

PROBABILITY THEORY

EXPONENTIAL FUNCTIONAL OF LÉVY PROCESSES: GENERALIZED WEIERSTRASS PRODUCTS AND WIENER-HOPF FACTORIZATION

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ABSTRACT. In this note, we state a representation of the Mellin transform of the exponential functional of Lévy processes in terms of generalized Weierstrass products. As by-product, we obtain a multiplicative Wiener-Hopf factorization generalizing previous results obtained by the authors in [14] as well as smoothness properties of its distribution.

RÉSUMÉ. Fonctionnelle exponentielle des processus de Lévy : produits de Weierstrass généralisés et factorisation de Wiener-Hopf. Dans cette note, nous énonçons une représentation de la transformée de Mellin de la fonctionnelle exponentielle des processus de Lévy sous la forme de produits de Weierstrass généralisés. Nous en déduisons une factorisation multiplicative de Wiener-Hopf généralisant un résultat obtenu récemment par les auteurs dans [14] ainsi que des propriétés de régularités pour sa loi.

1. Introduction

Let $\xi = (\xi_t)_{t \geq 0}$ be a possibly killed real-valued Lévy process with a positive mean if it is conservative. That means that ξ is a process with stationary and independent increments with $m = \mathbb{E}[\xi_1] > 0$ if not killed at an independent exponential time. We refer to the excellent monographs [1] and [15] for background. The law of ξ is characterized by its characteristic exponent, i.e. $\ln \mathbb{E}[e^{z\xi_t}] = \Psi(z)t$ where $\Psi : i\mathbb{R} \rightarrow \mathbb{C}$ admits, in our context, the following Lévy-Khintchine representation

$$(1.1) \quad \Psi(z) = \frac{\sigma^2}{2}z^2 + bz + \int_{\mathbb{R}} (e^{zr} - 1 - zr\mathbb{1}_{\{|r|<1\}}) \Pi(dr) - q,$$

where $q \geq 0$ is the killing rate, $\sigma \geq 0$, $b \in \mathbb{R}$, $m = b + \int_{|r|>1} r\Pi(dr) \in (0, \infty]$ if $q = 0$, and, the Lévy measure Π satisfies the integrability condition $\int_{\mathbb{R}} (1 \wedge r^2) \Pi(dr) < +\infty$. We use the convention that $\xi_t = \infty$ for any $t \geq \mathbf{e}_q$, where \mathbf{e}_q stands for an independent (of ξ) exponential variable of parameter $q > 0$. The aim of this note is to describe the distribution of the positive random variable

$$I_\Psi = \int_0^\infty e^{-\xi_t} dt$$

which is called the exponential functional of the Lévy process ξ . More specifically, we provide a representation on a strip of the Mellin transform of this variable in terms of generalized Weierstrass products. We deduce from this result a multiplicative Wiener-Hopf factorization as well as some smoothness properties of its distribution. We refer to the book of Yor [16] and the paper of Bertoin and Yor [4] for references on the topic and for motivations for studying the law of I_Ψ . We would also like to mention that from these publications, there have been numerous investigations of fine distributional properties of this random variable revealing new interesting connections with several fields. For instance, we indicate that its analysis is intimately connected with the study of special functions, see e.g. Kuznetsov and Pardo [8] for Barnes multiple gamma functions and also Patie [12], Patie and Savov [14] for generalization of hypergeometric functions. Hirsch and Yor [7] established an interesting connection between the exponential functional of some subordinators and

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the class of multiplicative infinitely divisible random variables. Haas and Rivero [6] study Yaglom limits of positive self-similar Markov processes and extreme value theory by means of specific properties of I_Ψ . Pardo et al. [11] and Patie and Savov [14] derived a first Wiener-Hopf type factorization for its distribution by resorting to probabilistic devices. Finally, in [13], where the proofs of the results stated in the next Section can be found, the law of the variable I_Ψ turns out to be a key object in the development of the spectral theory of a class of non-selfadjoint invariant Feller semigroups.

2. Main results

We shall provide a representation of the Mellin transform of the positive random variable I_Ψ which we denote, for some $z \in \mathbb{C}$, as follows

$$\mathcal{M}_{I_\Psi}(z) = \mathbb{E} [I_\Psi^{z-1}].$$

Before stating the main result and its main consequences, we introduce a few further notation. First, by defining the following subspaces of the negative of continuous negative definite functions

$$\mathcal{N} = \{\Psi \text{ of the form (1.1)}\},$$

we have $I_\Psi < +\infty$ a.s. if and only if $\Psi \in \mathcal{N}$. Indeed, this is plain if $q > 0$ and if $q = 0$, by the strong law of large numbers for Lévy processes we have the equivalence

$$I_\Psi < +\infty \text{ a.s.} \iff m > 0.$$

We have that $\Psi \in \mathcal{N}$ admits an analytical extension in the strip $\mathbb{C}_{(a,b)} = \{z \in \mathbb{C}; a < \Re(z) < b\}$ with $a < 0 < b$, if and only if $|\mathbb{E}[e^{z\xi_1}]| < \infty$, for all $z \in \mathbb{C}_{(a,b)}$. Under this condition, the restriction of Ψ on the real interval (a, b) is clearly convex and zero free on $(0, b)$. Next, with the usual convention $\inf\{\emptyset\} = \infty$, we set $\theta_\Psi = \inf_{u>0}\{\Psi(-u) = 0\} \in (0, \infty]$, $a_\Psi = \sup_{u>0}\{\Psi \text{ is analytical on } \mathbb{C}_{(-u,0)}\} \in [0, \infty)$ and

$$d_\Psi = a_\Psi \wedge \theta_\Psi \in [0, \infty).$$

We shall also need the space of Bernstein functions \mathcal{B} defined as the set of functions $\phi : \mathbb{C}_{(0,\infty)} \rightarrow \mathbb{C}$ which admit the representation

$$(2.1) \quad \phi(z) = \kappa + \delta z + \int_0^\infty (1 - e^{-zr})\mu(dr)$$

where $\kappa, \delta \geq 0$ and μ is a Lévy measure such that $\int_0^\infty (1 \wedge r)\mu(dr) < \infty$. It is easy to see that if $\phi \in \mathcal{B}$ then $-\phi \in \mathcal{N}$ and, in such a case, we simply write I_ϕ for $I_{-\phi}$. We proceed by recalling that the analytical form of the Wiener-Hopf factorization of Lévy processes leads, for any $\Psi \in \mathcal{N}$,

$$\Psi(z) = -\phi_+(-z)\phi_-(z), \quad z \in i\mathbb{R},$$

where $\phi_\pm \in \mathcal{B}$ with $\phi_+(0) = 0$ if $q = 0$ and $\phi_\pm(0) > 0$ otherwise. We point out that in the case $q > 0$, the parameters of the Wiener-Hopf factors depend on q . In particular, in this case we write simply, in the representation (2.1), with the obvious notation, $\mu_\pm(dr) = \int_0^\infty e^{-qr_1}\mu_{\pm,q}(dr_1, dr)$, where $\mu_{\pm,q}(dr_1, dr)$ is the Lévy measure of the bivariate ascending and descending ladder height and time processes. Finally, for a function $\phi : \mathbb{C} \rightarrow \mathbb{C}$, we write formally the generalized Weierstrass product

$$W_\phi(z) = \frac{e^{-\gamma_\phi z}}{\phi(z)} \prod_{k=1}^\infty \frac{\phi(k)}{\phi(k+z)} e^{\frac{\phi'(k)}{\phi(k)}z}$$

where

$$\gamma_\phi = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{\phi'(k)}{\phi(k)} - \log \phi(n) \right).$$

We observe that if $\phi(z) = z$, then W_ϕ corresponds to the Weierstrass product representation of the Gamma function Γ , valid on $\mathbb{C}/\{0, -1, -2, \dots\}$, and γ_ϕ is the Euler-Mascheroni constant, see e.g. [9], justifying both the terminology and notation. We are now ready to state our main result which provides an explicit representation, in terms of generalized Weierstrass products, of the Mellin transform of I_Ψ for general Lévy processes.

Theorem 2.1. *For any $\phi \in \mathcal{B}$, we have $|\gamma_\phi| < \infty$. Moreover, for any $\Psi \in \mathcal{N}$, we have*

$$(2.2) \quad \mathcal{M}_{I_\Psi}(z) = \phi_-(0)\Gamma(z) \frac{W_{\phi_-}(1-z)}{W_{\phi_+}(z)}, \quad z \in \mathbb{C}_{(0, d_\Psi+1)},$$

where the product W_{ϕ_+} (resp. W_{ϕ_-}) is absolutely convergent on $\mathbb{C}_{(0, \infty)}$ (resp. $\mathbb{C}_{(-d_{\phi_-}, \infty)}$).

We point out that besides its probabilistic nature, this result seems to have some analytical interests. Indeed, on the one hand, it reveals that the class of Bernstein functions appears to be a natural set to generalize the Weierstrass product of the Gamma function. In this vein, we mention that writing $\phi(z) = \frac{\Gamma(z+\alpha)}{\Gamma(z)}$, one has $\phi \in \mathcal{B}$ for any $0 < \alpha < 1$ and one recovers the Weierstrass product representation of double Gamma functions, see e.g. [8]. On the other hand, in the case $0 < m < \infty$, we are able to show, by developing a precised study of the asymptotic decay of the Mellin transform \mathcal{M}_{I_Ψ} on imaginary lines in its strip of definition, that the right-hand side of (2.2) is the unique solution of the functional equation, derived by Maulik and Zwart [10],

$$(2.3) \quad \mathcal{M}_{I_\Psi}(z+1) = \frac{-z}{\Psi(-z)} \mathcal{M}_{I_\Psi}(z), \quad \mathcal{M}_{I_\Psi}(1) = 1,$$

which holds on the domain $\{z \in \mathbb{C}; \Psi(\Re(-z)) \leq 0\}$. We emphasize that our representation remains valid in the case $m = \infty$ although, to the best of our knowledge, there does not exist any characterization of the Mellin transform in this case.

By Mellin identification, we obtain as a straightforward consequence of the representation (2.2), the following factorization of the distribution of the exponential functional.

Corollary 2.2. *For any $\Psi \in \mathcal{N}$, we have the following multiplicative Wiener-Hopf factorization*

$$(2.4) \quad I_\Psi \stackrel{d}{=} I_{\phi_+} \times X_{\phi_-}$$

where \times stands for the product of two independent random variables and X_{ϕ_-} is a positive variable whose distribution is determinate by its negative entire moments as follows, $\mathbb{E}[X_{\phi_-}^{-1}] = \phi_-(0)$, and for any $n \geq 2$, $\mathbb{E}[X_{\phi_-}^{-n}] = \phi_-(0) \prod_{k=1}^{n-1} \phi_-(k)$. Finally, we have $\psi(z) = z\phi_-(z) \in \{\Psi \in \mathcal{N}; q = 0 \text{ and } \Pi(dr)\mathbb{I}_{\{r>0\}} \equiv 0\}$ together with the identity in distribution

$$(2.5) \quad X_{\phi_-} \stackrel{d}{=} I_\psi$$

if and only if the Lévy measure μ_- in the representation (2.1) of ϕ_- has a non-increasing density.

It is worth mentioning that recently Hirsch and Yor [7] provide an infinite product representation, which differs from our Weierstrass product, of the Mellin transform of the variable I_{ϕ_+} when $q = 0$, which is stated valid on the positive real line. They also have a similar type of representation for the positive moments of a variable, introduced by Bertoin and Yor [3], whose distribution is closely connected to the one $X_{\phi_-}^{-1}$ when $\phi_-(0) = 0$ a situation excluded in our work. We also point out that the factorization (2.4) is a generalization of the result obtained first by Pardo et al. [11] and improved in Patie and Savov [14] under the sufficient conditions that either the Lévy measure $\Pi(r)\mathbb{I}_{\{r<0\}}$ or both Lévy measures in the representation (2.1) of ϕ_+ and ϕ_- , have a decreasing density. In fact, in the aforementioned papers the identity (2.4) was obtained with $X_{\phi_-} \stackrel{d}{=} I_\psi$, and, the methodologies developed therein stem heavily on the connection between the distribution of the exponential functional and the stationary measure of a family of Markov processes and additional specific properties of the exponential functional. It is not clear to us how these techniques could be used to derive both the general version of the factorization (2.4) and the necessary condition for the identity (2.5) to hold. We also point out that, in [13], this Wiener-Hopf factorization allows us to derive general intertwining relationships between the Feller semigroups of positive self-similar Markov processes, extending the work of Carmona et al. [5]. For at least this purpose, it would be interesting to give an interpretation of X_{ϕ_-} , in the most general situation, in terms of the underlying Lévy process ξ . We would also like to take the opportunity to raise the question whether one can provide a pathwise interpretation of the multiplicative Wiener-Hopf factorization (2.4).

In [13], relying on the representation (2.2), as mentioned above, we develop a detailed study of the asymptotic behavior of the modulus of the Mellin transform $\mathcal{M}_{I_\Psi}(z)$ along the imaginary lines. In particular, we provide sufficient conditions for its asymptotic decay to be nearly exponential, that is decay faster than any inverse polynomials. These estimates combined with Mellin inversion theorem allows to deduce smoothness properties of the distribution of I_Ψ . We recall that, in the case $q = 0$, Bertoin et al. [2], see also [14] for the case $q > 0$, have shown that the distribution of the positive variable I_Ψ for any $\Psi \in \mathcal{N}$ is absolutely continuous with a density which we denote by p_ψ . To state our last results, we write, for any $\phi \in \mathcal{B}$ and $b \geq 0$,

$$H_\phi(b) = \int_0^\infty \ln \left(\frac{|\phi(b(y+i))|}{\phi(by)} \right) dy,$$

$$\underline{H}_\phi = \liminf_{b \rightarrow \infty} H_\phi(b) \text{ and } \overline{H}_\phi = \limsup_{b \rightarrow \infty} H_\phi(b).$$

Corollary 2.3. *The following assertions hold true.*

1. We have for all $b \geq 0$, $0 \leq H_{\phi_\pm}(b) \leq \frac{\pi}{2}$.

2. $p_\Psi \in C_0^\infty(\mathbb{R}^+)$ if for any $n \in \mathbb{N}$

$$(2.6) \quad \limsup_{b \rightarrow \infty} b^n e^{-b(\frac{\pi}{2} - H_{\phi_+}(b) + H_{\phi_-}(b))} = 0.$$

3. (2.6) holds if, for instance, at least one the following conditions is satisfied.

(a) $\delta_- > 0$, where δ_- is the drift term of ϕ_- .

(b) For all λ big enough and some $\alpha > 0$, $\limsup_{x \rightarrow 0} \frac{\overline{\Pi}_-(\lambda x)}{\overline{\Pi}_-(x)} < \frac{1}{\lambda^{1+\alpha}}$, where $\overline{\Pi}_-$ is the tail of the measure $\Pi(r)\mathbb{I}_{\{r < 0\}}$.

(c) $\exists \epsilon > 0$ such that $\liminf_{x \rightarrow 0} \overline{\mu}_-(x)x^\epsilon > 0$ where $\overline{\mu}_-$ is the tail of μ_- the measure associated to ϕ_- .

(d) If μ_- has a decreasing density g_- and either $\liminf_{b \rightarrow \infty} \frac{g_-(\frac{1}{b})}{\phi_-(b)b} > 0$ or $\phi_-(b)\ln(b) = o(g_-(\frac{1}{b}))$ as $b \rightarrow \infty$.

4. Finally, if $0 < z_0 = \underline{H}_{\phi_-} + \frac{\pi}{2} - \overline{H}_{\phi_+} < \pi$ then the mapping $z \mapsto p_\Psi(\frac{1}{z})$ is analytical in the sector $\arg z < z_0$. This is, for instance, true if the condition (3a), (3b) or the first condition in (3d) above holds.

Note that the condition on $\overline{\Pi}_-$ in (3b) is very natural if we have a process of paths of unbounded variation. Since $\overline{\Pi}_-(r) \stackrel{0}{\sim} r^{-1}$ at worst we see that the ratio is justified if we assume a bit heavier tails. We conclude this note by pointing out that if $\Pi \equiv 0$, which corresponds to the Brownian motion case with positive drift m , i.e. $\Psi(z) = \frac{z^2}{2} + mz$, then $\overline{H}_{\phi_-} = \underline{H}_{\phi_+} = \frac{\pi}{2}$ and hence $z_0 = \frac{\pi}{2}$ in (4). Indeed, it is well known that in this case $p_\Psi(\frac{1}{z}) = \frac{z^m}{\Gamma(m)} e^{-z}$ and the Mellin transform of I_Ψ is $\frac{\Gamma(m+1-z)}{\Gamma(m)}$ whose exponential rate of decay along complex lines of the type $a + ib$ is precisely of the rate of $\frac{\pi}{2}|b|$. However, the rate of decay of the Mellin transform does not describe the sector of analyticity of $p_\Psi(\frac{1}{z})$ which is obviously the whole complex plane in the case m is an integer. Such phenomena could also be observed for any $\Psi \in \mathcal{N}$ with $\overline{\Pi}_- \equiv 0$ and $\int_0^1 r\Pi(dr) = \infty$ for which we also know that $p_\Psi(\frac{1}{z})$ is an entire function, see [12].

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