n, k\}$. Ring graphs provide a simple structural framework that can be exploited to attain a clear characterization of how number of agents n and the locality coefficient k impacts the performance metrics of interest.

We begin with the following theorem which identifies how the price of anarchy in (4) depends on the number of nodes n and the locality coefficient k .

Theorem 3.2: Consider a graph coloring game $\{n, k\}$ over a ring graph. If n/k is sufficiently large, then the price of anarchy satisfies

$$\gamma_1 \cdot \left(\frac{n}{\sqrt{k}} \right) \geq \text{PoA}(n, k) \geq \gamma_2 \cdot \left(\frac{n}{k} \right) \quad (6)$$

where $\gamma_1, \gamma_2 > 0$ are constants that do not depend on n or k .³

Theorem 3.2 demonstrates an intuitive property regarding the relationship between information and efficiency of equilibrium in graph coloring problems. In particular, the upper bound in (6) demonstrates that increasing the information to the nodes, i.e., increasing the locality coefficient k , improves the efficiency of the resulting equilibrium coloring profiles.

Proof: The following analysis will use addition and subtraction over the vertices with the understanding that all operations are modulo n .

We begin by establishing the lower bound in (6). In order to do so, we will explicitly construct an equilibrium coloring profile that satisfies the given bound. To that end, consider a ring graph with $n > 6$ vertices and a locality coefficient $k < n/2$. Define

$$\alpha = \begin{cases} k+2 & \text{if } k \text{ is even,} \\ k+1 & \text{if } k \text{ is odd.} \end{cases}$$

Partition the ring graph into $z = \lfloor \frac{n}{\alpha} \rfloor$ different sections S_1, \dots, S_z of contiguous vertices where the sections S_1, \dots, S_{z-1} each consist of α vertices and the remaining section S_z consists of $n - (z-1)\alpha \geq \alpha$ vertices. See Figure 1 for an illustration.

For each $i \leq z-1$, denote the vertices in each section S_i by a local index $\{1, \dots, \alpha\}$ where $f_i(m) \in \{1, \dots, n\}$ captures the global index of each node $m \in \{1, \dots, \alpha\}$ in section S_i . Likewise, denote the vertices in section S_z by the local index $\{1, \dots, n - (z-1)\alpha\}$ and define $f_z(\cdot)$ in a similar fashion. For each section S_i , $i \leq z$, consider the alternating coloring assignment $(c_1, c_2, \dots, c_{\alpha-1}, c_\alpha) = (x_i, y_i, \dots, x_i, y_i)$ where the colors $x_i, y_i \in \mathcal{C}$ are unique to section S_i . The section S_z could end in either a x_i or y_i depending upon whether the number of nodes n is odd or even. Note that the proposed coloring profile, which we denote by c^{ne} , is indeed a proper coloring profile which uses $2z$ different colors.

We will now show that c^{ne} is in fact an equilibrium. For simplicity, we will focus solely on the case where k is even as similar arguments can be constructed for the odd case. First, note that $U_i(c^{\text{ne}}) \geq k/2 + 1$ for all vertices $i \in N$. In

³The term n/k is sufficiently large for any n and k satisfying $k \geq 4$ and $n \geq k^{3/2}$.

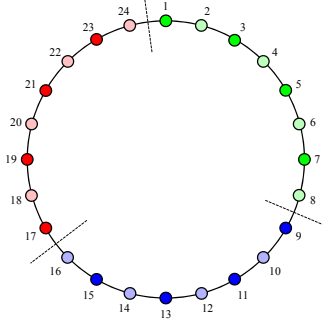


Fig. 1: Example of an equilibrium coloring when $n = 24$ and $k = 6$. There are three sections of 8 contiguous vertices. Each section uses two colors.

fact, for any $q \in S_i$, $i \leq z - 1$, we actually have

$$U_{f_i(q)}(c^{\text{ne}}) = \frac{k}{2} + 1,$$

as any node can observe all nodes choosing its same color. Since we are restricting our attention to proper colorings, this means that any agent can only deviate to alternative coloring choices not in its section. Since the locality coefficient is k , each node $i \in S_j$, $j \in \{1, \dots, z\}$, is able to observe the colors of at most k nodes in either set S_{j-1} or S_{j+1} . Because of the alternating colors in each section, we have that for any color $c'_i \in \mathcal{C}$

$$U_i(c'_i, c^{\text{ne}}) \leq \frac{k}{2} + 1.$$

Hence, c^{ne} is in fact an equilibrium as no agent has an incentive to deviate from the color c_i^{ne} . Figure 1 provides an illustration of one such coloring profile.

Since the optimal coloring profile uses either 2 or 3 colors, depending on whether n is even or odd, a lower bound on the price of anarchy is

$$\text{PoA}(n, k) \geq \frac{2}{3} \left\lfloor \frac{n}{k+2} \right\rfloor \geq \frac{n}{6k},$$

where the last inequality holds whenever $k \geq 2$ and $n \geq 2k$.

We now move on to a proof of the upper bound for the number of colors used in any equilibrium c^{ne} . To do this, we focus on two different classes of colors used in c^{ne} , which we refer to as *global* and *local* colors respectively. These color types are defined as follows:

- A global color $c_g \in \mathcal{C}$ in a coloring profile $c \in \mathcal{C}^n$ is any color that populates the entire ring graph. In particular, given any contiguous k nodes $\{i+1, i+2, \dots, i+k\}$, there exists at least one node $j \in \{i+1, i+2, \dots, i+k\}$ such that $c_j = c_g$.
- A local color is any color that is not a global color. Accordingly, given any coloring profile $c \in \mathcal{C}^n$, each local color $c_\ell \in \mathcal{C}$ has both a starting index and ending index, denoted by $\underline{\ell}$ and $\bar{\ell}$ respectively, where $c_\ell = c_{\bar{\ell}} = c_\ell$ and $c_j \neq c_\ell$ for any $j \in \{\underline{\ell} - k, \dots, \underline{\ell} - 1\} \cup \{\bar{\ell} + 1, \dots, \bar{\ell} + k\}$.

The number of colors use in an equilibrium coloring profile is

$$|c^{\text{ne}}| = |c^{\text{ne}}|_g + |c^{\text{ne}}|_\ell, \quad (7)$$

where $|\cdot|_g$ and $|\cdot|_\ell$ capture the number of local and global

colors in the coloring profile c^{ne} . Trivially, we have that $|c^{\text{ne}}|_g \leq k$. The remaining part of this proof will focus on establishing an upper bound on the number of local colors, $|c^{\text{ne}}|_\ell$.

To construct an upper bound on $|c^{\text{ne}}|_\ell$, we focus on identifying how many local colors can have a starting index in a segment of contiguous vertices of length k , which for simplicity we represent by the indices $\{1, 2, \dots, k-1, k\}$. Suppose m local colors, expressed by $\mathcal{C}^m = \{c^1, \dots, c^m\}$, are introduced in this segment with starting indices satisfying $1 \leq \underline{\ell}_1 < \dots < \underline{\ell}_m < k$. Given that the locality coefficient is k , for any two agents $i, i' \in \{\underline{\ell}_1, \dots, \underline{\ell}_m\}$ with $i < i' < i+k$, we have the property that

$$\begin{aligned} U_i(c^{\text{ne}}) &= \sum_{j=i-k}^{i+k} I\{c_j = c_i^{\text{ne}}\}, \\ &= \sum_{j=i}^{i+k} I\{c_j = c_i^{\text{ne}}\}, \\ &\leq \sum_{j=i'-k}^{i'+k} I\{c_j = c_i^{\text{ne}}\}. \end{aligned}$$

Hence, if agent i' was able to switch to color c_i^{ne} , i.e., $c_i^{\text{ne}} \notin \{c_{i'-1}^{\text{ne}}, c_{i'}^{\text{ne}}\}$, then the utility of agent i' must satisfy $U_{i'}(c^{\text{ne}}) > U_i(c^{\text{ne}})$ or else c^{ne} would not be an equilibrium coloring profile. We will use this fact to prove an upper bound on the number of local colors used in a equilibrium coloring profile.

To that end, for each node $\ell_i \in \{\underline{\ell}_1, \dots, \underline{\ell}_m\}$, define the following specific set of available coloring choices as

$$\mathcal{C}^{(i)} = \{c^1, \dots, c^{i-1}\} \setminus \{c_{i-1}^{\text{ne}}, c_{i+1}^{\text{ne}}\}. \quad (8)$$

Accordingly $\mathcal{C}^{(i)}$ defines a partial set of colors, i.e., all colors started before node i in this segment, that node i could potentially switch to. If c^{ne} represents an equilibrium coloring profile, then we know that for each node $i \in \{\underline{\ell}_1, \dots, \underline{\ell}_m\}$ and any coloring choice $c^j \in \mathcal{C}^{(i)}$ with starting index $\underline{\ell}_j < i$,

$$U_i(c^{\text{ne}}) \geq U_{\underline{\ell}_j}(c^{\text{ne}}) + 1.$$

Hence, for any $\underline{\ell}_j \in \{\underline{\ell}_1, \dots, \underline{\ell}_m\}$, ($j > 2$), there exist a subset of nodes $S_j \subset \{\underline{\ell}_1, \dots, \underline{\ell}_{j-1}\}$ satisfying $|S_j| \geq j-2$, such that

$$U_{\underline{\ell}_j}(c^{\text{ne}}) \geq U_s(c^{\text{ne}}) + 1, \forall s \in S_j.$$

Following this logic, we can bound the sum of the agents utilities as

$$\begin{aligned} \sum_{i=1}^m U_{\underline{\ell}_i}(c^{\text{ne}}) &\geq \sum_{i=1}^{\lfloor \frac{m}{3} \rfloor} i \\ &= \frac{(\lfloor \frac{m}{3} \rfloor)(\lfloor \frac{m}{3} \rfloor + 1)}{2} \\ &> \frac{\binom{m}{6} \binom{m}{3}}{2} = \frac{m^2}{36}. \end{aligned} \quad (9)$$

Since $1 \leq \underline{\ell}_1 < \dots < \underline{\ell}_m < k$, we know that the agents' utilities rely solely on the colors of the nodes in the section

$\{1, 2, \dots, 2k-1, 2k\}$. Furthermore, since $c_{\ell_i} \neq c_{\ell_j}$ for any distinct $\ell_i \neq \ell_j \in \{\underline{\ell}_1, \dots, \underline{\ell}_m\}$, we have that

$$\sum_{i=1}^m U_{\ell_i}(c^{\text{ne}}) \leq 2k. \quad (10)$$

Combining (9) with (10) gives us that

$$m < 8\sqrt{k}. \quad (11)$$

The inequality given in (11) implies an upper bound on the number of local colors that can be introduced in any contiguous segment of k nodes in a given equilibrium coloring profile. Accordingly, we have

$$|c^{\text{ne}}|_{\ell} \leq \left\lceil \frac{n}{k} \right\rceil (8\sqrt{k}) \leq \frac{2n}{k} (8\sqrt{k}) = 16 \frac{n}{\sqrt{k}}.$$

Therefore, we have that

$$|c^{\text{ne}}| = |c^{\text{ne}}|_g + |c^{\text{ne}}|_{\ell} \leq k + 16 \frac{n}{\sqrt{k}} \leq 17 \frac{n}{\sqrt{k}}$$

where the last inequality holds for the case when $n \geq k^{3/2}$. Hence, the price of anarchy satisfies

$$\text{PoA}(n, k) \leq \frac{17}{2} \left(\frac{n}{\sqrt{k}} \right),$$

which completes the proof. \blacksquare

C. Length of better reply paths for ring graphs

We now turn our attention to characterizing the length of the better reply paths for graph coloring games over ring graphs. To that end, let $\beta(n, k)$ denote the family of better reply paths associated with the graph coloring game $\{n, k\}$. Here, different better reply paths can be a result of different initial conditions or different agent deviations. Given a specific better reply path $\beta \in \beta(n, k)$, we denote the length of the better reply bath by $|\beta|$. The following theorem provides both upper bounds and lower bounds on the length of the better reply paths in the set $\beta(n, k)$.

Theorem 3.3: Consider a graph coloring game $\{n, k\}$ over a ring graph. If n/k is sufficiently large, then the length of the better reply paths in $\beta(n, k)$ can be bounded as

$$\gamma_3 \cdot (nk) \geq \max_{\beta \in \beta(n, k)} |\beta| \geq \gamma_4 \cdot (n\sqrt{k}) \quad (12)$$

where $\gamma_3, \gamma_4 > 0$ are constants that do not depend on n or k .⁴

Theorem 3.2 demonstrates that increasing the locality coefficient improves the efficiency of the resulting equilibrium coloring profiles. Accordingly, it would appear that a reasonable approach for improving system behavior would be to provide the nodes with additional information regarding the color choice of other nodes. Theorem 3.3 shows that there is a consequence associated with this approach to improving the efficiency of the resulting equilibrium colorings – a degradation in the underlying convergence rates. In particular, the lower bound in (12) shows that increasing the level of information to the nodes degrades the length of the

worst-case better reply path. Here, the length of the worst-case better reply path is a reasonable proxy for understanding the convergence rates of a better reply process [15].

Proof: Before beginning with the proof, recall that a better reply path is a sequence of coloring profiles c^1, c^2, \dots, c^m where for each $y \in \{1, \dots, m-1\}$ there exists a node i with a color $c_i \in \mathcal{C}$ such that $c^{y+1} = (c_i, c_{-i}^y)$ and $U_i(c^{y+1}) > U_i(c^y)$. Furthermore, due to the nature of the proposed coloring game, this strict inequality also implies $U_i(c^{y+1}) - U_i(c^y) \geq 1$.

The proof of Theorem 3.3 relies heavily on the fact that the proposed coloring game is a potential game with the potential function given in (5). The existence of this potential function implies that the potential associated with any two successive coloring profiles in a better reply process satisfies

$$\phi(c^{y+1}) - \phi(c^y) = U_i(c^{y+1}) - U_i(c^y) \geq 1.$$

Accordingly, if a better reply path β can be constructed that goes from a coloring profile c to c' , i.e., $\beta = \{c, c^0, c^1, \dots, c^m, c'\}$, then the length of this better reply path satisfies

$$|\beta| \leq \phi(c') - \phi(c).$$

The length of this path will be precisely equal to $\phi(c') - \phi(c)$ if the path consisted solely of deviations where the payoff difference associated with the deviating player was 1. Both of these properties will be used extensively in the forthcoming proof.

We begin with the upper bound. Using the arguments constructed above, an upper bound can be derived by

$$\begin{aligned} \max_{\beta \in \beta(n, k)} |\beta| &\leq \max_{c^+, c^- \in \mathcal{C}^n} (\phi(c^+) - \phi(c^-)) \\ &= \max_{c^+} \phi(c^+) - \min_{c^-} \phi(c^-) \\ &\leq n(k+1) - nk. \end{aligned}$$

We now move on to the lower bound which involves specifying a particular better reply path that has the desired number of steps. Throughout, we focus on the case where k is even for simplicity as similar arguments can be constructed for the odd case. Consider a contiguous sequence of k nodes, denoted by $\{1, \dots, k\}$ where each node i begins with a unique colors c_i , i.e., $c_i \neq c_j$ for any $i, j \leq k$, $i \neq j$. To avoid dealing with the constraints associated with neighboring colors, we will focus on deriving a better reply path over just the $k/2$ odd numbered nodes $\{1, 3, \dots, k-1\}$ which, with an abuse of notation, we will renumber as $\{1, 2, \dots, k/2\}$ for simplicity. Lastly, let $|c|$ be defined as the sorted number of each color used in the coloring profile c for the nodes $\{1, 2, \dots, k/2\}$. That is, $|c| = (x_1, \dots, x_{k/2})$ where each $x_i \in \{0, 1, \dots, k/2\}$ denotes the number of nodes in $\{1, 2, \dots, k/2\}$ choosing the i -th most common color, the tuple is sorted in the sense that $x_i \geq x_{i+1}$ for all i , and lastly $x_1 + \dots + x_{k/2} = k/2$. Note that for the initial coloring profile, say c^0 , we have $|c^0| = (1, 1, \dots, 1, 1)$. Alternatively, if all agents chose the same color in the color profile c , then $|c| = (k/2, 0, \dots, 0, 0)$. Lastly, let $|c|_j$ denote the j -th term of the tuple.

⁴The term n/k is sufficiently large for any n and k satisfying $k \geq 10$ and $n \geq 2k$.

Consider a better reply path $\beta = \{c^0, c^1, \dots\}$ constructed according to the following process. Let c^m be the current coloring profile. Suppose $|c^m|_i = |c^m|_j$ for some $i \neq j$. This means that there exists two nodes, say i and j , such that $U_i(c^m) = U_j(c^m)$ and $c_i^m \neq c_j^m$. Consider a new coloring profile of the form $c^{m+1} = (c_i = c_j^m, c_{-i}^m)$ where node i deviated to node j 's color. Note that such a move is possible since there are no neighboring constraints as we are only dealing with odd number vertices. Given this move, we have

$$U_i(c^{m+1}) - U_i(c^m) = 1.$$

Continue repeating this process until no such moves exist and denote this final coloring profile as c^z . Note that this process must eventual stop since potential game prohibit cyclic behavior in any better reply path.

We now focus on analyzing properties of this coloring profile c^z . First, note that for distinct i, j , $|c^z|_i = |c^z|_j$ only for the case when $|c^z|_i = |c^z|_j = 0$. Focusing on the nodes' utilities, we have

$$\sum_{i=1}^{k/2} U_i(c^z) = \sum_{i=1}^{k/2} (|c^z|_i)^2 \geq \sum_{i=1}^{\lfloor \sqrt{k/2} \rfloor} i^2, \quad (13)$$

where (13) follows from

$$\begin{aligned} \sum_{i=1}^{\lfloor \sqrt{k/2} \rfloor} i &= \frac{\lfloor \sqrt{k/2} \rfloor (\lfloor \sqrt{k/2} \rfloor + 1)}{2}, \\ &\leq \frac{\sqrt{k/2} (\sqrt{k/2} + 1)}{2}, \\ &\leq \frac{k}{2}. \end{aligned}$$

Define $\alpha = \lfloor \sqrt{k/2} \rfloor$. Since each unilateral deviation involves a payoff improvement of 1 for the deviating player, the length of this better reply path for $\alpha \geq 4$, which is equivalent to $k \geq 10$, is bounded by

$$\begin{aligned} \phi(c^z) - \phi(c) &\geq \frac{1}{2} \left(\sum_{i=1}^{k/2} U_i(c^z) - \sum_{i=1}^{k/2} U_i(c^0) \right), \\ &\geq \frac{1}{2} \left(\sum_{i=1}^{\alpha} i^2 - \frac{k}{2} \right), \\ &\geq \frac{1}{2} \left(\sum_{i=1}^{\alpha} i^2 - (\alpha + 1)^2 \right), \\ &\geq \frac{1}{2} \sum_{i=1}^{\alpha-2} i^2 = \frac{1}{2} \sum_{i=1}^{q\alpha} i^2, \end{aligned}$$

where $q = 1 - 2/\alpha \geq 1/2$. Continuing this derivation gives us

$$\phi(c^z) - \phi(c) \geq \frac{(q\alpha + 1)(2q\alpha + 1)q\alpha}{6} \geq \frac{(q\alpha)^3}{3} \geq \frac{k^{3/2}}{96}.$$

Note that this path was purely constructed over the odd numbered nodes of a single contiguous segment of k nodes. Building on this construction, one can establish a better reply path merely by stringing together such move sequences for

both the odd and even nodes of each segment of length k , as each sub-segment is operating on a distinct set of colors. The length of this better reply process must satisfy

$$|\beta| \geq 2 \lfloor \frac{n}{k} \rfloor \frac{k^{3/2}}{96} \geq \frac{n\sqrt{k}}{96},$$

which completes the proof. \blacksquare

IV. CONCLUSION

In this paper we studied the impact of information on a simple distributed graph coloring problem. The findings demonstrated that while increasing the information available to the nodes led to improvements in the efficiency of the resulting equilibrium coloring profiles, these improvements came at the expense of the underlying convergence rates. Hence, there is an apparent tradeoff between efficiency and convergence rates with information serving as the tradeoff mechanism. Whether this apparent tradeoff between efficiency and convergence rates is fundamental to multiagent coordination remains an open and interesting question and a focus of future work.

REFERENCES

- [1] F. Bullo, R. Carli, and P. Frasca. Gossip coverage control for robotic networks: Dynamical systems on the space of partitions. *SIAM Journal on Control and Optimization*, 50:419–447, January 2012.
- [2] A. Ganguli, J. Cortes, and F. Bullo. Distributed coverage of nonconvex environments. In V. Saligrama, editor, *Networked Sensing Information and Control (Proceedings of the NSF Workshop on Future Directions in Systems Research for Networked Sensing, May 2006, Boston, MA)*, Incis, pages 289–305. sv, 2007.
- [3] A. Ganguli, S. Susca, S. Martinez, F. Bullo, and J. Cortes. On collective motion in sensor networks: sample problems and distributed algorithms. In *Proceedings of the 44th IEEE Conference on Decision and Control*, pages 4239–4244, Seville, Spain, December 2005.
- [4] M. Kearns, S. Suri, and N. Montfort. An experimental study of the coloring problem on human subject networks. *Science*, 313(5788):824–827, 2006.
- [5] A. Kirkoryan and M. A. Belabbas. Decentralized stabilization with symmetric topologies. In *2014 IEEE 53rd Annual Conference on Decision and Control (CDC)*, pages 1347–1352, Dec 2014.
- [6] C. Langbort and J. Delvenne. Distributed design methods for linear quadratic control and their limitations. *Automatic Control, IEEE Transactions on*, 55(9):2085–2093, Sept 2010.
- [7] A. Mahajan, N.C. Martins, M.C. Rotkowitz, and S. Yuksel. Information structures in optimal decentralized control. In *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, pages 1291–1306, Dec 2012.
- [8] J.R. Marden. The role of information in multiagent coordination. In *2014 IEEE 53rd Annual Conference on Decision and Control (CDC)*, pages 445–450, Dec 2014.
- [9] S. Martinez, J. Cortes, and F. Bullo. Motion coordination with distributed information. *Control Systems Magazine*, 27(4):75–88, 2007.
- [10] D. Monderer and L. Shapley. Potential games. *Games and Economic Behavior*, 14:124–143, 1996.
- [11] Nisan N, T. Roughgarden, E. Tardos, and V. Vazirani. *Algorithmic game theory*, volume 1. Cambridge University Press Cambridge, 2007.
- [12] P. Panagopoulou and P. Spirakis. A game theoretic approach for efficient graph coloring. In S.-H. Hong, N. Nagamochi, and T. Fukunaga, editors, *Lecture notes in computer science*, pages 183–195. Springer-Verlag, 2008.
- [13] R.S. Tomar, S. Singh, S. Verma, and G.S. Tomar. A novel abc optimization algorithm for graph coloring problem. In *Computational Intelligence and Communication Networks (CICN), 2013 5th International Conference on*, pages 257–261, Sept 2013.
- [14] B. Touri and J. Shamma. Global games with noisy sharing of information. In *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, pages 4473–4478, Dec 2014.
- [15] H. P. Young. *Strategic Learning and its Limits*. Oxford University Press, 2005.