Characterizing distribution rules for cost sharing games

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***Abstract*—We consider the problem of designing the distribution rule used to share "welfare" (cost or revenue) among individually strategic agents. There are many distribution rules known to guarantee the existence of a (pure Nash) equilibrium in this setting, e.g., the Shapley value and its weighted variants; however a characterization of the space of distribution rules that yield the existence of a Nash equilibrium is unknown. Our work provides a step towards such a characterization. We prove that when the welfare function is strictly submodular, a budget-balanced distribution rule guarantees equilibrium existence for all games (i.e., all possible sets of resources, agent action sets, etc.) if and only if it is a weighted Shapley value.**

I. Introduction

How should the cost incurred (revenue generated) by a set of self-interested agents be shared among them? This fundamental question has led to a large literature in economics over the last decades [24], [20], [28], [21], [19], and more recently in computer science [8], [2], [5], [6], [16]. A classic framework within which to study this question is that of cost sharing games, in which there is a set of "agents" making strategic choices of which "resources" to utilize. Each resource generates a "welfare" (cost or revenue) depending on the set of agents that choose the resource. The focus is on finding *budget-balanced* distribution rules that provide "stable" and/or "fair" allocations, which is traditionally formalized by the concept of the *core* – the set of feasible distribution rules that guarantee a stable grand coalition.

Recently, there is an emerging focus on weaker notions of "stability" and, in particular, a (pure Nash) equilibrium for the agents, which is our focus in this work. This focus is driven by applications such as network-cost sharing [2], [6] where individually strategic behavior is commonly assumed. Additionally, the notion of an equilibrium is natural if cost sharing distribution rules are used to design utilities for distributed agents in the context of a game-theoretic approach to distributed control [16], [9].

Existing literature on cost sharing games provides several distribution rules that guarantee equilibrium existence [24], [23], [27], [26]. Perhaps, the most famous such distribution rule is the Shapley value [24], which guarantees the existence of a Nash equilibrium in any game, and for certain classes of games such as convex games, is always in the core. A generalization of the Shapley value which exhibits the same properties is the weighted Shapley value [23].

Although (weighted) Shapley value distribution rules guarantee equilibrium existence, this is only one of many desirable properties. Perhaps the next most important property is that these equilibria should be "efficient" in the sense of maximizing the social welfare. In order to provide the tools necessary to optimize efficiency (and other properties) while still always ensuring equilibrium existence, researchers have recently sought to provide characterizations of the class of distribution rules that guarantee equilibrium existence. Providing such a characterization is the goal of this paper.

The first step toward this goal is the work of Chen, Roughgarden, & Valiant [6], who prove that the only budget-balanced distribution rules that guarantee equilibrium existence in all cost sharing games are weighted Shapley value distribution rules. Following on [6], Marden & Wierman [17] provide the parallel characterization in the context of revenue sharing games. Though the characterizations in [6] and [17] seem general, they actually provide only a worst-case characterization. In particular, the proofs in [6], [17] consist of exhibiting a specific "worst-case" welfare function which requires that weighted Shapley value distribution rules be used. Thus, the question of characterizing the space of distribution rules for any specific welfare function remains open. In practice, it is exactly this issue that is important: when designing a distribution rule, one *knows* the specific welfare function for the situation. In such a situation, there may be distribution rules other than weighted Shapley value rules that also guarantee the existence of an equilibrium. In particular, recent work has shown that such settings do exist [16].

In this paper, we seek to provide a more detailed characterization of the space of distribution rules by understanding, *for specific welfare functions*, what rules guarantee the existence of a (pure Nash) equilibrium. In particular, our focus is on the class of submodular welfare functions. Submodular welfare functions are quite common and are found, for example, in power control and coverage problems in sensor networks [4], [16], wireless access point assignment and frequency selection [11], and influence maximization [12].

Our main result (Theorem 2) states that, given any strictly submodular welfare function, only weighted Shapley value distribution rules guarantee equilibrium existence in all games. Thus, we show, perhaps surprisingly, that the result of [6] holds much more generally. In particular, we show that it is not the existence of some worst-case welfare function which limits the design of "desirable" distribution rules to weighted Shapley values. In fact, even under practical (submodular) welfare functions, the design of distribution rules is constrained to the weighted Shapley value. This characterization means that in many practical settings, it is possible to optimize other desirable properties (such as the "efficiency" of the equilibrium) within the class of weighted Shapley value distribution rules.

II. Model

In this work, we consider a simple, but general, model of a "welfare" (cost or revenue) sharing game, where there is a set of self-interested agents/players $N = \{1, \ldots, n\}$ who each choose from a set $R = \{r_1, \ldots, r_m\}$, the resources to which to allocate themselves. Each agent $i \in N$ is capable of selecting potentially multiple resources in R; therefore, we say that agent i has action set $\mathcal{A}_i \subseteq 2^R$. The resulting action profile, or (joint) allocation, is a tuple $a = (a_1, \ldots, a_n) \in \mathcal{A}$ where the set of all possible allocations is denoted by $\mathcal{A} = \mathcal{A}_1 \times \ldots \times \mathcal{A}_n$. We occasionally denote an allocation a by (a_i, a_{-i}) where a_{-i} denotes the allocation of all agents except agent i.

Each allocation generates a welfare, $\mathcal{W}(a)$, which needs to be shared completely among the agents, i.e., the allocation is *budget-balanced*. In this work, we assume $\mathcal{W}(a)$ is *(linearly) separable* and *scalable* across resources, i.e., $\mathcal{W}(a) = \sum_{r\in R} v_r W(\{a\}_r)$ where $\{a\}_r = \{i \in N : r \in a_i\}$ is the set of agents that are allocated to resource r in a, $v_r \in \mathbb{R}_{++}$ is the local scaling factor of resource r, and $W : 2^N \to \mathbb{R}$ is the welfare function that is scaled at each resource. These are standard assumptions, e.g., see [6], [16], and are quite general. Note that this model incorporates both revenue and cost sharing games, since we allow for the welfare function W to be either positive or negative.

The manner in which the welfare is shared among the agents determines the utility function $U_i : \mathcal{A} \to \mathbb{R}$ that agent i seeks to maximize. Because the welfare function is assumed to be separable and scalable, it is natural that the utility functions should follow suit. The motivation for scalability is self-evident. To motivate separability, note that this corresponds to welfare garnered from each resource being distributed among only the agents allocated to that resource, which is most often appropriate, e.g., in profit sharing. This results in $U_i(a) = \sum_{r\in a_i} v_r f(i, \{a\}_r)$, where $f : N \times 2^N \to \mathbb{R}$ is the distribution rule, i.e., $f(i, S)$ is the portion of the welfare allocated to agent $i \in S$ when sharing with S. Recall that we require f to be budget-balanced, which means that for any player set $S \subseteq N$, $\sum_{i\in S} f(i, S) = W(S)$.

To summarize, we can specify a welfare sharing game G using the tuple $G = \left(N, R, \{\mathcal{A}_i\}_{i\in N}, \{v_r\}_{r\in R}, f, W\right)$, where the design of f is the focus of this paper.

The primary goals when designing f are to guarantee (i) equilibrium existence, and (ii) equilibrium efficiency. Our focus in this work is entirely on (i) and we consider pure Nash equilibria; however it should be noted that other equilibrium concepts are also of interest [25], [1], [13]. Recall that a *(pure Nash) equilibrium* is an action profile $a^* \in \mathcal{A}$ such that for each player i, $U_i(a_i^*, a_{-i}^*) = \max_{a_i\in\mathcal{A}_i} U_i(a_i, a_{-i}^*)$.

The Shapley value [24], one of the oldest and most commonly studied distribution rule in the cost sharing literature, is defined as

$$f^{SV}(i, S) = \sum_{T\subseteq S\setminus\{i\}} \frac{(|T|)!(|S| - |T| - 1)!}{|S|!}(W(S \cup \{i\}) - W(S)).$$

The importance of the Shapley value is that it is budget-balanced and guarantees equilibrium existence in any game, regardless of its parameters. Further, it has many other desirable properties, e.g., it results in the game being a so-called "potential game" [26]. However, it has one key drawback – computing it is often intractable since it requires the calculation of exponentially many marginal contributions [7].

There are generalizations of the Shapley value that maintain its properties. In particular, the weighted Shapley value [23], which is defined as ($\boldsymbol{\omega}$ is the vector of the player-specific weights)

$$f^{WSV}(i, S; \boldsymbol{\omega}) = \sum_{T\subseteq S: i\in T} \frac{\omega_i}{\sum_{j\in T}\omega_j} \left(\sum_{R\subseteq T} (-1)^{|T|-|R|} W(R)\right),$$

also guarantees equilibrium existence in any game.

III. Results and Discussion

In the previous section, we discussed examples of budget-balanced distribution rules that guarantee equilibrium existence in "welfare" (cost or revenue) sharing games. The goal of this paper is to characterize the space of all such distribution rules.

Towards this end, this paper builds on the recent work of Chen, Roughgarden, & Valiant [6] and Marden & Wierman [17], which takes the first steps toward providing such a characterization. The following result combines the main contributions of [6] and [17] into one statement. Let $\mathcal{G}(N, f)$ denote the class of all games with a fixed player set N and budget-balanced distribution rule f. Note that this is a very general class; in particular, it includes games with arbitrary action sets and an arbitrary welfare function.

Theorem 1 *[6], [17] All games in $\mathcal{G}(N, f)$ possess a pure Nash equilibrium if and only if f is a weighted Shapley value.*

Less formally, Theorem 1 states that if one wants to use a distribution rule that guarantees equilibrium existence for all possible welfare functions and for all possible action sets, then one is limited to the class of weighted Shapley value distribution rules. Though this result seems quite general, it is shown by exhibiting a specific "worst-case" welfare function for which this limitation holds. In reality, when designing a distribution rule, one *knows* the specific welfare function for the situation, and Theorem 1 claims nothing in this case. In particular, there may be other distribution rules that guarantee equilibrium existence for all games, and recent work has shown that there are settings where this is the case [16].

Our main result shows that even for a fixed strictly submodular welfare function, the conclusion of Theorem 1 is still valid, i.e., weighted Shapley distribution rules are the only ones which guarantee equilibrium existence.

More specifically, let $\mathcal{G}(N, f, W)$ denote the class of all games with a fixed player set N, budget-balanced base distribution rule f, and welfare function W. Note that $\mathcal{G}(N, f, W) \subsetneq \mathcal{G}(N, f)$ because we have fixed the welfare function W (though arbitrary action sets are still allowed). We focus on welfare functions W that are *strictly submodular*, i.e., for player sets $X, Y \subseteq N$, $W(X)+W(Y) > W(X\cup Y)+W(X\cap Y)$. A variety of problems such as power control and coverage problems in sensor networks [4], [16], wireless access point assignment and frequency selection [11], and influence maximization [12] all have submodular welfare functions.

We can now state our main result.

Theorem 2 *Let W be a strictly submodular welfare function. All games in $\mathcal{G}(N, f, W)$ possess a pure Nash equilibrium if and only if f is a weighted Shapley value.*

Though Theorem 2 and Theorem 1 are superficially very similar, Theorem 2 is much stronger. The key contrast between Theorems 1 and 2 is that Theorem 1 states that there exists a welfare function for which the distribution rule is required to be a weighted Shapley value in order to guarantee equilibrium existence, while Theorem 2 states that, *for any strictly submodular welfare function*, the distribution rule must be a weighted Shapley value to guarantee equilibrium existence. To highlight this, consider that the proof of Theorem 1 exhibits a welfare function, and shows the result for that specific case, while the proof of Theorem 2 allows working with an arbitrary strictly submodular welfare function.

One subtle implication of Theorem 2 is that if one hopes to use a distribution rule that always guarantees equilibrium existence in games with a strictly submodular welfare function, then one is limited to working within the class of "potential games", since weighted Shapley value distribution rules result in potential games [26]. This is, perhaps, surprising since a priori potential games are often thought to be a small, special case of games. However, this is useful since there are many well understood learning dynamics which guarantee convergence to equilibria in potential games [3], [14], [15].

Theorem 2 also has some negative implications. First, the limitation to weighted Shapley value distribution rules means that one is forced to use distribution rules which are often intractable [7], as discussed earlier. Second, Marden & Wierman [17] show that there are efficiency limits that hold for any weighted Shapley value distribution rule. In particular, under any weighted Shapley value distribution rule there exists a game where the best equilibrium has welfare that is a multiplicative factor of two worse than the optimal welfare.

We do not have space to provide the complete proof of Theorem 2, so we sketch an outline, highlighting the proof technique and the key steps involved, in the following. A complete proof is provided in the appendix.

Proof Sketch of Theorem 2:

First, note that we only need to prove one direction since it is well known that a weighted Shapley value distribution rule is budget-balanced and guarantees equilibrium existence in any resource allocation game [22], [10], [18]. Thus, in the remainder of this section, we discuss the proof technique for the other direction – for budget-balanced distribution rules that are not weighted Shapley values, there exists a game for which no equilibrium exists.

The general outline of the proof is as follows. We establish several necessary conditions for a budget-balanced distribution rule f that guarantees the existence of an equilibrium for all games in $\mathcal{G}(N, f, W)$. Effectively, these necessary conditions eliminate any budget-balanced distribution rule that is not a weighted Shapley value and hence give us our desired result. We establish these conditions by a series of counterexamples which amount to choosing a resource set R, the respective values $\{v_r\}_{r\in R}$, and the associated action sets $\{\mathcal{A}_i\}_{i\in N}$.

A key technique of the proof is that instead of working with W directly, we define a basis of simple welfare functions, and represent W using this basis, i.e., any W is equivalent to a linear combination of the basis welfare functions. The basis we use is the following class of T-welfare functions. For every player subset $T \subseteq N$, a T-welfare function is defined as:

$$W^T(S) = \begin{cases} 1, & T \subseteq S; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

It can be shown that the set of all T-welfare functions forms a basis for the set of all welfare functions, i.e., given any welfare function W, there exists a set $\mathcal{T} \subseteq 2^N$, and a sequence $Q = \{q_T\}_{T\in\mathcal{T}}$ of non-zero weights indexed by $\mathcal{T}$, such that:

$$W = \sum_{T\in\mathcal{T}} q_T W^T$$

From here on, we denote a welfare function by $(\mathcal{T}, Q)$, so the class of games defined in the theorem is denoted as $\mathcal{G}(N, f, \mathcal{T}, Q)$. It is useful to think of the sets in $\mathcal{T}$ as being "coalitions" of players that contribute to the welfare function, and the corresponding coefficients in Q as being their respective contributions. Also, for simplicity, we assume that $W(\emptyset) = 0$ and therefore, $\emptyset \notin \mathcal{T}$.

Warmup, a single T-welfare function: To build some intuition, we first present the proof outline for an isolated T-welfare function, i.e., for a W where $|\mathcal{T}| = 1$ above. This step consists of establishing the following necessary conditions for a budget-balanced f for any subset $S \subseteq N$ of players:

(a) If the coalition T is not formed in S ($T \nsubseteq S$), then f does not allocate any utility to the players in S.
(b) If the coalition T is formed in S ($T \subseteq S$), then f distributes the resulting welfare only among the contributing players (players in T). Therefore, players in $S - T$, if any, get nothing.
(c) If the coalition T is formed in S ($T \subseteq S$), then f distributes the welfare among players in T as if all other players (players in $S - T$) were absent.

It is easy to see that any budget-balanced f that satisfies the above conditions is completely specified by $|T| - 1$ values, namely the values of $f(i, T)$ for any $|T| - 1$ players in T. Further, it can be shown that $f(i, T)f(j, T) \geq 0$ for all $i, j \in T$. The proof is complete by observing that such an f is indeed a weighted Shapley value distribution rule, where the weight of player $i \in T$ is given by $\omega_i = \frac{f(i,T)}{q_T}$, and the weights of the other players are arbitrary.

To provide an idea of the proof technique we use to establish the above necessary conditions, in the following we prove (a). *Proof of (a):* Formally, we need to prove that any budget-balanced f that guarantees equilibrium existence in any game $G \in \mathcal{G}(N, f, \{T\}, \{q_T\})$ satisfies $f(i, S) = 0$ for all $i \in S$ when $T \nsubseteq S \subseteq N$. We do so by induction on $|S|$. The base case, where $|S| = 1$ is trivially true, because from budget balance, we get that for any player $i \in N$,

$$f(i, \{i\}) = \begin{cases} q_T & , T = \{i\} \\ 0 & , \text{otherwise} \end{cases}$$

For the induction hypothesis, let us assume that $(\forall\ S)$ $(\forall\ i \in S)$ $f(i, S) = 0$, where $T \nsubseteq S \subseteq N$ and $|S| = z$, for some integer z satisfying $0 < z < |N|$. Now, for $|S| = z + 1$, assume the contrary, that $f(i, S) \neq 0$ for some $i \in S$, where $T \nsubseteq S \subseteq N$ and $|S| = z+1$. Since f is budget-balanced, there has to be at least one other player $j \in S$ that also satisfies this condition, such that $f(i, S)f(j, S) < 0$, that is, $f(i, S)$ and $f(j, S)$ have opposite signs. Without loss of generality, assume that $f(i, S) < 0$ and $f(j, S) > 0$. From the induction hypothesis, we know that $f(i, S - \{j\}) = f(j, S - \{i\}) = 0$.

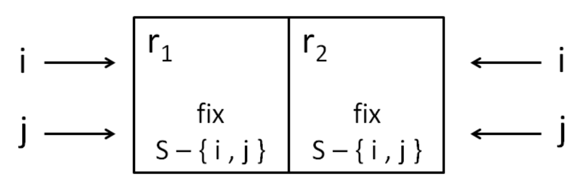

Fig. 1. Counter-example for (a)

Now consider the game illustrated in Fig. 1 with resource set $R = \{r_1, r_2\}$ and local resource coefficients $v_{r_1} = v_{r_2} > 0$. Players i and j have the same action sets – they can each choose either r_1 or r_2. All other players in S have a fixed action – they choose both resources. This is essentially a game between players i and j. It is easy to see that none of the four possible action profiles is a Nash equilibrium, which is a contradiction. For example, (r_1, r_1) is not a Nash equilibrium since $f(i, S - \{j\}) > f(i, S)$, and so player i has an incentive to deviate to r_2. This completes the inductive argument. □

General welfare functions: We are now ready to outline the sequence of necessary conditions that make up the core of the proof for a general welfare function, $(\mathcal{T}, Q)$. Before continuing, we need to introduce some more notation. For any subset $S \subseteq N$, we denote by $\mathcal{T}(S)$ the set of all sets in $\mathcal{T}$ that are contained in S. That is, $\mathcal{T}(S) = \{T \in \mathcal{T} \mid T \subseteq S\}$. In other words, $\mathcal{T}(S)$ is the set of contributing coalitions in S. Also, let $\mathcal{I}(S)$ denote the set of players within $\mathcal{T}(S)$, that is, $\mathcal{I}(S) = \bigcup \mathcal{T}(S)$. In other words, $\mathcal{I}(S)$ is the set of contributing players in S.

The first three conditions mirror those stated above for an isolated T-welfare function – for any subset $S \subseteq N$ of players,

(a) If no coalition is formed in S ($\mathcal{T}(S) = \emptyset$), then f does not allocate any utility to the players in S.
(b) If any coalition is formed in S ($\mathcal{T}(S) \neq \emptyset$), then f distributes the welfare only among players in $\mathcal{I}(S)$.
(c) If any coalition is formed in S ($\mathcal{T}(S) \neq \emptyset$), f distributes the resulting welfare among players in $\mathcal{I}(S)$ as if all other players (players in $S - \mathcal{I}(S)$) were absent.

It turns out that in the general case, these three conditions are not enough to complete the proof.

We next show that the conditions above imply that f can now be represented as a linear combination of basis weighted Shapley value distribution rules – for every player subset $T \subseteq N$, define a basis T-distribution rule (parameterized by a positive vector $\boldsymbol{\omega}^T = \left(\omega_i^T\right)_{i \in T}$) as follows:

$$f^T(i, S; \boldsymbol{\omega}^T) = \begin{cases} \frac{\omega_i^T}{\sum_{k \in T} \omega_k^T} & , \ i \in T \text{ and } T \subseteq S \\ 0 & , \ \text{otherwise} \end{cases} \tag{2}$$

It is easy to see that (2) is the weighted Shapley distribution rule for its corresponding T-welfare function defined in (1), where the player weights are given by $\boldsymbol{\omega}^T$. Formally, we show that for any budget-balanced distribution rule f that guarantees equilibrium existence in any game $G \in \mathcal{G}(N, f, \mathcal{T}, Q)$, there exists a sequence of weight-vectors $\Omega = \left\{\boldsymbol{\omega}^T\right\}_{T \in \mathcal{T}}$ such that

$$f = \sum_{T \in \mathcal{T}} q_T f^T \tag{3}$$

where each f^T is a weighted Shapley value distribution rule for its corresponding T-welfare function, with weight vector $\boldsymbol{\omega}^T$. Therefore, f is completely specified by a sequence of weight vectors Ω.

Note that this is not yet enough to guarantee equivalence to a weighted Shapley value distribution rule, since a pair of players can have "inconsistent" weights in different coalitions. Thus, our final steps focus on deriving necessary consistency conditions for the weight vector Ω.

To guarantee consistency of the weights, we first show that if there is a pair of players common to two coalitions, then their weights in those two coalitions must be "consistent", i.e., they should be linearly dependent. More formally, we show that for every pair of subsets $T_1, T_2 \in \mathcal{T}$, every pair of players (i, j) such that $\{i, j\} \subseteq T_1 \cap T_2$,

$$\frac{\omega_i^{T_1}}{\omega_i^{T_2}} = \frac{\omega_j^{T_1}}{\omega_j^{T_2}}$$

Finally, we show that all weight vectors in Ω must be consistent, i.e., there exists a global weight vector $\boldsymbol{\omega} = (\omega_k)_{k \in N}$, such that, for every $T \in \mathcal{T}$, $\boldsymbol{\omega}^T$ and $\boldsymbol{\omega}$ restricted to T are linearly dependent. Note that we use the fact that the welfare function is strictly submodular only for proving this final step.

From the form (3) for the distribution rule f, and the form (2) for the basis T-distribution rule, it is clear that scaling the local weight vectors by a constant does not affect the distribution rule. Therefore, it follows from our final result that any budget-balanced distribution rule f that guarantees the existence of an equilibrium in any game $G \in \mathcal{G}(N, f, W)$, where W is strictly submodular, is completely specified by a global weight vector $\boldsymbol{\omega}$. The proof is complete by observing that, at this stage, f, in the form (3), is exactly the weighted Shapley value distribution rule for the welfare function $(\mathcal{T}, Q)$, with weight vector $\boldsymbol{\omega}$. □

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Appendix

Proof of Theorem 2: First, note that we only need to prove one direction since it is well known that a weighted Shapley value distribution rule is budget-balanced and guarantees equilibrium existence in any resource allocation game [22], [10], [18]. Thus, in the remainder of this section, we present the proof for the other direction – for budget-balanced distribution rules that are not weighted Shapley values, there exists a game for which no equilibrium exists.

The general outline of the proof is as follows. We establish several necessary conditions for a budget-balanced distribution rule f that guarantees the existence of an equilibrium for all games in $\mathcal{G}(N, f, W)$. Effectively, these necessary conditions eliminate any budget-balanced distribution rule that is not a weighted Shapley value and hence give us our desired result. We establish these conditions by a series of counterexamples which amount to choosing a resource set R, the respective values $\{v_r\}_{r \in R}$, and the associated action sets $\{\mathcal{A}_i\}_{i \in N}$.

A key technique of the proof is that instead of working with W directly, we define a basis of simple welfare functions, and represent W using this basis, i.e., any W is equivalent to a linear combination of the basis welfare functions. The basis we use is the following class of T-welfare functions. For every player subset $T \subseteq N$, a T-welfare function is defined as:

$$W^T(S) = \begin{cases} 1, & T \subseteq S; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

It can be shown[1] that the set of all T-welfare functions forms a basis for the set of all welfare functions, i.e., given any welfare function W, there exists a set $\mathcal{T} \subseteq 2^N$, and a sequence $Q = \{q_T\}_{T \in \mathcal{T}}$ of non-zero weights indexed by $\mathcal{T}$, such that:

$$W = \sum_{T \in \mathcal{T}} q_T W^T$$

From here on, we denote a welfare function by $(\mathcal{T}, Q)$, so the class of games defined in the theorem is denoted as $\mathcal{G}(N, f, \mathcal{T}, Q)$. It is useful to think of the sets in $\mathcal{T}$ as being "coalitions" of players that contribute to the welfare function, and the corresponding coefficients in Q as being their respective contributions. Also, for simplicity[2], we assume that $W(\emptyset) = 0$ and therefore, $\emptyset \notin \mathcal{T}$.

Before continuing, we need to introduce some more notation. For any subset $S \subseteq N$, we denote by $\mathcal{T}(S)$ the set of all sets in $\mathcal{T}$ that are contained in S. That is, $\mathcal{T}(S) = \{T \in \mathcal{T} \mid T \subseteq S\}$. In other words, $\mathcal{T}(S)$ is the set of contributing coalitions in S. Also, let $\mathcal{I}(S)$ denote the set of players within $\mathcal{T}(S)$, that is, $\mathcal{I}(S) = \bigcup \mathcal{T}(S)$. In other words, $\mathcal{I}(S)$ is the set of contributing players in S. We also define $\mathcal{T}^{\min}$ and $\mathcal{T}^{\max}$, to be the subsets of $\mathcal{T}$ containing all the minimal and maximal sets[3] respectively in $\mathcal{T}$. We adopt analogous definitions for $\mathcal{T}^{\min}(S)$ and $\mathcal{T}^{\max}(S)$. We also use the notation $C(\mathcal{T})$ to denote the minimal cover $\cup_{T \in \mathcal{T}} T$ of a collection of sets $\mathcal{T}$.

Our first lemma formalizes the following intuitive necessary condition – for any subset $S \subseteq N$ of players, if no contributing coalition is formed in S, then f does not allocate any utility to the players in S.

[1]Shapley [23] showed this, he calls these basis functions "inclusion functions", and the set $\mathcal{T}$ the "spectrum" of W.

[2]This is merely a normalization and therefore, no generality is lost.

[3]A set is minimal (respectively, maximal) in a collection if there is no other set that is a strict subset (respectively, superset) of itself.

Lemma 1 *Let f be a budget-balanced distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$. Let $S \subseteq N$ be a nonempty set with $\mathcal{T}(S) = \emptyset$. Then, for any player $i \in S$, $f(i, S) = 0$.*

Proof: The proof is by induction on $|S|$. The base case, where $|S| = 1$ is trivially true, because from budget-balance, we have that for any player $i \in N$,

$$f(i, \{i\}) = \begin{cases} q_{\{i\}} & , \{i\} \in \mathcal{T} \\ 0 & , \text{otherwise} \end{cases}$$

Our induction hypothesis is the following statement. Any budget-balanced distribution rule f that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$ satisfies $f(i, S) = 0$ for all $i \in S$, where $S \subseteq N$, $\mathcal{T}(S) = \emptyset$ and $|S| = z$, for some integer z satisfying $0 < z < |N|$.

Now, we prove that if the induction hypothesis is true, then it is still true when z is replaced by $z + 1$. Assume the contrary, that there is a budget-balanced distribution rule f that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$, but satisfies $f(i, S) \neq 0$ for some $i \in S$, where $S \subseteq N$, $\mathcal{T}(S) = \emptyset$ and $|S| = z+1$. Since f is budget-balanced, there has to be at least one other player $j \in S$ that also satisfies this condition, such that $f(i, S)f(j, S) < 0$, that is, $f(i, S)$ and $f(j, S)$ have opposite signs. Without loss of generality, assume that $f(i, S) < 0$ and $f(j, S) > 0$. From the induction hypothesis, we know that $f(i, S - \{j\}) = f(j, S - \{i\}) = 0$.

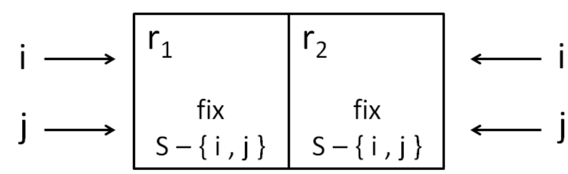

Fig. 2. Counter-example for Lemma 1

Now consider the following game illustrated in Figure 2, with resource set $R = \{r_1, r_2\}$ and local resource coefficients $v_{r_1} = v_{r_2} > 0$. Players i and j have the same action sets – they can each choose either r_1 or r_2. All other players in S have a fixed action – they choose both resources. This is essentially a game between players i and j. It is easy to see that none of the four possible action profiles is a Nash equilibrium, which is a contradiction. For example, (r_1, r_1) is not a Nash equilibrium since $f(i, S - \{j\}) > f(i, S)$, and so player i has an incentive to deviate to r_2. Similarly, (r_2, r_1) is not a Nash equilibrium since $f(j, S) > f(j, S - \{i\})$, and so player j has an incentive to deviate to r_2. This completes the inductive argument. □

The next lemma formalizes the intuitive necessary condition that for any subset $S \subseteq N$ of players, f distributes welfare only among the contributing players, i.e., the players in $\mathcal{I}(S)$.

Lemma 2 *Let f be a budget-balanced distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$. Let $S \subseteq N$ be a nonempty set. Then, $f(i, S) = 0$ for all players $i \notin \mathcal{I}(S)$.*

Proof: Since this is vacuously true for $\mathcal{I}(S) = N$, let us assume that $\mathcal{I}(S) \subsetneq N$. The proof is by induction on $|\mathcal{T}(S)|$. The base case, where $|\mathcal{T}(S)| = 0$ is vacuously true (this case reduces to Lemma 1). Our induction hypothesis is be the following statement. Any budget-balanced distribution rule f that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$ satisfies $f(i, S) = 0$ for all $i \notin \mathcal{I}(S)$, where $S \subseteq N$ and $|\mathcal{T}(S)| \leq z$, for some integer z satisfying $0 \leq z < |\mathcal{T}|$.

Now, we prove that if the induction hypothesis is true, then it is still true when z is replaced by $z + 1$. Our argument for this is again inductive, this time, the induction is on $|S - \mathcal{I}(S)|$. The base case, where $S = \mathcal{I}(S)$, is vacuously true. Our induction hypothesis is the following statement. Any budget-balanced distribution rule f that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$ satisfies $f(i, S) = 0$ for all $i \in S - \mathcal{I}(S)$, where $S \subseteq N$, $|\mathcal{T}(S)| = z + 1$, and $|S - \mathcal{I}(S)| = y$, for some integer y satisfying $0 \leq y < |N| - |\mathcal{I}(S)|$.

Now, we prove that if both the outer and inner induction hypotheses are true, then the inner induction hypothesis is still true when y is replaced by $y + 1$. Assume the contrary, that there is a budget-balanced distribution rule f that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$, but satisfies $f(i, S) \neq 0$ for some $i \in S - \mathcal{I}(S)$, where $S \subseteq N$, $|\mathcal{T}(S)| = z + 1$, and $|S - \mathcal{I}(S)| = y + 1$. Since f is budget-balanced, we have,

$$f(i, S) + \sum_{k \in S - \{i\}} f(k, S) = \sum_{k \in S - \{i\}} f(k, S - \{i\})$$
$$\Rightarrow \quad f(i, S) = \sum_{k \in S - \{i\}} \left(f(k, S - \{i\}) - f(k, S) \right)$$

Since $f(i, S)$ is nonzero, it is clear that at least one of the difference terms on the right hand side is either strictly positive or strictly negative, according to whether $f(i, S)$ is strictly positive or strictly negative. That is, there is some $j \in S - \{i\}$ such that

$$f(i, S) \left(f(j, S - \{i\}) - f(j, S) \right) > 0$$

We also know that[4] $f(i, S - \{j\}) = 0$. So, the above inequality can be rewritten as,

$$\left(f(i, S) - f(i, S - \{j\}) \right) \left(f(j, S) - f(j, S - \{i\}) \right) < 0$$

First, let us consider the case where $f(i, S) - f(i, S - \{j\}) < 0$ and $f(j, S) - f(j, S - \{i\}) > 0$. For this case, the exact same game that was illustrated in Figure 2, and the argument for non-existence of equilibrium therein (proof of Lemma 1) serves as a counterexample here too. The proof for the other case, where $f(i, S) - f(i, S - \{j\}) > 0$ and $f(j, S) - f(j, S - \{i\}) < 0$ is symmetric.[5] This completes the inductive argument. □

Our third lemma formalizes the intuitive necessary condition that for any subset $S \subseteq N$ of players, f distributes welfare among the contributing players (players in $\mathcal{I}(S)$) as if all other players (players in $S - \mathcal{I}(S)$) were absent. In other words, the manner in which f distributes welfare among contributing players is independent of which other (non-contributing) players are present.

Lemma 3 *Let f be a budget-balanced distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$. Let $S \subseteq N$ be a nonempty set. Then, for all players $i \in \mathcal{I}(S)$, $f(i, S) = f(i, \mathcal{I}(S))$.*

[4]If $j \in \mathcal{I}(S)$, we know this from the outer induction hypothesis. If $j \notin \mathcal{I}(S)$, we know this from the inner induction hypothesis.

[5]The best response cycle is just reversed.

Proof: The proof is by induction on $|\mathcal{T}(S)|$. The base case, where $|\mathcal{T}(S)| = 0$, is vacuously true. Our induction hypothesis is the following statement. Any budget-balanced distribution rule f that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$ satisfies $f(i, S) = f(i, \mathcal{I}(S))$ for all $i \in \mathcal{I}(S)$, where $S \subseteq N$ and $|\mathcal{T}(S)| \leq z$, for some integer z satisfying $0 \leq z < |\mathcal{T}|$.

Now, we prove that if the induction hypothesis is true, then it is still true when z is replaced by $z + 1$. Assume the contrary, that is, suppose that there is a budget-balanced distribution rule f that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$, but satisfies $f(i, S) \neq f(i, \mathcal{I}(S)$ for some $i \in \mathcal{I}(S)$, and some $S \subseteq N$, such that $S \supsetneq \mathcal{I}(S)$ and $|\mathcal{T}(S)| = z + 1$. Without loss of generality, let us assume that,

$$f(i, S) < f(i, \mathcal{I}(S)) \tag{5}$$

Because f is budget-balanced, we have,

$$\sum_{k \in S} f(k, S) = \sum_{k \in \mathcal{I}(S)} f(k, \mathcal{I}(S))$$

From Lemma 2, we know that $f(k, S) = 0$ for all $k \in S - \mathcal{I}(S)$. Therefore, we get,

$$\sum_{k \in \mathcal{I}(S)} f(k, S) = \sum_{k \in \mathcal{I}(S)} f(k, \mathcal{I}(S))$$

So, there exists a player $j \in \mathcal{I}(S)$ such that,

$$f(j, S) > f(j, \mathcal{I}(S)) \tag{6}$$

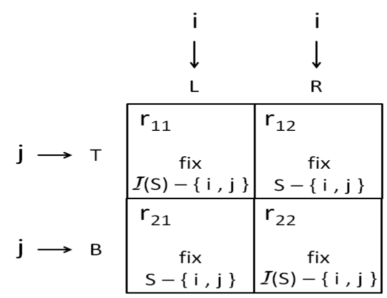

Fig. 3. Counter-example for Lemma 3

Now consider the following game illustrated in Figure 3, with resource set $R = \{r_{11}, r_{12}, r_{21}, r_{22}\}$ and local resource coefficients $v_{r_{11}} = v_{r_{12}} = v_{r_{21}} = v_{r_{22}} > 0$. Player i is the column player and player j is the row player. That is, their action sets are given by $\mathcal{A}_i = \{L = (r_{11}, r_{21}), R = (r_{12}, r_{22})\}$ and $\mathcal{A}_j = \{T = (r_{11}, r_{12}), B = (r_{21}, r_{22})\}$. All other players in $\mathcal{I}(S)$ have a fixed action – they choose all four resources. That is, $\mathcal{A}_k = \{(r_{11}, r_{12}, r_{21}, r_{22})\}$ for $k \in \mathcal{I}(S) - \{i, j\}$. Finally, for all remaining players $k \in S - \mathcal{I}(S)$, $\mathcal{A}_k = \{(r_{12}, r_{21})\}$. This is essentially a game between players i and j. The joint action set can therefore be represented as $\mathcal{A} = \{LT, LB, RT, RB\}$. We now show that none of these four action profiles is a Nash equilibrium, which would be a contradiction:

(1) LT would not be a Nash equilibrium if player j has an incentive to switch from T to B. This would happen if

$$f(j, S) + f(j, \mathcal{I}(S) - \{i\}) > f(j, \mathcal{I}(S)) + f(j, S - \{i\})$$

But this is true, because of (6), and the fact that[6] $f(j, \mathcal{I}(S) - \{i\}) = f(j, S - \{i\})$.

[6]First, observe that $\mathcal{I}(\mathcal{I}(S) - \{i\}) = \mathcal{I}(S - \{i\})$, and then, since $i \in \mathcal{I}(S)$, $|\mathcal{T}(S - \{i\})| < |\mathcal{T}(S)|$, so the induction hypothesis applies.

(2) LB would not be a Nash equilibrium if player i has an incentive to switch from L to R. This would happen if

$$f(i, \mathcal{I}(S)) + f(i, S - \{j\}) > f(i, S) + f(i, \mathcal{I}(S) - \{j\})$$

But this is true, by our assumption in (5), and the fact that[7] $f(i, S - \{j\}) = f(i, \mathcal{I}(S) - \{j\})$.

(3) The action profiles RB and RT are not Nash equilibria either, because in these action profiles, players j and i respectively have incentives to deviate – the arguments are essentially identical to Case (1) and (2) above, respectively.

This completes the inductive argument. □

We next show that the three necessary conditions above (Lemmas 1-3) imply that f can now be represented as a linear combination of basis weighted Shapley value distribution rules – for every player subset $T \subseteq N$, define a basis T-distribution rule (parameterized by a positive vector $\boldsymbol{\omega}^T = \left(\omega_i^T\right)_{i \in T}$) as follows:

$$f^T(i, S; \boldsymbol{\omega}^T) = \begin{cases} \frac{\omega_i^T}{\sum_{k \in T} \omega_k^T} & , \ i \in T \text{ and } T \subseteq S \\ 0 & , \ \text{otherwise} \end{cases} \quad (7)$$

It is easy to see that (7) is the weighted Shapley distribution rule for its corresponding T-welfare function defined in (4), where the player weights are given by $\boldsymbol{\omega}^T$.

Lemma 4 *Let f be a budget-balanced distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$. Then, there exists a sequence of weight-vectors $\Omega = \{\boldsymbol{\omega}^T\}_{T \in \mathcal{T}}$ such that*

$$f = \sum_{T \in \mathcal{T}} q_T f^T \quad (8)$$

where each f^T is a weighted Shapley value distribution rule for its corresponding T-welfare function, with weight vector $\boldsymbol{\omega}^T$.

Proof: First, we introduce the following partition of $\mathcal{T}$, which we call its min-decomposition. Begin with $\mathcal{T}$. Set $\mathcal{T}_1 = \mathcal{T}^{\min}$ and remove the elements of this set from $\mathcal{T}$. Repeat this process (the next iteration would set $\mathcal{T}_2 = (\mathcal{T} - \mathcal{T}_1)^{\min}$) until the partition is complete. Let this partition be $\{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_\ell\}$.

Now for the proof. Let f be a budget-balanced distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N, f, W)$. From Lemmas 1-3, it follows that to specify such a distribution rule f, one only has to specify the values $f(i, S)$ for those subsets S for which $S = \mathcal{I}(S)$ (that is, those subsets S that are unions of one or more coalitions in $\mathcal{T}$). Given an f specified in such a manner, we proceed to derive the basis distribution rules f^T that appear in (8), for each $T \in \mathcal{T}$ as follows. First, for all $T \in \mathcal{T}_1$, for all $i \in T$, set $f^T(i, T) = \frac{1}{q_T} f(i, T)$. Then, recursively for $z > 1$, for all $T \in \mathcal{T}_z$, for all $i \in T$, set

$$f^T(i, T) = \frac{1}{q_T} \left(f(i, T) - \sum_{\substack{T' \subsetneq T \\ T' \in \mathcal{T} \\ i \in T'}} q_{T'} f^{T'}(i, T') \right) \quad (9)$$

We show now that f^T as defined above is budget-balanced, i.e., $\sum_{i \in T} f^T(i, T) = 1$. This will be useful later in the proof. This can be shown by induction on z (the partition level that contains T in the min-decomposition of $\mathcal{T}$). The base case, where $z = 1$ is true by definition and the budget-balance of f. Assume that f^T is budget-balanced for all $T \in \mathcal{T}_m$ for $1 \leq m \leq z$. For any $T \in \mathcal{T}_{z+1}$, from the recursive definition in (9) we have,

$$\begin{aligned}
\sum_{i \in T} f^T(i, T) &= \frac{1}{q_T} \left(\sum_{i \in T} f(i, T) - \sum_{i \in T} \sum_{\substack{T' \subsetneq T \\ T' \in \mathcal{T} \\ i \in T'}} q_{T'} f^{T'}(i, T') \right) \\
&= \frac{1}{q_T} \left(W(T) - \sum_{\substack{T' \subsetneq T \\ T' \in \mathcal{T}}} \sum_{i \in T'} q_{T'} f^{T'}(i, T') \right) \\
&= \frac{1}{q_T} \left(W(T) - \sum_{\substack{T' \subsetneq T \\ T' \in \mathcal{T}}} q_{T'} \right) \\
&= \frac{1}{q_T} (q_T) = 1
\end{aligned}$$

where we have used the budget-balance of f, followed by the induction hypothesis.

Note that the definition (9) immediately implies that for each $T \in \mathcal{T}$,

$$f(i, T) = \sum_{\substack{T' \subseteq T \\ i \in T'}} q_{T'} f^{T'}(i, T') \quad (10)$$

This means that f already satisfies (8) for all subsets $S \in \mathcal{T}$. We still need to show that f satisfies (8) for all subsets S that are unions of two or more coalitions in $\mathcal{T}$. So, for any subset $S \subseteq N$, such that $S = \mathcal{I}(S)$, we prove this by induction on $|\mathcal{T}(S)|$. The base case, where $|\mathcal{T}(S)| = 1$ is trivial, from (10). Suppose f satisfies (8) for all subsets $S \subseteq N$, such that $S = \mathcal{I}(S)$, with $|\mathcal{T}(S)| \leq z$ for some integer z satisfying $1 \leq z < |\mathcal{T}|$. Consider the case where $|\mathcal{T}(S)| = z + 1$. Assume the contrary, i.e.,

$$f(i, S) < \sum_{T \in \mathcal{T}(S)} q_T f^T(i, T) \quad (11)$$

for some $i \in S$.[8] Since f and f^T are budget-balanced, it must be that for some $j \in S$,

$$f(j, S) > \sum_{T \in \mathcal{T}(S)} q_T f^T(j, T) \quad (12)$$

As with the other proofs, we present a counter-example game where the above two inequalities would suffice to show that a Nash equilibrium doesn't exist. This time, it is more complicated, and so, we describe the intuition first. We'd like to have a counter-example of the form in Figure 3, where, to begin

[7] First, observe that $\mathcal{I}(\mathcal{I}(S) - \{j\}) = \mathcal{I}(S - \{j\})$, and then, since $j \in \mathcal{I}(S)$, $|\mathcal{T}(S - \{j\})| < |\mathcal{T}(S)|$, so the induction hypothesis applies.

[8] The '<' sign in the assumption is without loss of generality.

with, when player j selects T, the difference in the utilities of player i between choosing R and L is exactly

$$\sum_{T \in \mathcal{T}(S)} q_T f^T(i,T) - f(i,S),$$

which we can argue is positive due to (11). Note that this cannot be accomplished by simply having the whole set S in the top-left box and all the individual coalitions $T \in \mathcal{T}(S)$ in the top-right box, because $f(i,T)$ is not the same as $q_T f^T(i,T)$, since $f(i,T)$ includes utilities to i from all sub-coalitions of T as well, so there would be a lot of repetition. This suggests the following idea of an inclusion-exclusion method of adding resources to the two boxes. First, add the sets in $\mathcal{T}^{\max}(S)$ to the top-right box. Then, to compensate for the repetition in this box, add all the sets that cover all possible two-way intersections between distinct sets in $\mathcal{T}^{\max}(S)$ to the top-left box. To compensate for the repetition in this box, add all the sets that cover all possible three-way intersections between distinct sets in $\mathcal{T}^{\max}(S)$ to the top-right box. And so on (this process will obviously terminate). Then, to complete the symmetric lower half, copy all the resource sets from the top-left box into the bottom-right box and the top-right box into the bottom-left box.

Now, we present the formal details. Let $\mathcal{T}^{\max}(S) = \{T_1(S), T_2(S), \ldots, T_{\ell(S)}(S)\}$. For $1 \le k \le \ell(S)$, define $\mathcal{P}_k(S) = \mathcal{T}(T_k(S))$; this is the collection of all coalitions that will appear in the expansion of $f(\cdot, T_k(S))$. For $1 \le k \le \ell(S)$, define the collection of minimal covers of all possible distinct k-way intersections,

$$C(k;S) = \left\{ C\left(\bigcap_{m=1}^{k} \mathcal{P}_{y_m}(S) \right) \right\}_{\substack{1 \le y_m \le \ell(S) \\ y_1 \ne y_2 \ne \ldots \ne y_k}}$$

Let $c(k;S) = |C(k;S)|$.

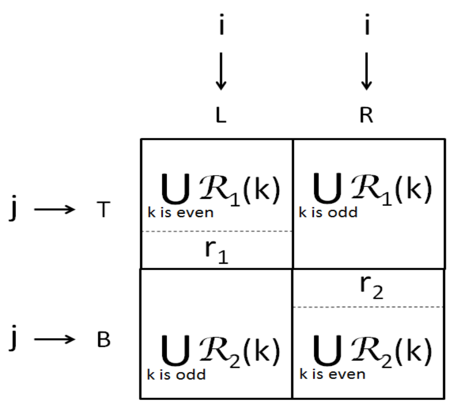

Fig. 4. Counter-example for Lemma 4

Our counter-example, illustrated in Figure 4, has the following set of resources, $\mathcal{R} = \{r_1, r_2\} \bigcup_{k=1}^{\ell(S)} \mathcal{R}_1(k) \bigcup_{k=1}^{\ell(S)} \mathcal{R}_2(k)$, where $\mathcal{R}_y(k) = \{r_{y1}(k), r_{y2}(k), \ldots, r_{yc(k;S)}(k)\}$ for $y = 1, 2$. The local resource coefficients are all identical positive numbers. The action set for player i is given by $\mathcal{A}_i = \{L, R\}$, where

$$L = \{r_1\} \bigcup_{k \text{ is even}} \mathcal{R}_1(k) \bigcup_{k \text{ is odd}} \mathcal{R}_2(k),$$

$$R = \{r_2\} \bigcup_{k \text{ is odd}} \mathcal{R}_1(k) \bigcup_{k \text{ is even}} \mathcal{R}_2(k)$$

Player j has the action set $\mathcal{A}_j = \{T, B\}$, where

$$T = \{r_1\} \bigcup_k \mathcal{R}_1(k),$$

$$B = \{r_2\} \bigcup_k \mathcal{R}_2(k)$$

All other players have a single action, implied by the following fixtures – for all k, for all $1 \le y \le c(k;S)$, fix players in $C_y(k;S) - \{i,j\}$ at resources $r_{1y}(k)$, $r_{2y}(k)$, r_1 and r_2. This is essentially a game between players i and j. The joint action set can therefore be represented as $\mathcal{A} = \{LT, LB, RT, RB\}$. We now show that none of these four action profiles is a Nash equilibrium, which would be a contradiction:

(1) LT would not be a Nash equilibrium if player i has an incentive to switch from L to R. This would happen if

$$\begin{aligned} &\sum_{k \text{ is odd}} \sum_{y=1}^{c(k;S)} f(i, C_y(k;S)) + f(i, S - \{j\}) \\ &+ \sum_{k \text{ is even}} \sum_{y=1}^{c(k;S)} f(i, C_y(k;S) - \{j\}) \\ > &\sum_{k \text{ is even}} \sum_{y=1}^{c(k;S)} f(i, C_y(k;S)) + f(i, S) \\ &+ \sum_{k \text{ is odd}} \sum_{y=1}^{c(k;S)} f(i, C_y(k;S) - \{j\}) \end{aligned} \tag{13}$$

The inclusion-exclusion principle, coupled with (10), gives the following:

$$\sum_{k \text{ is odd}} \sum_{y=1}^{c(k;S)} f(i, C_y(k;S)) - \sum_{k \text{ is even}} \sum_{y=1}^{c(k;S)} f(i, C_y(k;S)) = \sum_{T \in \mathcal{T}(S)} q_T f^T(i,T)$$

$$\sum_{k \text{ is odd}} \sum_{y=1}^{c(k;S)} f(i, C_y(k;S) - \{j\}) - \sum_{k \text{ is even}} \sum_{y=1}^{c(k;S)} f(i, C_y(k;S) - \{j\}) = \sum_{T \in \mathcal{T}(S - \{j\})} q_T f^T(i,T)$$

Now, rearranging the terms in (13) and applying the above, we get

$$\sum_{T \in \mathcal{T}(S)} q_T f^T(i,T) - f(i,S) > \sum_{T \in \mathcal{T}(S - \{j\})} q_T f^T(i,T) - f(i, S - \{j\})$$

But this is true, because of (11), and the fact that the right hand side vanishes due to the induction hypothesis.[9]

(2) By an argument analogous to the one above, it can be shown that RT is not a Nash equilibrium, because player j has an incentive to switch from T to B.

(3) The action profiles RB and LB are not Nash equilibria either, because in these action profiles, players i and j respectively have incentives to deviate – the arguments are essentially identical to Case (1) and (2) above, respectively.

This completes the inductive argument.

[9]Note that $|\mathcal{T}(S - \{j\})| < |\mathcal{T}(S)|$.

Finally, we show that each f^T as defined in (9), already shown to be budget-balanced, corresponds to a weighted Shapley value distribution rule on its corresponding W^T for some vector of weights $\boldsymbol{\omega}^T$. To show this, we first show that f^T is positive. We prove this by contradiction. Suppose there is some player $i \in T$ such that $f^T(i,T) < 0$. Since f^T is budget-balanced, there has to be some other player $j \in T$ such that $f^T(j,T) > 0$. Without loss of generality,[10] assume $q_T > 0$. Let $\mathcal{T}_{ij}(T)$ denote the set of all sub-coalitions in T (excluding T) containing both i and j. The counter-example construction here utilizes the same inclusion-exclusion technique that was employed before. In fact, the counter-example is exactly the same as before, but replace S by T, and $\mathcal{T}^{\max}(S)$ by $\mathcal{T}_{ij}^{\max}(T)$. The arguments for none of the action profiles being Nash equilibria are analogous:

(1) LT would not be a Nash equilibrium if player i has an incentive to switch from L to R. This would happen if (after applying the inclusion-exclusion principle),

$$\sum_{T' \in \bigcup_{P \in \mathcal{T}_{ij}(T)} \mathcal{T}(P)} q_{T'} f^{T'}(i,T') - \sum_{T' \in \mathcal{T}(T)} q_{T'} f^{T'}(i,T')$$
$$> \sum_{T' \in \bigcup_{P \in \mathcal{T}_{ij}(T)} \mathcal{T}(P-\{j\})} q_{T'} f^{T'}(i,T') - \sum_{T' \in \mathcal{T}(T-\{j\})} q_{T'} f^{T'}(i,T')$$

which simplifies to $q_T f^T(i,T) < 0$, which is true by assumption.

(2) By an argument analogous to the one above, it can be shown that RT is not a Nash equilibrium, because player j has an incentive to switch from T to B (the condition in this case would simplify to $q_T f^T(j,T) > 0$.

(3) The action profiles RB and LB are not Nash equilibria either, because in these action profiles, players i and j respectively have incentives to deviate – the arguments are essentially identical to Case (1) and (2) above, respectively.

This completes the inductive argument.

Now, since f^T is positive and budget-balanced, it is completely specified by $|T|$ positive values $f^T(i,T)$ for all $i \in T$, that sum to 1. Define a weight vector $\boldsymbol{\omega}^T$ by

$$\omega_i^T = \begin{cases} f^T(i,T) & , \ i \in T \\ \text{arbitrary} & , \ \text{otherwise} \end{cases}$$

Then, the weighted Shapley value distribution rule f^{WSV} on W^T, with weight vector $\boldsymbol{\omega}^T$), for $i \in T$, is given by

$$\begin{aligned} f^{WSV}(i,T;\boldsymbol{\omega}^T) &= \sum_{S \subseteq T: i \in S} \frac{\omega_i^T}{\sum_{j \in S} \omega_j^T} \left(\sum_{R \subseteq S} (-1)^{|S|-|R|} W^T(R) \right) \\ &= \frac{\omega_i^T}{\sum_{j \in T} \omega_j^T} \left(\sum_{R \subseteq T} (-1)^{|T|-|R|} W^T(R) \right) \\ &= \frac{\omega_i^T}{\sum_{j \in T} \omega_j^T} W^T(T) \\ &= \frac{\omega_i^T}{\sum_{j \in T} \omega_j^T} = \omega_i^T = f^T(i,T) \end{aligned}$$

This concludes the proof. □

[10] If $q_T < 0$, interchange the labels of i and j and the same proof would work.

Notice that f is now completely specified by a sequence of weight vectors $\Omega = \{\boldsymbol{\omega}^T\}_{T \in \mathcal{T}}$, but we are not done yet, since a pair of players can have "inconsistent" weights in different coalitions. Thus, our final steps focus on deriving necessary consistency conditions for the weight vector Ω.

To guarantee consistency of the weights, we first show that if there is a pair of players common to two coalitions, then their weights in those two coalitions must be "consistent", i.e., their ratios must be the same in both coalitions. This is formalized by the following lemma.

Lemma 5 *Let f be a budget-balanced distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N,f,W)$. Then, when written in the form (8), for every pair of players (i,j), every pair of subsets $T_1, T_2 \in \mathcal{T}$ with $\{i,j\} \subseteq T_1 \cap T_2$,*

$$\frac{\omega_i^{T_1}}{\omega_i^{T_2}} = \frac{\omega_j^{T_1}}{\omega_j^{T_2}}$$

Proof: For each pair of players (i,j), let $\mathcal{T}_{ij}$ denote the collection of all coalitions in $\mathcal{T}$ that contain both i and j. Also, let $\mathcal{T}_{ij}(S)$ denote the collection of all coalitions in $\mathcal{T}$ that are present in S and contain both i and j. Let T be a minimal coalition in $\mathcal{T}_{ij}$. Given a collection $\mathcal{T}' \subseteq \mathcal{T}$ of coalitions, we informally use the notation $f(i,\mathcal{T}')$ to denote $\sum_{P \in \mathcal{T}'} q_P f^P(i,P)$. For example, this means that $f(i,S)$ and $f(i,\mathcal{T}(S))$ denote the same quantity. We now state a quick result that will be useful within the proof.

Lemma 6 *Let $W = (\mathcal{T},\mathcal{Q})$ be a welfare function. For any subset $S \subseteq N$, any two players $i,j \in S$, we have, $f(i,\mathcal{T}_{ij}(S))f(i,\mathcal{T}_{ij}(S)) \geq 0$.*

Proof: The proof is by contradiction. Suppose, there exists a subset $S \subseteq N$, and two players $i,j \in S$, such that $f(i,\mathcal{T}_{ij}(S))f(i,\mathcal{T}_{ij}(S)) < 0$. Without loss of generality, assume that $f(i,\mathcal{T}_{ij}(S)) < 0$ and $f(i,\mathcal{T}_{ij}(S)) > 0$. This means that $f(i,S) < f(i,S-\{j\})$ and $f(j,S) > f(j,S-\{i\})$. Now, the exact same game that was illustrated in Figure 2, and the argument for non-existence of equilibrium therein (proof of Lemma 1) serves as a counterexample here too. □

We now proceed to prove Lemma 5 – for any set $S \in \mathcal{T}_{ij}$,

$$\frac{\omega_i^S}{\omega_i^T} = \frac{\omega_j^S}{\omega_j^T} \tag{14}$$

The proof is by induction on the partition level that contains S in the $\min$-decomposition of $\mathcal{T}_{ij}$. Given any coalition $S \neq T$, assume that all "lower" coalitions already satisfy (14). Assume the contrary, that is, S doesn't satisfy (14). We consider the following cases:

(a) $f(i,\mathcal{T}_{ij}(S))f(i,\mathcal{T}_{ij}(T)) > 0$. Note that in this case, by Lemma 6, it follows that $f(j,\mathcal{T}_{ij}(S))f(j,\mathcal{T}_{ij}(T)) > 0$. Consider the game (with four resources) in Figure 5. Note that unlike all previous counter-examples, we now have different positive local resource coefficients – v_1 for resources r_{11} and r_{22}, and v_2 for resources r_{12} and r_{21}. This is essentially a game between players i and j (all other players have a single action, specified through the fixtures in the resources indicated on the figure). The joint action set can therefore be represented as $\mathcal{A} = \{TL, TR, BL, BR\}$. We will now show that we can pick

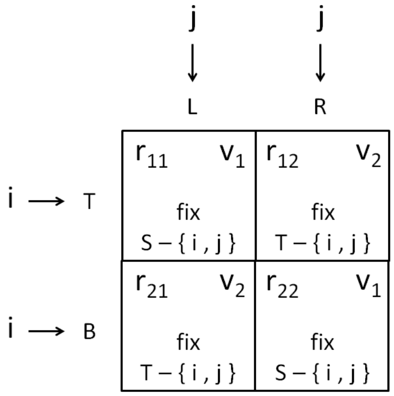

Fig. 5. Counter-example for Lemma 5(a)

the constants v_1, v_2 in such a way that none of these four action profiles will be Nash equilibria.

In action profiles TL and BR, j will have an incentive to deviate if:

$$v_2 f(j,T) + v_1 f(j, S-\{i\}) > v_1 f(j,S) + v_2 f(j, T-\{i\})$$
$$\iff \quad v_2 f(j, \mathcal{T}_{ij}(T)) > v_1 f(j, \mathcal{T}_{ij}(S))$$

In action profiles TR and BL, i will have an incentive to deviate if:

$$v_2 f(i, T-\{j\}) + v_1 f(i,S) > v_1 f(i, S-\{j\}) + v_2 f(i,T)$$
$$\iff \quad v_i f(i, \mathcal{T}_{ij}(S)) > v_2 f(i, \mathcal{T}_{ij}(T))$$

Without loss of generality, assume $f(i, \mathcal{T}_{ij}(S)) > 0$ and $f(i, \mathcal{T}_{ij}(T)) > 0$.[11] One again, from Lemma 6, this means $f(j, \mathcal{T}_{ij}(S)) > 0$ and $f(j, \mathcal{T}_{ij}(T)) > 0$. Then, the two inequalities above give:

$$\frac{f(i, \mathcal{T}_{ij}(T))}{f(i, \mathcal{T}_{ij}(S))} < \frac{v_1}{v_2} < \frac{f(j, \mathcal{T}_{ij}(T))}{f(j, \mathcal{T}_{ij}(S))}$$

It is possible to find $v_1 > 0$ and $v_2 > 0$ to satisfy this inequality if and only if:

$$\frac{f(i, \mathcal{T}_{ij}(S))}{f(i, \mathcal{T}_{ij}(T))} \neq \frac{f(j, \mathcal{T}_{ij}(S))}{f(j, \mathcal{T}_{ij}(T))} \iff \frac{f(i, \mathcal{T}_{ij}(S))}{\frac{\omega_i^T}{\sum_{k\in T}\omega_k^T} q_T} \neq \frac{f(j, \mathcal{T}_{ij}(S))}{\frac{\omega_j^T}{\sum_{k\in T}\omega_k^T} q_T}$$

$$\iff \quad \frac{\frac{\omega_i^S}{\sum_{k\in S}\omega_k^S} q_S + \sum_{T' \in \mathcal{T}_{ij}(S)-\{S\}} \left(\frac{\omega_i^{T'}}{\sum_{k\in T'}\omega_k^{T'}} q_{T'} \right)}{\omega_i^T}$$

$$\neq \frac{\frac{\omega_j^S}{\sum_{k\in S}\omega_k^S} q_S + \sum_{T' \in \mathcal{T}_{ij}(S)-\{S\}} \left(\frac{\omega_j^{T'}}{\sum_{k\in T'}\omega_k^{T'}} q_{T'} \right)}{\omega_j^T}$$

$$\iff \quad \frac{\omega_i^S}{\omega_i^T} \neq \frac{\omega_j^S}{\omega_j^T}$$

which is true, by our assumption that S doesn't satisfy (14). Here, the first equivalence is due to the fact that T is a minimal coalition in $\mathcal{T}_{ij}$, and therefore, $\mathcal{T}_{ij}(T) = \{T\}$. The second equivalence is due to canceling common terms and expanding numerators on both sides. The third and final equivalence is due to the induction hypothesis, that all "lower" coalitions T' in $\mathcal{T}_{ij}$ satisfy (14), and a lot of resulting cancelations.

[11]For the other case, when $f(i, \mathcal{T}_{ij}(S)) < 0$ and $f(i, \mathcal{T}_{ij}(T)) < 0$, the same arguments apply, this time, the deviations are reversed.

(b) $f(i, \mathcal{T}_{ij}(S)) f(i, \mathcal{T}_{ij}(T)) < 0$. Note that in this case, by Lemma 6, it follows that $f(j, \mathcal{T}_{ij}(S)) f(j, \mathcal{T}_{ij}(T)) < 0$.

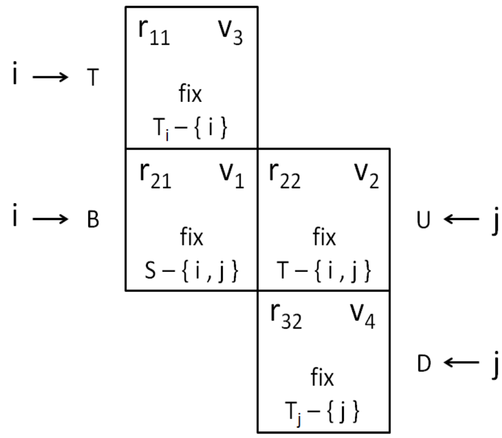

Fig. 6. Counter-example for Lemma 5(b)

Consider the game (with four resources) in Figure 6. This time, we have four different positive local resource coefficients as indicated. Also, let $T_i, T_j \in \mathcal{T}$ be some two coalitions that contain i and j respectively. This is essentially a game between players i and j (all other players have a single action, specified through the fixtures in the resources indicated on the figure). The joint action set can therefore be represented as $\mathcal{A} = \{TU, BU, BD, TD\}$. We will now show that we can pick the constants v_1, v_2, v_3, v_4, and coalitions T_i, T_j in such a way that none of these four action profiles will be Nash equilibria.

In action profile TU and BD, i will have an incentive to deviate if:

$$v_1 f(i,S) + v_2 f(i,T) > v_3 f(i, T_i)$$
$$> v_1 f(i, S-\{j\}) + v_2 f(i, T-\{j\})$$
$$\implies \quad v_1 f(i, \mathcal{T}_{ij}(S)) + v_2 f(i, \mathcal{T}_{ij}(T)) > 0$$

In action profiles BU and TD, j will have an incentive to deviate if:

$$v_1 f(j,S) + v_2 f(j,T) < v_4 f(j, T_j)$$
$$< v_1 f(j, S-\{i\}) + v_2 f(j, T-\{i\})$$
$$\implies \quad v_1 f(j, \mathcal{T}_{ij}(S)) + v_2 f(j, \mathcal{T}_{ij}(T)) < 0$$

Without loss of generality, assume $f(i, \mathcal{T}_{ij}(S)) > 0$ and $f(i, \mathcal{T}_{ij}(T)) < 0$.[12] One again, from Lemma 6, this means $f(j, \mathcal{T}_{ij}(S)) > 0$ and $f(j, \mathcal{T}_{ij}(T)) < 0$. Then, the two inequalities above give:

$$-\frac{f(i, \mathcal{T}_{ij}(T))}{f(i, \mathcal{T}_{ij}(S))} < \frac{v_1}{v_2} < -\frac{f(j, \mathcal{T}_{ij}(T))}{f(j, \mathcal{T}_{ij}(S))}$$

It is possible to find $v_1 > 0$ and $v_2 > 0$ to satisfy this inequality if and only if:

$$\frac{f(i, \mathcal{T}_{ij}(S))}{f(i, \mathcal{T}_{ij}(T))} \neq \frac{f(j, \mathcal{T}_{ij}(S))}{f(j, \mathcal{T}_{ij}(T))}$$

which is true, by our assumption that S doesn't satisfy (14) by the exact same argument in the previous case.

[12]For the other case, when $f(i, \mathcal{T}_{ij}(S)) < 0$ and $f(i, \mathcal{T}_{ij}(T)) > 0$, the same arguments apply, this time, the deviations are reversed.

We are not done yet – we still need to argue that given any S, we can find coalitions T_i, T_j and $v_3 > 0$, $v_4 > 0$ such that:

$$\begin{aligned} v_1 f(i,S) + v_2 f(i,T) &> v_3 f(i,T_i) \\ &> v_1 f(i, S-\{j\}) + v_2 f(i, T-\{j\}) \text{ and} \\ v_1 f(j,S) + v_2 f(j,T) &< v_4 f(j,T_j) \\ &< v_1 f(j, S-\{i\}) + v_2 f(j, T-\{i\}) \end{aligned}$$

Consider the first inequality. If the left hand side is positive, then at least one of $f(i,S)$, $f(i,T)$ must be positive, so choose T_i to be S or T accordingly and adjust v_3 to satisfy the first inequality. The same argument applies when the left hand side is negative. In an analogous manner, T_j and v_4 can be chosen to satisfy the second inequality.

This concludes the inductive argument. □

Finally, we show that all weight vectors in Ω must be consistent, i.e., there exists a global weight vector $\boldsymbol{\omega} = (\omega_k)_{k \in N}$, such that, for every $T \in \mathcal{T}$, $\boldsymbol{\omega}^T$ and $\boldsymbol{\omega}$ restricted to T are linearly dependent. To show this, we first show a powerful sufficient condition, for a special class of welfare functions, for the collection Ω of weight vectors to be consistent.

Lemma 7 *Let $W = (\mathcal{T}, \mathcal{Q})$ be such that $\mathcal{T}$ contains all subsets of N of size 2. In particular, all strictly submodular welfare functions satisfy this property. Suppose Ω is a collection of weight vectors, such that for any three players $i,j,k \in N$, the collection $\left\{\boldsymbol{\omega}^{\{i,j\}}, \boldsymbol{\omega}^{\{j,k\}}, \boldsymbol{\omega}^{\{i,k\}}\right\}$ of three weight vectors is consistent. Then Ω is consistent.*

Proof: We are given that for every three players $i,j,k \in N$, $\left\{\boldsymbol{\omega}^{\{i,j\}}, \boldsymbol{\omega}^{\{j,k\}}, \boldsymbol{\omega}^{\{i,k\}}\right\}$ is a consistent collection of weight vectors. In other words,

$$\frac{\omega_i^{\{i,j\}}}{\omega_j^{\{i,j\}}} \frac{\omega_j^{\{j,k\}}}{\omega_k^{\{j,k\}}} \frac{\omega_k^{\{i,k\}}}{\omega_i^{\{i,k\}}} = 1 \tag{15}$$

Construct the following global weight vector $\boldsymbol{\omega} = (\omega_k)_{k \in N}$ as follows. Set $(\omega_1, \omega_2) = \boldsymbol{\omega}^{\{1,2\}}$. Then, for $3 \le i \le n$, recursively set

$$\omega_i = \frac{\omega_i^{\{i-1,i\}}}{\omega_{i-1}^{\{i-1,i\}}} \omega_{i-1}$$

We now prove an important property of this construction in a quick lemma:

Lemma 8 *With $\boldsymbol{\omega}$ constructed as above, for any two players $i,j \in N$,*

$$\frac{\omega_i^{\{i,j\}}}{\omega_j^{\{i,j\}}} = \frac{\omega_i}{\omega_j} \tag{16}$$

Proof: Equivalently, we show that for all $\ell \in \{1,2,\ldots,n-1\}$, for all $i \in \{1,2,\ldots,n-\ell\}$, (16) holds for players i and $j = i+\ell$. The proof is by induction on ℓ. The base case, where $\ell = 1$ is trivially true, by construction of $\boldsymbol{\omega}$. Assume this is true for some ℓ, then for $\ell+1$, consider, for any $i \in \{1,2,\ldots,n-\ell-1\}$, the three players i, $j = i+\ell$, and $k = i+\ell+1$. From the induction hypothesis, we have:

$$\frac{\omega_i^{\{i,j\}}}{\omega_j^{\{i,j\}}} = \frac{\omega_i}{\omega_j}$$

From the construction of $\boldsymbol{\omega}$, we have:

$$\frac{\omega_j^{\{j,k\}}}{\omega_k^{\{j,k\}}} = \frac{\omega_j}{\omega_k}$$

Plugging the above two equations into (15), we have:

$$\frac{\omega_i}{\omega_j} \frac{\omega_j}{\omega_k} \frac{\omega_k^{\{i,k\}}}{\omega_i^{\{i,k\}}} = 1 \implies \frac{\omega_i^{\{i,k\}}}{\omega_k^{\{i,k\}}} = \frac{\omega_i}{\omega_k}$$

This completes the inductive argument. □

To complete the proof of Lemma 7, consider any coalition $T \in \mathcal{T}$, and any two players $i,j \in T$. Then, we have:

$$\frac{\omega_i^T}{\omega_j^T} = \frac{\omega_i^{\{i,j\}}}{\omega_j^{\{i,j\}}} = \frac{\omega_i}{\omega_j}$$

The first equality is due to Lemma 5, and the second is due to Lemma 8. Hence, $\boldsymbol{\omega}^T$ and $\boldsymbol{\omega}$ restricted to T are linearly dependent. This concludes the proof. □

Armed with this sufficient condition[13] for consistency, all we need to show now is that for every three players $i,j,k \in N$, the collection $\left\{\boldsymbol{\omega}^{\{i,j\}}, \boldsymbol{\omega}^{\{j,k\}}, \boldsymbol{\omega}^{\{i,j\}}\right\}$ of three weight vectors is consistent. We formalize this in our final lemma:

Lemma 9 *Let W be a strictly submodular welfare function. Let f be a budget-balanced distribution rule that guarantees the existence of an equilibrium in all games $G \in \mathcal{G}(N,f,W)$. Then, when written in the form (8), for any three players $i,j,k \in N$, the collection $\left\{\boldsymbol{\omega}^{\{i,j\}}, \boldsymbol{\omega}^{\{j,k\}}, \boldsymbol{\omega}^{\{i,k\}}\right\}$ of three weight vectors is consistent, i.e.,*

$$\frac{\omega_i^{\{i,j\}}}{\omega_j^{\{i,j\}}} \frac{\omega_j^{\{j,k\}}}{\omega_k^{\{j,k\}}} \frac{\omega_k^{\{i,k\}}}{\omega_i^{\{i,k\}}} = 1 \tag{17}$$

Proof: From the form (8) for the distribution rule f, and the form (2) for the basis T-distribution rule, it is clear that scaling the local weight vectors by a constant does not affect the distribution rule. So, without loss of generality, assume $\boldsymbol{\omega}^{\{i,j\}} = (\omega_i, \omega_j)$, $\boldsymbol{\omega}^{\{j,k\}} = (\omega_j, \omega_k)$, and $\boldsymbol{\omega}^{\{i,k\}} = (\omega_i', \omega_k)$. Also, without loss of generality, assume $\omega_i' > \omega_i$.[14] For simplicity, denote $q_{\{i,j\}}$, $q_{\{j,k\}}$, $q_{\{i,k\}}$ by q_1, q_2, q_3 respectively. We are given that W is strictly submodular, and therefore, $q_1 < 0$, $q_2 < 0$, $q_3 < 0$. To prove the theorem, we need to show that $\omega_i' = \omega_i$. Assume the contrary.

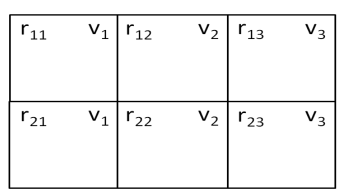

Fig. 7. Counter-example for Lemma 9

Figure (7) depicts a game with six resources. The local resource coefficients are as shown in the figure. Each of the three players has two possible actions: $\mathcal{A}_i = \{L_i = (r_{11}, r_{13}), R_i = (r_{21}, r_{23})\}$, $\mathcal{A}_j = \{L_j = (r_{11}, r_{22}), R_j = (r_{12}, r_{21})\}$, $\mathcal{A}_k = \{L_k = (r_{13}, r_{22}), R_k = (r_{12}, r_{23})\}$.

[13] Notice that this is also (obviously) a necessary condition.

[14] Otherwise, simply switch labels of players j and k.

There are eight possible action profiles, and we now show that it is possible to choose positive v_1, v_2, v_3 such that none of them are Nash equilibria.

(1) Regardless of the values of v_1, v_2, v_3 (as long as they are positive), it is clear that (L_i, L_j, L_k) and (R_i, R_j, R_k) are not Nash equilibria, since q_1, q_2, q_3 are all negative, so players prefer being alone to being with another player.

(2) In action profiles (L_i, L_j, R_k) and (R_i, R_j, L_k), player j will have an incentive to deviate, if

$$\frac{v_2 q_2}{\omega_j + \omega_k} > \frac{v_1 q_1}{\omega_i + \omega_j}$$

(3) In action profiles (L_i, R_j, R_k) and (R_i, L_j, L_k), player k will have an incentive to deviate, if

$$\frac{v_3 q_3}{\omega_i' + \omega_k} > \frac{v_2 q_2}{\omega_j + \omega_k}$$

(4) In action profiles (L_i, R_j, L_k) and (R_i, L_j, R_k), player i will have an incentive to deviate, if

$$\frac{v_1 q_1}{\omega_i + \omega_j}\omega_i > \frac{v_3 q_3}{\omega_i' + \omega_k}\omega_i'$$

Now, to simplify the calculations, let $v_1' = \frac{v_1 q_1}{\omega_i + \omega_j}$, $v_2' = \frac{v_2 q_2}{\omega_j + \omega_k}$, $v_3' = \frac{v_3 q_3}{\omega_i' + \omega_k}$. Then, all we need to show is that we can find $v_1' < 0$, $v_2' < 0$, $v_3' < 0$, such that:

$$v_3' > v_2' > v_1' \quad \text{and} \quad v_1'\omega_i > v_3'\omega_i'$$
$$\implies \quad 1 < \frac{v_1'}{v_3'} < \frac{\omega_i'}{\omega_i}$$

Clearly, this is feasible, because by assumption, $\omega_i' > \omega_i$. Once v_1' and v_3' are chosen, since they are distinct, v_2' can also be chosen to lie between them. This completes the proof. □

As already argued above, scaling the local weight vectors by a constant does not affect the distribution rule. Therefore, it follows from Lemmas 7 and 9, that any budget-balanced distribution rule f that guarantees the existence of an equilibrium in any game $G \in \mathcal{G}(N, f, W)$, where W is strictly submodular, is completely specified by a global weight vector $\boldsymbol{\omega}$. The proof is complete by observing that, at this stage, f, in the form (8), is exactly the weighted Shapley value distribution rule for the welfare function $(\mathcal{T}, Q)$, with weight vector $\boldsymbol{\omega}$. □