Stable Utility Design for Distributed Resource Allocation*

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Abstract— **The framework of resource allocation games is becoming an increasingly popular modeling choice for distributed control and optimization. In recent years, this approach has evolved into the paradigm of *game-theoretic control*, which consists of first modeling the interaction between the distributed agents as a strategic form game, and then designing *local* utility functions for these agents such that the resulting game possesses a stable outcome (e.g., a pure Nash equilibrium) that is efficient (e.g., good "price of anarchy" properties). One then appeals to the large, existing literature on *learning in games* for distributed algorithms for agents that guarantee convergence to such an equilibrium. An important first problem is to obtain a characterization of stable utility designs, that is, those that guarantee equilibrium existence for a large class of games. Recent work has explored this question in the general, multi-selection context, that is, when agents are allowed to choose more than one resource at a time, showing that the only stable utility designs are the so-called "weighted Shapley values". It remains an open problem to obtain a similar characterization in the *single-selection* context, which several practical problems such as vehicle target assignment, sensor coverage, etc. fall into. We survey recent work in the multi-selection scenario, and show that even though other utility designs become stable for specific single-selection applications, perhaps surprisingly, in a broader context, the limitation to "weighted Shapley value" utility design continues to prevail.**

I. INTRODUCTION

Resource allocation is a fundamental problem that is at the core of several application domains ranging from socioeconomic systems to systems engineering. A persistent example in the computer science literature is that of routing data through a shared-link network, where the global objective is to minimize the average delay [1], [2]. Another example in multiagent systems is the problem of deploying sensors in a given mission space where the global objective is to maximize the area covered and/or the quality of coverage [3]. The central objective in all these problems is to allocate resources to optimize some global objective. Increasingly, these problems need to be solved in a distributed, decentralized manner, especially in large scale engineering systems.

Game-theoretic control has emerged as a promising approach for distributed resource allocation (see [4] and references therein). This approach is motivated by the fact that the underlying decision making architecture in economic systems and distributed engineering systems is identical. That is, local decisions based on local information result in emergent global behavior. Game theoretic control consists of two distinct steps. First, the interactions of autonomous agents are modeled within the framework of a strategic form game where the agents are modeled as independent decision making entities. This involves specifying decision makers ("players"), their respective choices ("action sets"), and a local utility function for each agent, so that the resulting game has an equilibrium (e.g., pure Nash equilibria), which takes on the role of a stable operating point. The second step is to specify a local "learning rule" for the agents according to which they process available information to make individual decisions that collectively steer the system towards an equilibrium of the game. The goal is to complete these two design steps, referred to as *utility design* and *learning design* respectively, in order to ensure that the emergent global behavior is desirable [4].

Our focus in this paper is on utility design for separable resource allocation problems, where "separable" means that the global objective function can be decomposed into local objective functions for each resource. A key constraint is that the design must be "local", meaning that an agent's utility can only depend on the resources selected, the objective at each resource, and other agents that selected the same resources. Therefore, designing local utility functions reduces to the problem of defining a "distribution rule" that specifies how the welfare garnered from each resource is distributed to the players who have chosen that particular resource.

A fundamental research problem in utility design is to characterize the space of distribution rules that guarantee equilibrium existence in resource allocation games. Such a characterization would provide a structured search space while optimizing for distribution rule(s) whose equilibria have the best efficiency properties (how well the equilibrium outcome performs in relation to the globally optimal outcome) and/or are "easy" to converge to. First, [5], [6] showed the existence of a special, worst-case, welfare function for which, any *budget-balanced* distribution rule (one that completely distributes the welfare garnered among the agents without surplus or deficit) that guarantees the existence of a pure Nash equilibrium in any resource allocation game must be equivalent to a "weighted Shapley value".[1] Then, [7] generalized this, showing that the weighted Shapley value characterization holds for *any* welfare function, and this holds true even if the budget-balance constraint is dropped.

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[1]The (weighted) Shapley value, defined later in (4), is a game-theoretic solution concept of enormous importance in the economics literature.

The above work does not, however, investigate characterizations of stable distribution rules when agents are restricted to choosing only a single resource. In such *single-selection* settings, distribution rules other than weighted Shapley values may also guarantee equilibrium existence. In fact, recent work suggests that the landscape of single-selection resource allocation games could be vastly different. In particular, [8] shows that for coverage games such as vehicle target assignment, a much simpler distribution rule called "proportional share" guarantees equilibrium existence. This compels the question: Is the emergence of such alternate stable distribution rules simply for a few "isolated" welfare functions related to coverage-type problems, or is it a broader phenomenon to be expected for all welfare functions?

Our goal in this paper is to explore this question by investigating characterizations of the space of distribution rules that guarantee equilibrium existence in *single-selection* resource allocation games for *all* welfare functions. Since single-selection games are just a special subclass of multi-selection games, weighted Shapley values continue to guarantee equilibrium existence. However, other feasible distribution rules may also guarantee equilibrium existence, and characterizing such rules is important, since (i) weighted Shapley values cannot be computed in polynomial time; therefore, simpler distribution rules such as proportional share are practically appealing, and (ii) recent work [2] has shown that there are settings where distribution rules other than weighted Shapley values result in more efficient equilibria.

Outline and Contributions: We present our formal model in Section II. For concreteness, we describe the vehicle target assignment problem as an illustrative example in Section III, where we compare and contrast two common budget-balanced distribution rules, namely, weighted Shapley value and proportional share distribution rules. Doing so exposes a stark contrast between a general result limiting utility design to weighted Shapley values in the multi-selection scenario and a special result that identifies other stable utility designs in specific single-selection scenarios.

Motivated by this observation, and the lack of any general characterization results for single-selection games, we investigate whether the extra "flexibility" in utility design obtained by restricting the structure of the action sets to single-selection, for the specific setting of coverage games, is a phenomenon that holds more generally. Our results, presented in Section IV, provide the first complete characterizations for budget-balanced[2] distribution rules that guarantee equilibrium existence for single-selection resource allocation games. In particular, we show that:

1) The only linear budget-balanced distribution rules that guarantee equilibrium existence in all single-selection games, for *all* welfare functions, are weighted Shapley values (Theorem 1).
2) Given *any* linear welfare function with no dummy players,[3] the only budget-balanced distribution rules that guarantee equilibrium existence in all single-selection games are weighted Shapley values (Theorem 2).
3) Given *any* welfare function, the only budget-balanced distribution rules that guarantee equilibrium existence in all *two-player* single-selection games are weighted Shapley values (Theorem 3).

Thus, perhaps surprisingly, we show that the limitation to weighted Shapley value utility design (and the accompanying limitation to (weighted) potential games) persists in a broader sense, even when restricted to single-selection action sets.

II. MODEL

We consider a simple, but general, model of a resource allocation game, where there is a set of self-interested agents/players $N = \{1, \ldots, n\}$ $(n > 1)$ that each select a collection of resources from a set $R = \{r_1, \ldots, r_m\}$ $(m > 1)$. That is, each agent $i \in N$ is capable of selecting potentially multiple resources in R (*multi-selection* scenario); therefore, we say that agent i has an action set $\mathcal{A}_i \subseteq 2^R$. If, on the other hand, agents are restricted to selecting only a single resource, resulting in their action sets consisting of only singleton actions, then we are in the *single-selection* scenario. The resulting action profile, or (joint) allocation, is a tuple $a = (a_1, \ldots, a_n) \in \mathcal{A}$ where the set of all possible allocations is denoted by $\mathcal{A} = \mathcal{A}_1 \times \ldots \times \mathcal{A}_n$. We occasionally denote an action profile a by (a_i, a_{-i}) where $a_{-i} \in \mathcal{A}_{-i}$ denotes the actions of all agents except agent i.

Each allocation generates a "welfare", $\mathcal{W}(a)$, which corresponds to the global objective function of the optimization problem that a system manager would seek to maximize. We assume $\mathcal{W}(a)$ is *(linearly) separable* across resources, i.e., $\mathcal{W}(a) = \sum_{r \in R} W_r(\{a\}_r)$, where $\{a\}_r = \{i \in N : r \in a_i\}$ is the set of agents that are allocated to resource r in a, and $W_r : 2^N \to \mathbb{R}$ is the local welfare function at resource r. This is a standard assumption [12], [13], [5], [8], and is quite general. Without loss of generality, we assume $W_r(\emptyset) = 0$ for all $r \in R$. Let the space of all such welfare functions be denoted by $\mathbb{W} = \{W : 2^N \to \mathbb{R} \mid W(\emptyset) = 0\}$.

The goal is to design the utility functions $U_i : \mathcal{A} \to \mathbb{R}$ that the agents seek to maximize, often without exact knowledge of what the $\{W_r\}$ are, so that the design is applicable more generally. Because the welfare is assumed to be separable, it is natural that the utility functions should follow suit, i.e.,

$$U_i(a) = \sum_{r \in a_i} f(W_r)(i, \{a\}_r), \tag{1}$$

where $f(W_r) : \mathbb{W} \times N \times 2^N \to \mathbb{R}$ is the *local distribution rule*, i.e., $f(W_r)(i, S)$ is the portion of the local welfare W_r that is allocated to agent $i \in S$ when sharing with S. We refer to the operator f simply as the *distribution rule*. It maps a given welfare function W to the corresponding W-*specific distribution rule* $f(W)$. For completeness, we define $f(W)(i, S) := 0$ whenever $i \notin S$.

[2] Unlike economic systems where monetary incentives are involved, there is no obvious motivation for requiring budget-balance in designing utilities for distributed agents in engineering systems. However, a bevy of recent work (e.g., [9], [10], [11]) has shown that imposing budget constrains on utility design leads to good equilibrium efficiency guarantees.

[3] Note that such welfare functions, which we define precisely in Section IV, constitute a large class.

The requirement (1) on the functional form of an agent's utility function embodies the feature of *locality* that is often a critical constraint in distributed systems, i.e., each agent's utility function only depends on its selected resources and the agents that select the same resources.

In order to simplify the statement and discussion of our results, we impose a *scalability* constraint by assuming that there is just one *base welfare function* W, that is scaled at each resource r by a strictly positive coefficient, $v_r \in \mathbb{R}_{++}$. That is, $W_r = v_r W$, for all $r \in R$. Correspondingly, the distribution rule f should also be *scalable*, i.e., for any $W \in \mathbb{W}$ and any $c \in \mathbb{R}$, we have $f(cW) = cf(W)$. However, our results hold more generally.

A stricter requirement than scalability is *linearity*: A distribution rule f is said to be *linear* if, for any $(W_1, W_2) \in \mathbb{W}^2$ and any $(c_1, c_2) \in \mathbb{R}^2$, we have $f(c_1 W_1 + c_2 W_2) = c_1 f(W_1) + c_2 f(W_2)$.

A distribution rule f is said to be *budget-balanced* if, for any welfare function $W \in \mathbb{W}$ and any player set $S \subseteq N$, $\sum_{i \in S} f(W)(i, S) = W(S)$.

We represent a (separable and scalable) resource allocation game as:

$$G = \left(N, R, \{\mathcal{A}_i\}_{i \in N}, \{v_r\}_{r \in R}, W, f\right), \tag{2}$$

and the design of f is the focus of this paper.

The primary goals when designing the distribution rules are to guarantee (i) equilibrium existence, and (ii) equilibrium efficiency. Our focus in this work is entirely on (i) and we consider pure Nash equilibria; however, other equilibrium concepts are also of interest [14], [15], [16]. Recall that a *(pure Nash) equilibrium* is an outcome $a^* \in \mathcal{A}$ such that

$$(\forall\ i \in N) \quad U_i(a_i^*, a_{-i}^*) \ = \ \max_{a_i \in \mathcal{A}_i} U_i(a_i, a_{-i}^*).$$

III. AN ILLUSTRATIVE EXAMPLE: VEHICLE TARGET ASSIGNMENT

The vehicle target assignment problem [17] consists of a finite set of targets (resources) denoted by R and each target $r \in R$ has a relative worth $v_r > 0$. There are a finite number of vehicles (agents) denoted by $N = \{1, 2, \ldots, n\}$. The set of possible assignments for vehicle i is $\mathcal{A}_i \subseteq 2^R$, and $\mathcal{A} = \prod_{i \in N} \mathcal{A}_i$ represents the set of joint assignments. In general, the structure of $\mathcal{A}$ is not available to a system designer a priori. Lastly, each vehicle $i \in N$ is parameterized with an invariant success probability $0 \leq p_i \leq 1$ that indicates the probability vehicle i will successfully eliminate a target r given that $r \in a_i$. The benefit of a subset of agents $S \subseteq N$, $S \neq \emptyset$, being assigned to a target r is $W_r(S) = v_r W(S)$, where

$$W(S) = \left(1 - \prod_{i \in S} (1 - p_i)\right) \tag{3}$$

represents the joint probability of successfully eliminating target r. Accordingly, the target assignment problem is to find a joint assignment $a \in \mathcal{A}$ that maximizes the global objective $\mathcal{W}(a) = \sum_{r \in R} W_r\left(\{a\}_r\right)$, where $\{a\}_r = \{i \in N \ : \ r \in a_i\}$.

Note that this is a scalable and separable resource allocation problem that fits our model. Thinking of W_r as the local "welfare" garnered at resource r, designing local agent utilities of the form (1) for this problem reduces to designing the distribution rule f. Next, we discuss two common budget-balanced distribution rules for this problem, both parameterized by $\boldsymbol{\omega} \in \mathbb{R}_{++}^{|N|}$, a vector of strictly positive player weights.

The **weighted Shapley value** distribution rule [18], [19], denoted by $f_{WSV}[\boldsymbol{\omega}]$, is defined as:[4]

$$f_{WSV}[\boldsymbol{\omega}](W)(i, S) = \sum_{T \subseteq S : i \in T} \frac{\omega_i}{\sum_{j \in T} \omega_j} \left(\sum_{R \subseteq T} (-1)^{|T| - |R|} W(R) \right). \tag{4}$$

The importance of this distribution rule is that it is *universal* — it guarantees equilibrium existence in *any* resource allocation game.[5] Moreover, Gopalakrishnan et al. [7] recently established that when action sets are not restricted to be single-selection, the converse is true: given any welfare function $W \in \mathbb{W}$, any W-specific distribution rule (not necessarily budget-balanced) that guarantees equilibrium existence in all games *must* be a W'-specific weighted Shapley distribution rule, $f_{WSV}[\boldsymbol{\omega}](W')$, for some $W' \in \mathbb{W}$ and some weight-vector $\boldsymbol{\omega}$. However, the key drawback is that computing $f_{WSV}[\boldsymbol{\omega}](W)$ is often intractable, since it requires exponentially many calls to W.

The **proportional share** distribution rule, denoted by $f_{PR}[\boldsymbol{\omega}]$, is defined as:

$$f_{PR}[\boldsymbol{\omega}](W)(i, S) = \frac{\omega_i}{\sum_{j \in S} \omega_j} W(S). \tag{5}$$

In other words, the welfare is simply divided among the players in proportion to their weights. Note that this is very easy to compute, requiring just a single call to W. It also guarantees equilibrium existence for *single-selection* instances of the vehicle target assignment problem, when the player weights ω_i correspond to the vehicle success probabilities p_i [8].[6] However, it does not generally guarantee equilibrium existence in other settings, e.g., a welfare function that is different from the one in (3).

When the players are restricted to selecting only a single resource, the lack of a tight characterization (such as the one in [7] for the multi-selection scenario) leaves many questions unanswered: What other distribution rules guarantee a Nash equilibrium for the *single-selection* vehicle target assignment problem? How about distribution rules for welfare functions other than (3)? Are there *universal* distribution rules other than the weighted Shapley value that can guarantee a Nash equilibrium for *all single-selection* games?

[4] One way to interpret this distribution rule is as follows. For any given subset of players S, imagine these players arriving one at a time to the resource, according to some order π. Each player i can be thought of as contributing $W(P_i^\pi \cup \{i\}) - W(P_i^\pi)$ to the welfare $W(S)$, where P_i^π denotes the set of players in S that arrived before i in π. This is the "marginal contribution" of player i to the welfare, according to the order π. Then, the weighted Shapley value is the expected marginal contribution, where the expectation is over all $|S|!$ orders, according to a probability distribution determined by the weights $\boldsymbol{\omega}$.

[5] This is a consequence of the resulting games being so-called (weighted) "potential games", a special class of games for which there are many well understood learning dynamics that guarantee equilibrium convergence.

[6] This result is a special case of the following more general (partial) characterization of Marden and Wierman [8]: given *any* welfare function W, they outline three sufficient conditions for a W-specific distribution rule to guarantee equilibrium existence for all single-selection games.

IV. MAIN CONTRIBUTIONS

In this section, we attempt to bridge the characterization gap between the single-selection and multi-selection settings, by looking for complete characterizations of distribution rules that guarantee equilibrium existence for all *single-selection* games. Our first result provides just such a characterization – we show that the only such distribution rules that are linear and budget-balanced are weighted Shapley values.

Let $\mathcal{G}_s(N, f, W)$ denote the set of all single-selection resource allocation games G (see (2)) with player set N, welfare function W, and distribution rule f.

Theorem 1: For all base welfare functions $W \in \mathbb{W}$, all games in $\mathcal{G}_s(N, f, W)$ possess a pure Nash equilibrium for a linear budget-balanced f if and only if there exists a weight-vector $\boldsymbol{\omega}$ such that f is the weighted Shapley value distribution rule, $f_{WSV}[\boldsymbol{\omega}]$.

Proof: Note that we only need to prove one direction, since it is known that for any $W \in \mathbb{W}$, any weight-vector $\boldsymbol{\omega}$, all games in $\mathcal{G}_s(N, f_{WSV}[\boldsymbol{\omega}], W)$ have a pure Nash equilibrium [20]. Thus, we present the bulk of the proof – the other direction – proving that any linear budget-balanced distribution rule f that guarantees a pure Nash equilibrium for all games in $\mathcal{G}_s(N, f, W)$ for all $W \in \mathbb{W}$, must be a weighted Shapley value distribution rule $f_{WSV}[\boldsymbol{\omega}]$ for some weight-vector $\boldsymbol{\omega}$.

The proof is carried out by induction on the size of the player set, in four steps: (i) We derive a necessary condition for f to guarantee a Nash equilibrium, regardless of the welfare function. (ii) We show that in games with only two players $\{1, 2\}$, f is completely characterized by a single parameter γ, and is equivalent to $f_{WSV}[(\omega_1, \omega_2)]$ where γ is the weight-ratio ω_1/ω_2. (iii) We show that when there are more than two players, the pairwise weight-ratios obtained in the previous step are all "consistent". (iv) We show that for any set $S \subseteq N$, if $f(W)(\cdot, R)$ for all proper subsets $R \subsetneq S$ is equivalent to $f_{WSV}[\boldsymbol{\omega}](\cdot, R)$ for all $W \in \mathbb{W}$, then $f(W)(\cdot, S)$ is *uniquely* determined. However, since $f_{WSV}[\boldsymbol{\omega}](\cdot, S)$ is a feasible solution for $f(W)(\cdot, S)$, it must be that unique solution, concluding the proof.

For convenience, we assume that for all $S \subseteq N$, $f(W)(i, S)$ must vary with $W(S)$, but the proof can be generalized to cases where this does not hold.

Recall that $\mathbb{W}$ is a $2^n - 1$ dimensional vector space, where $n = |N|$. Let $\{S_1, S_2, \ldots, S_{2^n-1}\}$ be an ordering of the non-empty subsets of N, chosen such that $|S_j| \leq |S_k|$ $\forall j < k$. We can represent a welfare function $W \in \mathbb{W}$ as a vector $\overrightarrow{W} = [W(S_1) \;\; W(S_2) \;\; \ldots \;\; W(S_{2^n-1})]^T$, and the related player shares $f(W)(i, \cdot)$ as the vector $\overrightarrow{f}(W)(i, \cdot) = [f(W)(i, S_1) \;\; f(W)(i, S_2) \;\; \ldots \;\; f(W)(i, S_{2^n-1})]^T$.

Any linear distribution rule may now be interpreted as a set of n matrices $F = \{F^1, F^2, \ldots, F^n\}$, with $F^i\overrightarrow{W} = \overrightarrow{f}(W)(i, \cdot) \;\; \forall i \in N$.

Let $\{e_S\}$ be the standard basis for $\mathbb{W}$. We define $F^i_S = (e_S)^T F^i$ as the row of F^i associated with the set S and $F^i_{S,R} = (e_S)^T F^i e_R$ is element in the row associated with S and the column associated with R.

In the vector notation, the budget-balance condition becomes

$$\sum_{i \in N} F^i = I, \qquad (6)$$

where I is the identity matrix.

Next, we take a moment to reformulate the weighted Shapley Value distribution rule $f_{WSV}[\boldsymbol{\omega}]$ defined in (4) in terms of our vector notation. It can be shown that the term in the F^i matrix corresponding to the row for subset S and the column for subset R is

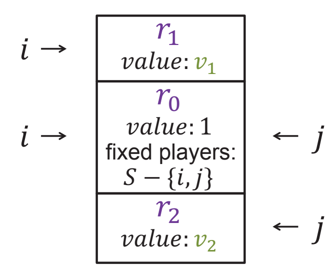
$$F^i_{S,R} = \omega_i (-1)^{|R|} \sum_{R \subseteq T \subseteq S} \frac{(-1)^{|T|}}{\sum_{j \in T} \omega_j}. \qquad (7)$$

A. A Necessary Condition

Lemma 1: If f is a distribution rule that guarantees equilibrium existence for all games $G \in \mathcal{G}_s(N, f, W)$ for all $W \in \mathbb{W}$, then, for all $S \subseteq N$, for all $i, j \in S$, $F^i_S - F^i_{S\setminus\{j\}} = \frac{c_i}{c_j}(F^j_S - F^j_{S\setminus\{i\}})$, where $c_i = F^i_{S,S}$ and $c_j = F^j_{S,S}$.

Proof: Consider the game in Figure 1a, with resource set $R = \{r_0, r_1, r_2\}$ and local resource coefficients $v_{r_1} = v_1$, $v_{r_2} = v_2$, and $v_{r_0} = 1$, and base welfare function W. Player i can choose between the resources r_1 and r_0, and player j can choose between the resources r_2 and r_0. All other players in S have a fixed action – they choose resource r_0. This is essentially a game between i and j with payoff matrix in Figure 1b.

(a) The game



(b) The payoff matrix

$i \backslash j$	r_2	r_0
r_1	$v_1 W(\{i\}), v_2 W(\{j\})$	$v_1 W(\{i\}), f(W)(j, S\setminus\{i\})$
r_0	$f(W)(i, S\setminus\{j\}), v_2 W(\{j\})$	$f(W)(i, S),\; f(W)(j, S)$

Fig. 1: Example

A counterclockwise best-response cycle results, when the following set of inequalities hold:

$$v_2 W(\{j\}) > f(W)(j, S),\;\; v_1 W(\{i\}) > f(W)(i, S\setminus\{j\})$$
$$f(W)(j, S\setminus\{i\}) > v_2 W(\{j\}),\;\; f(W)(i, S) > v_1 W(\{i\}).$$

Since W, v_1 and v_2 may be chosen arbitrarily, these reduce to

$$f(W)(j, S\setminus\{i\}) > f(W)(j, S),\;\; f(W)(i, S) > f(W)(i, S\setminus\{j\}).$$

Thus, a necessary condition for the existence of an equilibrium for all such games is that, for all $S \subseteq N$, for all $i, j \in S$,

$$f(W)(i,S) < f(W)(i,S\backslash\{j\}) \Rightarrow f(W)(j,S) \leq f(W)(j,S\backslash\{i\}).$$

In terms of our vector notation, this condition is

$$\langle F_S^i - F_{S\backslash\{j\}}^i, \overrightarrow{W}\rangle < 0 \Rightarrow \langle F_S^j - F_{S\backslash\{i\}}^j, \overrightarrow{W}\rangle \leq 0,$$

where $\langle\cdot,\cdot\rangle$ is the inner product. Since the choice of $\overrightarrow{W}$ is arbitrary, this condition is satisfied if and only if the vectors $F_S^i - F_{S\backslash\{j\}}^i$ and $F_S^j - F_{S\backslash\{i\}}^j$ point in the same direction, that is,

$$F_S^i - F_{S\backslash\{j\}}^i = c(F_S^j - F_{S\backslash\{i\}}^j),$$

for some $c \in \mathbb{R}$. Since $F_{S\backslash\{j\},S}^i = F_{S\backslash\{i\},S}^j = 0$, we must have $c = F_{S,S}^i / F_{S,S}^j$. ■

B. Pairwise Shares Must be Weighted Shapley Values

Lemma 2: If f is a linear budget-balanced distribution rule that guarantees equilibrium existence for all games $G \in \mathcal{G}_s(\{1,2\}, f, W)$ for all $W \in \mathbb{W}$, then there exist weights ω_1, ω_2, such that f is the weighted Shapley value distribution rule, $f_{WSV}[(\omega_1, \omega_2)]$.

Proof: Let the ordering on the non-empty subsets of $\{1,2\}$ be $\{\{1\},\{2\},\{1,2\}\}$. Any linear budget-balanced distribution rule matrices $F = \{F^1, F^2\}$ can be written as

$$F^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & -\beta & \frac{\gamma}{1+\gamma} \end{pmatrix}, \quad F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -\alpha & \beta & \frac{1}{1+\gamma} \end{pmatrix}$$

From Lemma 1 with $S = \{1,2\}$, we have

$$\begin{pmatrix} \alpha - 1 & -\beta & \frac{\gamma}{1+\gamma} \end{pmatrix} = \gamma \begin{pmatrix} -\alpha & \beta - 1 & \frac{1}{1+\gamma} \end{pmatrix}.$$

If we let γ be a free parameter, we may solve for $\alpha = \frac{1}{1+\gamma}$ and $\beta = \frac{\gamma}{1+\gamma}$, which is equivalent to the weighted Shapley value with weight-ratio $\dfrac{w_1}{w_2} = \gamma$. ■

C. Consistency of Pairwise Shares

Lemma 3: If γ_{ij} denotes the weight-ratio for the pairwise shares in the game with players i and j, then, $\forall i,j,k \in N$, $\gamma_{ij}\gamma_{jk} = \gamma_{ik}$.

Proof: Let $S = \{i,j,k\}$. Applying lemma 1 with the three possible pairs of players, we get

$$F_S^i - F_{S\backslash\{j\}}^i = c_{ij}(F_S^j - F_{S\backslash\{i\}}^j) \tag{8a}$$

$$F_S^j - F_{S\backslash\{k\}}^j = c_{jk}(F_S^k - F_{S\backslash\{j\}}^k) \tag{8b}$$

$$F_S^i - F_{S\backslash\{k\}}^i = c_{ik}(F_S^k - F_{S\backslash\{i\}}^k) \tag{8c}$$

Substituting equation (8b) and (8c) into equation (8a),

$$\begin{aligned} & F_{S\backslash\{k\}}^i + c_{ik}(F_S^k - F_{S\backslash\{i\}}^k) - F_{S\backslash\{j\}}^i \\ & \qquad = c_{ij}(F_{S\backslash\{k\}}^j + c_{jk}(F_S^k - F_{S\backslash\{j\}}^k) - F_{S\backslash\{i\}}^j) \\ \Rightarrow \quad & c_{ik}(F_S^k - F_{S\backslash\{i\}}^k) - F_{S\backslash\{j\}}^i + F_{S\backslash\{k\}}^i \\ & \qquad = c_{ij}c_{jk}(F_S^k - F_{S\backslash\{j\}}^k) - c_{ij}(F_{S\backslash\{i\}}^j - F_{S\backslash\{k\}}^j). \end{aligned}$$

Now in the elements corresponding to subset S, the only non-zero terms are in F_S^k and thus $c_{ij}c_{jk} = c_{ik}$, and with this we may cancel out the F_S^k terms on either side, yielding

$$F_{S\backslash\{j\}}^i - F_{S\backslash\{k\}}^i = c_{ik}(F_{S\backslash\{j\}}^k - F_{S\backslash\{i\}}^k) + c_{ij}(F_{S\backslash\{i\}}^j - F_{S\backslash\{k\}}^j)$$

This system of equations (obtained by element-wise comparison) may be solved to get $c_{ik} = \gamma_{ik}$ and $c_{ij} = \gamma_{ij}$. Furthermore, by combining the equations (8a-8c) differently, we get $c_{jk} = \gamma_{jk}$. This completes the proof. ■

D. The Inductive Step

Lemma 4: If f is a distribution rule that guarantees equilibrium existence for all games $G \in \mathcal{G}_s(N, f, W)$ for all $W \in \mathbb{W}$, and if $f(W)(\cdot, R) = f_{WSV}[\boldsymbol{\omega}](W)(\cdot, R)$ for all $W \in \mathbb{W}$ and all $R \subsetneq S$ for some $S \subseteq N$ (with $|S| > 2$), then the set of vectors $\{F_S^i : i \in S\}$ is uniquely determined by the set of vectors $\{F_{S\backslash\{j\}}^i : i,j \in S\}$.

Proof: First, for all subsets $S \subseteq N$, and all $i \in S$, we decompose the vectors F_S^i as $F_S^i = c_i v_i$, where the scalar coefficient $c_i = F_{S,S}^i$. We determine $\{c_i\}$ first, and then show how to construct $\{v_i\}$. Consider an arbitrary ordering of the players in S. For any pair of players $(i, i+1)$, we have, from Lemma 1,

$$v_i - v_{i+1} = \frac{F_{S\backslash\{i\}}^{i+1}}{c_{i+1}} - \frac{F_{S\backslash\{i+1\}}^i}{c_i}, \qquad \forall i \in S,$$

where player $|S| + 1$ is the same as player 1. Adding up these equations, we get

$$0 = \sum_{i \in S} \frac{1}{c_i}\left(F_{S\backslash\{i+1\}}^i - F_{S\backslash\{i-1\}}^i\right)$$

This is an over-constrained system of equations (obtained by element-wise comparison) for $\{c_i\}$, with $2^{|S|} - 2$ equations for $|S|$ variables. Along with the budget-balance condition $\sum\limits_{i \in S} c_i = 1$ (obtained from (6)), it can be shown that this system has the unique solution, $c_i = \frac{\omega_i}{\sum_{k \in S}\omega_k}$. Though we are only looking to prove the uniqueness of the coefficients $\{c_i\}$, we note that expression found matches that of the weighted Shapley value from (7).

We next determine the set of vectors $\{v_i\}$. To do this we return to Lemma 1, this time picking a fixed player $i \in S$ and finding all other $\{v_j\}$ in terms of v_i.

$$c_j v_j = c_j v_i + F_{S\backslash\{i\}}^j - F_{S\backslash\{j\}}^i \frac{c_j}{c_i} \qquad \forall i,j \in S.$$

From budget-balance (6), we have

$$e_S = \sum_{j \in S} c_j v_j = v_i + \sum_{j \in S}\left(F_{S\backslash\{i\}}^j - F_{S\backslash\{j\}}^i \frac{c_j}{c_i}\right),$$

Which allows us to solve for v_i as

$$v_i = e_S - \sum_{j \in S}\left(F_{S\backslash\{i\}}^j - F_{S\backslash\{j\}}^i \frac{c_j}{c_i}\right)$$

Once this is done for all players $i \in S$, we have uniquely extended the distribution rule. ■

TABLE I: Distribution Rules that Guarantee Equilibrium Existence

CLASS OF GAMES	FOR ALL WELFARE FUNCTIONS, BUDGET-BALANCED	FOR A GIVEN WELFARE FUNCTION
Multi-selection	ONLY weighted Shapley values (see [7])	ONLY weighted Shapley values (see [7])
Single-selection	**ONLY Weighted Shapley values (Theorem 1)**	Those satisfying three sufficient conditions (see [8])
		Proportional shares for coverage games (see [8])
		ONLY weighted shapley values for linear welfare functions with no dummy players (Theorem 2)
		ONLY weighted shapley values for any welfare function for two-player games (Theorem 3)

The importance of Theorem 1 is its implication that the restriction of budget-balanced distribution rules to weighted Shapley values persists broadly, even if the structure of the action sets is vastly simplified to be single-selection.

Next, we present a finer complete characterization, i.e., a complete characterization of W-specific distribution rules given a specific W.[7] Before proceeding, we quickly define $\mathbb{W}_\ell^*$, the class of all linear welfare functions with no dummy players. This is the set of all welfare functions $W \in \mathbb{W}$ which satisfy the following two properties:

1) *Linearity:* $(\forall S \subseteq N) \quad W(S) = \sum_{i \in S} W(\{i\})$.
2) *No Dummy Player:* $(\forall i \in N) \quad W(\{i\}) \neq 0$.

Theorem 2: Given any base welfare function $W \in \mathbb{W}_\ell^*$, all games in $\mathcal{G}_s(N, f, W)$ possess a pure Nash equilibrium for a budget-balanced f if and only if there exists a weight-vector $\boldsymbol{\omega}$ such that $f(W)$ is the W-specific weighted Shapley value distribution rule, $f_{WSV}[\boldsymbol{\omega}](W)$.

Our third result provides another finer complete characterization of budget-balanced distribution rules for *any* base welfare function, but just for the class of *two-player* single-selection games:

Theorem 3: For any base welfare function $W \in \mathbb{W}$, all games in $\mathcal{G}_s(\{1,2\}, f, W)$ possess a pure Nash equilibrium for a budget-balanced f if and only if there exists a weight-vector $\boldsymbol{\omega} = (\omega_1, \omega_2)$ such that $f(W)$ is the W-specific weighted Shapley value distribution rule, $f_{WSV}[\boldsymbol{\omega}](W)$.

Due to space constraints, we defer the proofs of Theorems 2 and 3 to an Appendix of the full version of this paper [21].

V. CONCLUSION AND FUTURE WORK

The limitation to weighted Shapley values for stable, budget-balanced utility design was first exposed by Chen et al. [5]. Theorems 1-3 shed new light on how *robust* this limitation is to restrictions on the structure of action sets (single/multi-selection). In particular, this limitation continues to hold even if the action sets are restricted to be single-selection, for a large class of welfare functions (Theorem 2). When further restricted to two-player games, the limitation holds for any welfare function (Theorem 3). Table I summarizes our results (in boldface) against previous results.

It might be of interest to explore deeper to find complete characterizations of equilibrium guaranteeing distribution rules for an arbitrary, nonlinear welfare function with single-selection action sets and three or more players (these are classes of games not covered by two finer characterization theorems). However, we expect this to be a severely challenging ordeal, because the parallel characterization in the multi-selection scenario is complex enough to begin with [7], starting from which one has to add more distribution rules depending on the specific welfare function at hand.

[7]The characterization of Marden and Wierman [8] is finer and W-specific, but is only a partial characterization.

REFERENCES

[1] E. Anshelevich, A. Dasgupta, E. Tardos, and T. Wexler, "Near-optimal network design with selfish agents," in *STOC*, 2003, pp. 511–520.

[2] V. Gkatzelis, K. Kollias, and T. Roughgarden, "Optimal cost-sharing in weighted congestion games," Available at http://www.stanford.edu/~kkollias/optimal.pdf, 2014.

[3] C. G. Cassandras and W. Li, "Sensor networks and cooperative control," *Eur. J. Control*, vol. 11, no. 4-5, pp. 436–463, 2005.

[4] R. Gopalakrishnan, J. R. Marden, and A. Wierman, "An architectural view of game theoretic control," *SIGMETRICS Perf. Eval. Rev.*, vol. 38, no. 3, pp. 31–36, 2011.

[5] H.-L. Chen, T. Roughgarden, and G. Valiant, "Designing network protocols for good equilibria," *SIAM J. Comput.*, vol. 39, no. 5, pp. 1799–1832, 2010.

[6] J. R. Marden and A. Wierman, "Overcoming the limitations of utility design for multiagent systems," *IEEE Trans. Autom. Control*, vol. 58, no. 6, pp. 1402–1415, 2013.

[7] R. Gopalakrishnan, J. R. Marden, and A. Wierman, "Potential games are necessary to ensure pure Nash equilibria in cost sharing games," *Math. Oper. Res.*, 2014, to appear.

[8] J. R. Marden and A. Wierman, "Distributed welfare games," *Oper. Res.*, vol. 61, no. 1, pp. 155–168, 2013.

[9] H. Moulin and S. Shenker, "Strategyproof sharing of submodular costs: Budget balance versus efficiency," *Econ. Theory*, vol. 18, no. 3, pp. 511–533, 2001.

[10] T. Roughgarden, "Intrinsic robustness of the price of anarchy," in *STOC*, 2009, pp. 513–522.

[11] H. Moulin, "An efficient and almost budget balanced cost sharing method," *Games and Economic Behavior*, vol. 70, no. 1, pp. 107–131, 2010.

[12] E. Anshelevich, A. Dasgupta, J. Kleinberg, É. Tardos, T. Wexler, and T. Roughgarden, "The price of stability for network design with fair cost allocation," in *FOCS*, 2004, pp. 295–304.

[13] C. Chekuri, J. Chuzhoy, L. Lewin-Eytan, J. Naor, and A. Orda, "Non-cooperative multicast and facility location games," *IEEE J. Sel. Areas Commun.*, vol. 25, no. 6, pp. 1193–1206, 2007.

[14] S. Adlakha, R. Johari, G. Weintraub, and A. Goldsmith, "Oblivious equilibrium: an approximation to large population dynamic games with concave utility," in *GameNets*, 2009, pp. 68–69.

[15] Y. Su and M. van der Schaar, "Conjectural equilibrium in multiuser power control games," *IEEE Trans. Signal Process.*, vol. 57, no. 9, pp. 3638–3650, 2009.

[16] J. R. Marden, "State based potential games," *Automatica*, vol. 48, no. 12, pp. 3075–3088, 2012.

[17] R. A. Murphey, "Target-based weapon target assignment problems," in *Nonlinear Assignment Problems*, ser. Combinatorial Optimization, P. M. Pardalos and L. S. Pitsoulis, Eds. Springer, 2000, vol. 7, pp. 39–53.

[18] L. S. Shapley, "Additive and non-additive set functions," Ph.D. dissertation, Department of Mathematics, Princeton University, 1953.

[19] ——, "A value for n-person games," in *Contributions to the theory of games*. Princeton University Press, 1953, vol. 2, pp. 307–317.

[20] S. Hart and A. Mas-Colell, "Potential, value, and consistency," *Econometrica*, vol. 57, no. 3, pp. 589–614, 1989.

[21] R. Gopalakrishnan, S. D. Nixon, and J. R. Marden, "Stable utility design for distributed resource allocation," Available at http://ecee.colorado.edu/marden/files/cdc2014full.pdf, 2014.