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On a martingale associated to generalized Ornstein–Uhlenbeck processes and an application to finance ☆

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Abstract

In this paper we study the two-dimensional joint distribution of the first passage time of a constant level by spectrally negative generalized Ornstein–Uhlenbeck processes and their primitive stopped at this first passage time. By using martingales techniques, we show an explicit expression of the Laplace transform of the distribution in terms of new special functions. Finally, we give an application in finance which consists of computing the Laplace transform of the price of an European call option on the maximum on the yield in the generalized Vasicek model. The stable case is studied in more detail.

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1. Introduction

Let $Z := (Z_t,\ t \geqslant 0)$ be a spectrally negative Lévy process starting from 0 given on a complete probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$ where the filtration $(\mathscr{F}_t)_{t\geqslant 0}$ satisfies the usual conditions. For any $\lambda > 0$, we define a generalized Ornstein–Uhlenbeck (for short GOU) process $X := (X_t,\ t \geqslant 0)$, starting from $x \in \mathbb{R}$, with backward driven Lévy process (for short BDLP) Z as the unique solution to the following stochastic differential equation:

$$\mathrm{d}X_t = -\lambda X_t\,\mathrm{d}t + \mathrm{d}Z_t, \quad X_0 = x. \tag{1.1}$$

These are a generalization of the classical Ornstein–Uhlenbeck process constructed by simply replacing the driving Brownian motion with a Lévy process. In this paper we are concerned with the positive random variables T_y and the functional I_t defined by

$$T_y = \inf\{s \geqslant 0;\ X_s > y\} \quad \text{and} \quad I_t = \int_0^t X_s\,\mathrm{d}s, \tag{1.2}$$

respectively. The Laplace transform of T_y is known from Hadjiev [9]. There is an important literature regarding the distribution of additive functionals, stopped at certain random times, of diffusions processes, see for instance the book of Borodin and Salminen [5] for a collection of explicit results. However, the law of such functionals for Markov processes with jumps are not known except in some special cases (e.g. the exponential functional of some Lévy processes, see Carmona et al. [6] and the Hilbert transform of Lévy processes see Fitzsimmons and Getoor [8] and Bertoin [2]). The explicit form of the joint distribution (T_y, I_{T_y}), when X is the classical Ornstein–Uhlenbeck, is given by Lachal [10]. Here, the author exploits the fact that the bivariate process $(I_t, X_t, t \geqslant 0)$ is a Markov process. We shall extend his result by providing the Laplace transform of this two-dimensional distribution in the general case, i.e. when X is a GOU process as defined above. We recall that first passage times problems for Markov processes are closely related to the finding of an appropriate martingale associated to the process. We shall provide a methodology which allows us to build up the martingale used to compute the joint Laplace transform we are looking for. In a second step we shall combine martingales and Markovian techniques to derive the Laplace–Fourier transform.

GOU processes have found many applications in several fields. They are widely used in finance today to model the stochastic volatility of a stock price process (see e.g. Barndorff-Nielsen and Shephard [1]) and for describing the dynamics of the instantaneous interest rate. The latter application, as a generalization of the Vasicek model, deserves particular attention, as these processes belong to the class of one factor affine term structure model. These are well known to be tractable, in the sense that it is easy to fit the entire yield curve by basically solving Riccati equations, see Duffie et al. [7] for a survey on affine processes. From the expression of the joint Laplace transform of (T_y, I_{T_y}), we provide an analytical formula for the price of an European call option on maximum on yields in the framework of GOU processes.

The remainder of the paper is organized as follows. In Section 2, we give some results about Lévy and GOU processes and recall some well-known facts about the first passage times above a constant level. In Section 3, we give an explicit form for the joint Laplace transform (T_y, I_{T_y}) in terms of new special functions. Section 4 is devoted to the special case of stable OU processes, that is when Z is a stable process. In the last section, we apply the previous results to the pricing of path dependent option on yields with a more detailed study of the stable Vasicek case.

2. Preliminaries and recalls

Throughout the rest of this paper $Z := (Z_t,\ t \geqslant 0)$ denotes a real-valued spectrally negative Lévy process starting from 0. It is a process with stationary and independent increments, whose Lévy measure ν charges only the negative real line $(\nu((0,\infty)) = 0)$. The Lévy measure being the compensator of the jumps measure of the process, Z has non-positive jumps. Due to the absence of positive jumps, it is possible to extend analytically the characteristic function of Z to the negative imaginary line. Thus, one characterizes this process by its so-called Laplace exponent $\psi : [0,\infty) \to (-\infty,\infty)$ which is specified by the identity

$$\mathbb{E}[\exp(uZ_t)] = \exp(t\psi(u)), \quad t, u \geqslant 0$$

and has the form

$$\psi(u) = bu + \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{0} (\mathrm{e}^{ur} - 1 - ur\chi(r))\nu(\mathrm{d}r), \tag{2.1}$$

where $\chi(r) := \mathbb{I}_{\{r>-1\}}$, $b \in \mathbb{R}$, $\sigma \geqslant 0$ and $\nu(.)$ is the Lévy measure on $(-\infty, 0]$ which satisfies the integrability condition $\int_{-\infty}^{0} (1 \wedge x^2)\,\nu(\mathrm{d}x) < \infty$. It is known that ψ is a convex function with $\lim_{u\to\infty} \psi(u) = +\infty$. We assume that the process X does not drift to $-\infty$, which is the case when $\psi'(0^+)$ is non-negative, see Bertoin [3, Chapter VII] for a thorough description of these processes.

We introduce the first passage time process $\tau := (\tau_z,\ z \geqslant 0)$ defined, for a fixed $z \geqslant 0$, by $\tau_z = \inf\{s \geqslant 0;\ Z_s > z\}$. Denoting by ϕ the inverse function of the continuous and increasing function ψ, the Laplace exponent of τ is given by, see Bertoin [3, Theorem VII.1],

$$\mathbb{E}[\exp(-u\tau_z), \tau_z < \infty] = \exp(-z\phi(u)). \tag{2.2}$$

We now review some well-known facts concerning GOU processes. By a variation of constant technique, the solution of (1.1) can be written in terms of Z as follows:

$$X_t = \mathrm{e}^{-\lambda t}\left(x + \int_0^t \mathrm{e}^{\lambda s}\,\mathrm{d}Z_s\right), \quad t \geqslant 0. \tag{2.3}$$

From this expression, it is easy to derive the Laplace exponent of X

$$\mathbb{E}_x[\exp(uX_t)] = \exp\left(\mathrm{e}^{-\lambda t}xu + \int_0^t \psi(\mathrm{e}^{-\lambda r}u)\,\mathrm{d}r\right), \quad u \geqslant 0,$$

where $\mathbb{E}_x$ being the expectation with respect to $\mathbb{P}_x$, the law of the process starting from x. From the representation (2.3), we also get that $X_t \longrightarrow \int_0^\infty e^{-\lambda s}\,dZ_s$ a.s. as t tends to ∞. Consequently, the Laplace transform of the limiting distribution of X, denoted by $\widehat{\rho}^X(u)$, $u \geqslant 0$, is given by

$$\widehat{\rho}^X(u) = \exp\left(\int_0^\infty \psi(e^{-\lambda r}u)\,dr\right), \tag{2.4}$$

whenever the Lévy measure satisfies the condition $\int_{r<-1} \log|r|\,v(dr) < \infty$, see Sato [16, Chapter III].

The process X is a Feller process. Its infinitesimal generator $\mathscr{A}$ is an integro-differential operator acting on $\mathscr{C}_c^2(\mathbb{R})$, the space of twice continuously differentiable functions with compact support. It is defined by

$$\mathscr{A}f(x) = \tfrac{1}{2}\sigma^2 f''(x) + (b-\lambda x)f'(x) + \int_{-\infty}^0 (f(x+r) - f(x) - f'(x)r\chi(r))v(dr).$$

To complete the description, we mention that X is a special semimartingale with triplet of predictable characteristics given by

$$\left(bt - \lambda\int_0^t X_s\,ds, \frac{1}{2}\sigma^2 t, v(dr)\,dt\right). \tag{2.5}$$

3. Study of the law of $(T_y, \int_0^{T_y} X_s\,ds)$

For $y > x$, we introduce the stopping time $T_y = \inf\{s \geqslant 0;\ X_s > y\}$. For the remainder of the paper, we shall impose the following condition.

Assumption 1. Either $\sigma > 0$ or $\int_{-1}^0 rv(dr) = \infty$ or $b - \int_{-1}^0 rv(dr) > \lambda y$.

Our aim in this section is to characterize the joint law of the couple $(T_y, \int_0^{T_y} X_s\,ds)$ through transform techniques. We shall start with computing the following joint Laplace transform:

$$\Pi_y^{(\gamma,\theta)}(x) := \mathbb{E}_x\left[\exp\left(-\gamma T_y + \theta\int_0^{T_y} X_s\,ds\right)\right]. \tag{3.1}$$

To this end, we introduce the GOU process, denoted by $X^{\frac{\theta}{\lambda}}$, with the triplet of predictable characteristics

$$\left(b't - \lambda\int_0^t X_s\,ds, \frac{\sigma^2}{2}t, e^{\frac{\theta}{\lambda}r}v(dr)\,dt\right),$$

$$\text{where } b' := b + \frac{\theta}{\lambda}\sigma^2 + \int_{-\infty}^{-1}(e^{\frac{\theta}{\lambda}r} - 1)rv(dr).$$

Before stating our main result we note two intermediate results.

Lemma 1. *For $\gamma, \theta > 0$ such that $\eta := \gamma - \psi(\frac{\theta}{\lambda}) > 0$, and $y > x$, we have*

$$\Pi_y^{(\gamma,\theta)}(x) = \mathrm{e}^{-\frac{\theta}{\lambda}(y-x)}\, \mathbb{E}_x\left[\exp\left(-\eta\, T_y^{(\frac{\theta}{\lambda})}\right)\right], \tag{3.2}$$

where $T_y^{(\frac{\theta}{\lambda})} = \inf\{s \geqslant 0;\ X_s^{\frac{\theta}{\lambda}} > y\}$.

Remark 2. We note that this lemma can easily be extended to compute the joint law of the couple $(T_y, \int_0^{T_y} \Lambda(X_s)\,\mathrm{d}s)$ where X is the solution to the SDE $\mathrm{d}X_t = \Lambda(X_t)\,\mathrm{d}t + \mathrm{d}Z_t$, $X_0 = x < y$, and where $\Lambda(x)$ is any locally integrable function on $\mathbb{R}$ and $T_y = \inf\{s \geqslant 0;\ X_s > y\}$ such that y is regular for itself.

Proof. Fix $y > x$. Exploiting the fact that X has non-positive jumps, we get

$$\int_0^{T_y} X_s\,\mathrm{d}s = \frac{1}{\lambda}(Z_{T_y} + x - y),$$

which yields

$$\Pi_y^{(\gamma,\theta)}(x) = \mathrm{e}^{-\frac{\theta}{\lambda}(y-x)}\, \mathbb{E}_x\left[\exp\left(-\gamma T_y + \frac{\theta}{\lambda} Z_{T_y}\right)\right]. \qquad \square$$

We recall that $\mathscr{F}_t = \sigma(Z_s, s \leqslant t)$ denotes the natural filtration of Z up to time t. We now consider the Girsanov's transform $\mathbb{P}^{(\xi)}$ of the probability measure $\mathbb{P}$ which is defined by $\mathrm{d}\mathbb{P}^{(\xi)}_{|\mathscr{F}_t} = \exp(\xi Z_t - t\psi(\xi))\,\mathrm{d}\mathbb{P}_{|\mathscr{F}_t}$, $t, \xi \geqslant 0$. Under $\mathbb{P}^{(\xi)}$, Z, denoted by $Z^{(\xi)}$, is again a Lévy process with the following Laplace exponent, for $u \geqslant 0$:

$$\begin{aligned}
\psi^{(\xi)}(u) :=&\ \log\left(\mathbb{E}\left[\exp\left(uZ_1^{(\xi)}\right)\right]\right) \\
=&\ \log(\mathbb{E}[\exp((u+\xi)Z_1)]) - \psi(\xi) \\
=&\ \left(b + \sigma^2\xi + \int_{-\infty}^{-1} (\mathrm{e}^{\xi r} - 1) r v(\mathrm{d}r)\right) u + \tfrac{1}{2}\sigma^2 u^2 \\
&+ \int_{-\infty}^{0} (\mathrm{e}^{ur} - 1 - ur\chi(r))\mathrm{e}^{\xi r} v(\mathrm{d}r).
\end{aligned}$$

By choosing $\xi = \frac{\theta}{\lambda}$ and using the representation (2.5), it is straightforward to deduce the triplet of predictable characteristics of the associated GOU process $X^{\frac{\theta}{\lambda}}$. We point out that $X^{\frac{\theta}{\lambda}}$ has again non-positive jumps, since the two probability measures are absolutely continuous. Finally, our relationship follows from the computations:

$$\begin{aligned}
\Pi_y^{(\gamma,\theta)}(x) &= \mathrm{e}^{-\frac{\theta}{\lambda}(y-x)}\, \mathbb{E}_x\left[\exp\left(-\gamma T_y + \frac{\theta}{\lambda} Z_{T_y}\right)\right] \\
&= \mathrm{e}^{-\frac{\theta}{\lambda}(y-x)}\, \mathbb{E}_x\left[\exp\left(-\left(\gamma - \psi\left(\frac{\theta}{\lambda}\right)\right) T_y + \frac{\theta}{\lambda} Z_{T_y} - \psi\left(\frac{\theta}{\lambda}\right) T_y\right)\right] \\
&= \mathrm{e}^{-\frac{\theta}{\lambda}(y-x)}\, \mathbb{E}_x\left[\exp\left(-\left(\gamma - \psi\left(\frac{\theta}{\lambda}\right)\right) T_y^{(\frac{\theta}{\lambda})}\right)\right].
\end{aligned}$$

We now recall the Laplace transform of the random variable T_y see Hadjiev [9] or Novikov [15].

Proposition 3. *For $y > x$, and $\gamma > 0$, we have*

$$\mathbb{E}_x[\exp(-\gamma T_y)] = \frac{H_{\frac{\gamma}{\lambda}}(x)}{H_{\frac{\gamma}{\lambda}}(y)}, \tag{3.3}$$

where $H_\nu(x) = \int_0^\infty \exp\left(xr - \frac{1}{\lambda}\int_1^r \psi(v)\,\frac{dv}{v}\right) r^{\nu-1}\,dr$.

Remark 4. For a fixed x, the Laplace transform is analytical on the domain $\{\eta \in \mathbb{C} : \Re(\eta) > 0\}$.

Remark 5. It is at this stage that Assumption 1 is required, see Novikov [15].

See Section 4 for a proof of this result in the case of stable BDLP. We are now ready to state the following:

Theorem 6. *For $\gamma, \theta > 0$ and $y > x$, we have*

$$\Pi_y^{(\gamma,\theta)}(x) = e^{-\frac{\theta}{\lambda}(y-x)} \frac{H_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(x)}{H_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(y)}, \tag{3.4}$$

where

$$H_{\nu,\beta}(x) = \int_0^\infty \exp\left(xr - \frac{1}{\lambda}\int_0^r \psi(v+\beta)\,\frac{dv}{v}\right) r^{\nu-1}\,dr. \tag{3.5}$$

Proof. Combining the results of the two previous lemmas and using the obvious notation, we obtain

$$\Pi_y^{(\gamma,\theta)}(x) = e^{-\frac{\theta}{\lambda}(y-x)} \frac{H^{(\frac{\theta}{\lambda})}_{\frac{\eta}{\lambda}}(x)}{H^{(\frac{\theta}{\lambda})}_{\frac{\eta}{\lambda}}(y)}.$$

Next, since

$$\begin{aligned} H^{(\frac{\theta}{\lambda})}_{\frac{\eta}{\lambda}}(x) &= \int_0^\infty \exp\left(xr - \frac{1}{\lambda}\int_1^r \psi^{(\frac{\theta}{\lambda})}(v)\,\frac{dv}{v}\right) r^{\frac{\eta}{\lambda}-1}\,dr \\ &= \int_0^\infty \exp\left(xr - \frac{1}{\lambda}\int_1^r \psi\left(v+\frac{\theta}{\lambda}\right)\frac{dv}{v}\right) r^{\frac{\gamma}{\lambda}-1}\,dr \\ &= \exp\left(\frac{1}{\lambda}\int_0^1 \psi\left(v+\frac{\theta}{\lambda}\right)\frac{dv}{v}\right) H_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(x), \end{aligned}$$

we obtain the identity $H^{(\frac{\theta}{\lambda})}_{\frac{\eta}{\lambda}}(x)/H^{(\frac{\theta}{\lambda})}_{\frac{\eta}{\lambda}}(y) = H_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(x)/H_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(y)$. By using the convexity of ψ and the fact that $\lim_{u\to\infty}\psi(u) = +\infty$, we have for a fixed $\theta > 0$ and a large u, $\psi(u+\theta) \geqslant \psi(u)$. Moreover, under the Assumption 1, Novikov [15] shows that $\lim_{u\to\infty} u^{-1}\int_0^u \psi(r) r^{-1}\,dr = +\infty$. Therefore, by following a line of reasoning similar to Novikov [15, Theorem 2] the proof is completed. □

In Section 4, the special case with stable BDLPs is studied in detail. In what follows, we provide the Laplace–Fourier transform of the joint distribution. We first show the following lemma:

Lemma 7. *The bivariate process* $(I_t, X_t,\ t\geqslant 0)$ *is a Markov process. Its infinitesimal generator is defined on* $\mathscr{C}_c^{2,1}(\mathbb{R}\times\mathbb{R})$ by

$$\mathscr{A}^* f(x,y) = \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x}(x,y) + (b-\lambda x)\frac{\partial f}{\partial x}(x,y) + x\frac{\partial f}{\partial y}(x,y) + \int_{-\infty}^{0}\left(f(x+r,y) - f(x,y) - \frac{\partial f}{\partial x}(x,y) r\chi(r)\right)\nu(\mathrm{d}r).$$

Proof. We start by recalling that, although the additive functional I_t is not Markovian, the bivariate process $(I_t, X_t,\ t\geqslant 0)$ is a strong Markov process, see Blumenthal and Getoor [4]. The second part of the lemma is a consequence of Itô's formula. Indeed, for any function $f\in\mathscr{C}_c^{2,1}(\mathbb{R}\times\mathbb{R})$, we have

$$\begin{aligned} f(X_t, I_t) &= f(x,0) + \int_0^t \frac{\partial f}{\partial x}(X_{s-}, I_s)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x}(X_s, I_s)\,\mathrm{d}\langle X^{\mathrm{c}}\rangle_s \\ &\quad + \int_0^t \frac{\partial f}{\partial y}(X_s, I_s)\,\mathrm{d}I_s + \sum_{0<s\leqslant t} f(X_s, I_s) - f(X_{s-}, I_s) - \frac{\partial f}{\partial x}(X_{s-}, I_s)\Delta X_s \\ &= f(x,0) - \lambda\int_0^t \frac{\partial f}{\partial x}(X_s, I_s) X_s\,\mathrm{d}s + \int_0^t \frac{\partial f}{\partial x}(X_{s-}, I_s)\,\mathrm{d}Z_s \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x}(X_s, I_s)\,\mathrm{d}\langle Z^{\mathrm{c}}\rangle_s + \int_0^t \frac{\partial f}{\partial y}(X_s, I_s) X_s\,\mathrm{d}s \\ &\quad + \sum_{0<s\leqslant t} f(X_{s-} + \Delta Z_s, I_s) - f(X_{s-}, I_s) - \frac{\partial f}{\partial x}(X_{s-}, I_s)\Delta Z_s, \end{aligned}$$

where X^{c} denotes the continuous martingale part of X. Finally, taking into consideration that $\mathrm{d}\langle Z^{\mathrm{c}}\rangle_s = \sigma^2\,\mathrm{d}s$, we obtain $\mathscr{A}^*$. $\square$

Corollary 8. *For* $\gamma,\theta,\lambda>0$ *and* $y>x$, *we have*

$$\Pi_y^{(\gamma,\mathrm{i}\theta)}(x) = \frac{\overline{H}_{\frac{\gamma}{\lambda},\frac{\mathrm{i}\theta}{\lambda}}(x)}{\overline{H}_{\frac{\gamma}{\lambda},\frac{\mathrm{i}\theta}{\lambda}}(y)}, \tag{3.6}$$

where $\overline{H}_{\nu,\beta}(x) = \mathrm{e}^{\beta x} H_{\nu,\beta}(x)$.

Proof. In order to simplify the notation in the proof we assume that $\sigma=0$. We consider the process $M := (M_t,\ t\geqslant 0)$ defined, for a fixed $t\geqslant 0$, by

$$M_t = \exp\left(-\gamma t + \mathrm{i}\theta\int_0^t X_s\,\mathrm{d}s\right)\overline{H}_{\frac{\gamma}{\lambda},\frac{\mathrm{i}\theta}{\lambda}}(X_t). \tag{3.7}$$

We shall prove that M is a complex martingale. From the integral representation (3.5), it follows that the function $H_{\nu,\beta}(x)$ is analytic in the domain $\Re(\nu)>0$, $\Re(\beta)>0$, $x\in\mathbb{R}$. Set $u(t,x,y) := \mathrm{e}^{-\gamma t + \mathrm{i}(\theta y + \frac{\theta}{\lambda}x)}$, $g(x) := H_{\frac{\gamma}{\lambda},\frac{\mathrm{i}\theta}{\lambda}}(x)$ and $f(t,x,y) := u(t,x,y)g(x)$. Thanks to the remark following Proposition 3, we see that g is a solution of the following integro-differential equation:

$$\mathscr{A}^{(\mathrm{i}\frac{\theta}{\lambda})}g(x) = \left(\gamma - \psi\left(\mathrm{i}\,\frac{\theta}{\lambda}\right)\right)g(x), \tag{3.8}$$

with

$$\mathscr{A}^{\xi}f(x) = (\bar{b} - \lambda x)f'(x) + \int_{-\infty}^{0}(f(x+r) - f(x) - f'(x)r\chi(r))\mathrm{e}^{\xi r}v(\mathrm{d}r),$$

where we recall that $\bar{b} := b + \int_{-\infty}^{0}(\mathrm{e}^{\xi r} - 1)r\chi(r)v(\mathrm{d}r)$. We observe that

$$\frac{\partial f}{\partial y}(t,x,y) = \mathrm{i}\theta u(t,x,y)g(x),$$
$$\frac{\partial f}{\partial x}(t,x,y) = u(t,x,y)\left(\mathrm{i}\,\frac{\theta}{\lambda}g(x) + g'(x)\right).$$

By applying the change of variables formula for processes with finite variation, we get

$$\begin{aligned}
&\mathrm{d}f(t,X_t,I_t)\\
&\quad= \left(\frac{\partial f}{\partial t}(t,X_t,I_t) - \lambda X_t\frac{\partial f}{\partial x}(t,X_t,I_t) + \frac{\partial f}{\partial y}(t,X_t,I_t)\right)\mathrm{d}t\\
&\qquad+\frac{\partial f}{\partial x}(t,X_{t-},I_t)\,\mathrm{d}Z_t + \int_{-\infty}^{0} f(x+r,y) - f(x,y) - \frac{\partial f}{\partial x}(x,y)rJ(\mathrm{d}r)\,\mathrm{d}t\\
&\quad= u(t,x,y)\left(\left(b + \int_{-\infty}^{0}(\mathrm{e}^{\frac{\theta}{\lambda}r} - 1)r\chi(r)v(\mathrm{d}r) - \lambda X_t\right)g'(x)\right.\\
&\qquad\times\int_{-\infty}^{0}(g(x+r) - g(x) - g'(x)r\chi(r))\mathrm{e}^{\frac{\theta}{\lambda}r}v(\mathrm{d}r)\\
&\qquad\left.+\left(-\gamma + \mathrm{i}b\frac{\theta}{\lambda} + \int_{-\infty}^{0}\left(\mathrm{e}^{\frac{\theta}{\lambda}r} - 1 - \mathrm{i}\,\frac{\theta}{\lambda}r\chi(r)\right)v(\mathrm{d}r)\right)g(x)\right)\mathrm{d}t + N_t,
\end{aligned}$$

where $(N_t,\ t\geqslant 0)$ is a $\mathscr{F}$-martingale. Consequently, by using the fact that g is a solution of Eq. (3.8), we have shown that $(M_t,\ t\geqslant 0)$ is also a purely discontinuous martingale with respect to the natural filtration of X.

Next, we derive the following estimates, for any $t\geqslant 0$:

$$\begin{aligned}
\mathbb{E}[|M_{T_y\wedge t}|] &\leqslant \mathbb{E}\left[\left|H_{\frac{\gamma}{\lambda},\frac{\mathrm{i}\theta}{\lambda}}(X_{T_y\wedge t})\right|\right]\\
&\leqslant \mathbb{E}\left[\left|H_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(X_{T_y\wedge t})\right|\right]\\
&\leqslant \mathbb{E}\left[\left|H_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(y)\right|\right] < \infty.
\end{aligned}$$

We complete the proof of the corollary by applying the Doob's optional sampling theorem at the bounded stopping time $T_y \wedge t$ and the dominated convergence theorem. □

In the sequel, we assume that the exponential moments of the BDLP Z are finite, that is

Assumption 2. $\int_{r<-1} \mathrm{e}^{vr} v(\mathrm{d}r) < \infty$ for every $v \in \mathbb{R}$.

Theorem 9. *For $\gamma, \theta, \lambda > 0$ and $y > x$ we have*

$$\Pi_y^{(\gamma,-\theta)}(x) = \mathrm{e}^{\frac{\theta}{\lambda}(y-x)} \frac{H_{\frac{\gamma}{\lambda},-\frac{\theta}{\lambda}}(x)}{H_{\frac{\gamma}{\lambda},-\frac{\theta}{\lambda}}(y)}. \tag{3.9}$$

Proof. It is well known that when the Lévy measure of Z satisfies the Assumption 2, its Laplace exponent is an entire function, see Skorohod [18]. Then we can follow the same route as for the proof of the Theorem 6, but using the martingale $(\exp(-\xi Z_t - t\psi(-\xi)),\ t \geqslant 0)$, for any $\xi > 0$, in the Girsanov transform. □

Remark 10. For a fixed $\delta > 0$, if we assume only that $\int_{r<-1} \mathrm{e}^{-\frac{\delta}{\lambda} r} v(\mathrm{d}r) < \infty$, then $\Pi^{(\gamma,-\theta)}$ is well defined for any $\theta < \delta$, since the Laplace exponent is analytic in a convex domain.

4. The stable case

We investigate the stable OU processes, i.e. the GOU processes with stable BDLPs, in more detail. We recall that a stable process $Z := (Z_t,\ t \geqslant 0)$ with index $\alpha \in (0, 2]$ is a Lévy process which enjoys the self-similarity property $(Z_{kt}, t \geqslant 0) \overset{(d)}{=} (k^{1/\alpha} Z_t, t \geqslant 0)$, for any $k > 0$.

If the stable process has non-positive jumps, excluding the negative of stable subordinator, its Laplace exponent is given, for $1 < \alpha \leqslant 2$, by

$$\psi(u) = cu^{\alpha}, \quad u \geqslant 0, \tag{4.1}$$

where $c = \tilde{c}\ |\cos(\frac{1}{2}\pi\alpha)|^{-1}$ and $\tilde{c} > 0$, see Sato [16, Example 46.7]. Finally, it is worth noting that if Z is a stable process, with index α, we have the following representation for X, for any $t \geqslant 0$,

$$X_t = \mathrm{e}^{-\lambda t}(x + \tilde{Z}_{\kappa(t)}), \tag{4.2}$$

where $\tilde{Z}$ is an α-stable Lévy process defined on the same probability space as Z and $\kappa(t) = \frac{\mathrm{e}^{\alpha\lambda t} - 1}{\alpha\lambda}$.

We now compute the Laplace transform of the first passage time of a constant level by the stable OU process ($\alpha \in (1, 2]$). As we have said, another proof exists of this result, see Hadjiev [9]. However, we shall describe a methodology which can be extended to more general self-similar Markov processes with one sided jumps and for which singleton is regular for itself. For instance, we refer to Lamperti [11] for a characterization of self-similar processes in $\mathbb{R}^+$, the so-called semi-stable processes. Our proof is based on the self-similarity property of Z. We shall proceed in two steps. First, we give the Mellin transform of the first passage time of the BDLP to a specific curve, see Shepp [17] and Yor [19] for self-similar diffusions with continuous paths and Novikov [14] for spectrally negative Lévy processes. Using a deterministic time change we then derive the Laplace transform of T_y. It is clear that the first passage time of a constant level of these processes inherits the self-similarity property. Consequently, a unique monotone and continuous function φ exists such that, for $\gamma > 0$

$$\mathbb{E}_x[\exp(-\gamma\tau_y)] = \frac{\varphi(\gamma^{1/\alpha}x)}{\varphi(\gamma^{1/\alpha}y)}, \tag{4.3}$$

where $x \lessgtr y$ depending on the side of the jumps of X. We recall that in the stable case $\varphi(x) = \mathrm{e}^{-c^{-1/\alpha}x}$. In order to emphasize the role played by the scaling property in the proof of the following result we shall keep the notation φ. We introduce the following positive random variable

$$\tau_y^d = \inf\{s \geqslant 0;\ Z_s > y(s+d)^{1/\alpha}\} \quad (y > x), \tag{4.4}$$

which is the first passage time of the process Z above the curve $y(t+d)^{\alpha}$.

Theorem 11. *For $1 < \alpha \leqslant 2$ and $m > 0$, the Mellin transform of the random variable τ_y^d is given by*

$$\mathbb{E}_x[(\tau_y^d + d)^{-m}, \tau_y^d < \infty] = d^{-m}\frac{H_m^{\alpha}(d^{-1/\alpha}x)}{H_m^{\alpha}(y)}, \tag{4.5}$$

where

$$H_m^{\alpha}(-x) = \int_0^{\infty} \varphi(-xr^{1/\alpha})\mathrm{e}^{-r}r^{m-1}\,\mathrm{d}r \tag{4.6}$$

$$= \alpha\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(\frac{k}{\alpha}+m)}{c^{k/\alpha}k!}x^k. \tag{4.7}$$

Proof. From (2.2) and the self-similarity of Z, it is clear that the process $(\mathrm{e}^{-\gamma t}\varphi(\gamma^{1/\alpha}Z_t),\ t \geqslant 0)$ is an $\mathscr{F}$-martingale. By an application of Doob's optional sampling theorem, we have (using the bounded stopping time $\tau_y^d \wedge t$ and then applying the dominated convergence theorem)

$$\mathbb{E}_x[\mathrm{e}^{-\gamma\tau_y^d}\varphi(\gamma^{1/\alpha}Z_{\tau_y^d})] = \varphi(\gamma^{1/\alpha}x), \tag{4.8}$$

where by integrating both sides of (4.8) by the measure $\mathrm{e}^{-d\gamma}\gamma^{m-1}\mathrm{d}\gamma$, and using Fubini's theorem we get

$$\mathbb{E}_x\left[\int_0^{\infty}\mathrm{e}^{-\gamma\tau_y^d}\varphi(\gamma^{1/\alpha}Z_{\tau_y^d})\mathrm{e}^{-d\gamma}\gamma^{m-1}\mathrm{d}\gamma\right] = \int_0^{\infty}\varphi(\gamma^{1/\alpha}x)\mathrm{e}^{-d\gamma}\gamma^{m-1}\mathrm{d}\gamma.$$

Using the fact that Z has non-positive jumps, it follows that $Z_{\tau_y^d} = y(\tau_y^d + d)^{1/\alpha}$. Thus,

$$\mathbb{E}_x\left[\int_0^{\infty}\mathrm{e}^{-\gamma(\tau_y^d+d)}\varphi\left(\gamma^{1/\alpha}y(\tau_y^d+d)^{1/\alpha}\right)\gamma^{m-1}\mathrm{d}\gamma\right] = d^{-m}H_m^{\alpha}(d^{-1/\alpha}x).$$

The change of variable $r=\gamma(\tau_y^d+d)$ yields

$$\mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-r}\varphi(r^{1/\alpha}y)r^{m-1}(\tau_y^d+d)^{-m}\,\mathrm{d}r\right]=d^{-m}H_m^\alpha(d^{-1/\alpha}x).$$

Thus, we have

$$\mathbb{E}_x[(\tau_y^d+d)^{-m}]=d^{-m}\frac{H_m^\alpha(d^{-1/\alpha}x)}{H_m^\alpha(y)}. \tag{4.9}$$

Next, note that

$$H_m^\alpha(-x)=\sum_{k=0}^\infty \frac{(-1)^k(c^{-1/\alpha}x)^k}{k!}\int_0^\infty \mathrm{e}^{-r}r^{m+\frac{k}{\alpha}-1}\,\mathrm{d}r.$$

The proof is then completed by using the identity $\Gamma(z)=\int_0^\infty \mathrm{e}^{-r}r^{z-1}\,\mathrm{d}r$, $\Re(z)>0$. □

For more information on the property of the function H, we refer to Novikov [14]. As a consequence we state the following result:

Theorem 12. *The Laplace transform of the random variable* T_y *is given by*

$$\mathbb{E}_x[\exp(-\gamma T_y)]=\frac{H^\alpha_{\frac{\gamma}{\alpha\lambda}}((\alpha\lambda)^{1/\alpha}x)}{H^\alpha_{\frac{\gamma}{\alpha\lambda}}((\alpha\lambda)^{1/\alpha}y)},\quad y>x. \tag{4.10}$$

Proof. Fix $y>x$. We have the following relationship between first passage times:

$$\begin{aligned}
T_y &= \inf\{s\geqslant 0;\ X_s>y\}\\
&= \inf\left\{s\geqslant 0;\ \mathrm{e}^{-\lambda t}\left(x+\int_0^t \mathrm{e}^{\lambda s}\,\mathrm{d}Z_s\right)>y\right\}\\
&= \inf\{s\geqslant 0;\ \mathrm{e}^{-\lambda t}(x+\tilde{Z}_{\tau(s)})>y\}\\
&= \kappa(\inf\{s\geqslant 0;\ x+\tilde{Z}_s>y(\alpha\lambda s+1)^{1/\alpha}\})\\
&= \kappa(T^{(\alpha\lambda)^{-1}}_{(\alpha\lambda)^{1/\alpha}y}),
\end{aligned}$$

where we have performed the deterministic time change $\kappa(t)=\tau^{-1}(t)$, i.e. $\kappa(t)=\frac{1}{\alpha\lambda}\ln(\alpha\lambda t+1)$. Therefore,

$$\begin{aligned}
\mathbb{E}_x[\exp(-\gamma T_y)] &= \mathbb{E}_x\left[\left(\alpha\lambda T^{(\alpha\lambda)^{-1}}_{(\alpha\lambda)^{1/\alpha}y}+1\right)^{-\frac{\gamma}{\lambda\alpha}}\right]\\
&= (\alpha\lambda)^{-\frac{\gamma}{\lambda\alpha}}\mathbb{E}_x\left[\left(T^{(\alpha\lambda)^{-1}}_{(\alpha\lambda)^{1/\alpha}y}+(\alpha\lambda)^{-1}\right)^{-\frac{\gamma}{\lambda\alpha}}\right]\\
&= \frac{H^\alpha_{\frac{\gamma}{\alpha\lambda}}((\alpha\lambda)^{1/\alpha}x)}{H^\alpha_{\frac{\gamma}{\alpha\lambda}}((\alpha\lambda)^{1/\alpha}y)}. \quad \square
\end{aligned}$$

Finally, we mention the expression of $\Pi_y^{(\gamma,\theta)}$ in the following:

Theorem 13. *For $\gamma > 0$, $\theta \in \mathbb{R}$ and $y > x$, we have*

$$\Pi_y^{(\gamma,\theta)}(x) = \mathrm{e}^{-\frac{\theta}{\lambda}(y-x)} \frac{H^{\alpha}_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(x)}{H^{\alpha}_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(y)}, \tag{4.11}$$

where $H^{\alpha}_{v,\beta}(x) = \int_0^\infty \exp\left(xr - \frac{c}{\lambda}\int_0^r (v+\beta)^\alpha \frac{\mathrm{d}v}{v}\right) r^{v-1}\,\mathrm{d}r$.

Remark 14. When Z is a Brownian with drift b (i.e. $\alpha = 2, c = \frac{1}{2}$), we obtain

$$\Pi_y^{(\gamma,\theta)}(x) = \mathrm{e}^{\lambda/2(x^2-y^2)-\lambda b(x-y)} \frac{D_{\bar{v}}(-\sqrt{2\lambda}(x - \frac{b}{\lambda} - \frac{\theta}{\lambda^2}))}{D_{\bar{v}}(-\sqrt{2\lambda}(y - \frac{b}{\lambda} - \frac{\theta}{\lambda^2}))}, \tag{4.12}$$

where $\bar{v} := \frac{\theta^2}{2\lambda^3} + \frac{b\theta}{\lambda^2} - \frac{\gamma}{\lambda}$ and $D_v(x) = \frac{\mathrm{e}^{-x^2/2}}{\Gamma(-v)}\int_0^\infty \exp\left(-xr - \frac{1}{2}r^2\right) r^{-v-1}\,\mathrm{d}r$ denotes the cylinder parabolic function, see e.g. Lebedev [12]. We also note that, by taking $b = 0$ in (4.12), we recover the result of Lachal [10].

5. Application to finance

We apply the results of the previous sections to the pricing of a European option on maximum on yields in the generalized Vasicek framework. We extend the results of Leblanc and Scaillet [13] by allowing jumps in the interest rate dynamics. We refer to their paper for the motivation and the description of the financial problems.

In our framework, that is when the interest rate dynamics is given as the solution to (1.1), it is an easy task to derive the current price of the discount bond

$$\begin{aligned} P_x(0,T) &:= \mathbb{E}_x\left[\exp\left(-\int_0^T X_s\,\mathrm{d}s\right)\right] \\ &= \exp(A(T)x + D(T)), \end{aligned} \tag{5.1}$$

where $A(t) = \frac{1}{\lambda}(1 - \mathrm{e}^{-\lambda t})$ and $D(t) = -\int_0^t \psi(A(r))\,\mathrm{d}r$, where ψ stands for the Laplace exponent of Z. The price of the option is given by

$$C^X(0,T^*,K;x,T) := \mathbb{E}_x\left[\mathrm{e}^{-\int_0^T X_s\,\mathrm{d}s}\left(\sup_{u\in[0,T^*]} X(u,T) - K\right)^+\right],$$

where $K \in \mathbb{R}^+$ (resp. $T^* \in \mathbb{R}^+$) denotes the strike (resp. the time to maturity). Next, we shall give a closed form expression for the Laplace transform with respect to time to maturity of this functional. For $\gamma > 0$, we introduce the notation

$$L_\gamma(0,K;x,T) := \int_0^\infty \mathrm{e}^{-\gamma T^*} C^X(0,T^*,K;x,T)\,\mathrm{d}T^*. \tag{5.2}$$

Proposition 15. *We assume that $\int_{r<-1} \mathrm{e}^{-\frac{1}{\lambda}r} v(\mathrm{d}r) < \infty$. Then, for $x \leqslant K$, we have*

$$L_\gamma(0,K;x,T) = H_{\frac{\gamma}{\lambda},-\frac{1}{\lambda}}(x)\int_K^\infty \mathrm{e}^{y/\lambda}\frac{P_\gamma(y)}{H_{\frac{\gamma}{\lambda},-\frac{1}{\lambda}}(y)}\,\mathrm{d}y, \tag{5.3}$$

where $P_\gamma(y) := \int_0^\infty \mathrm{e}^{-\gamma T} P_y(0,T)\,\mathrm{d}T$.

Proof. By using the strong Markov property of the process X we obtain

$$\begin{aligned} L_\gamma(0,K;x,T) &= \mathbb{E}_x\left[\int_K^\infty \mathrm{d}y \int_0^\infty \mathrm{d}T^* \exp\left(-\gamma T^* - \int_0^{T^*} X_s\,\mathrm{d}s\right)\mathbb{I}_{\{\sup_{[0,T^*]} r_u > y\}}\right] \\ &= \mathbb{E}_x\left[\int_K^\infty \mathrm{d}y \int_{T_y}^\infty \mathrm{d}T^* \exp\left(-\gamma T^* - \int_0^{T^*} X_s\,\mathrm{d}s\right)\right] \\ &= \mathbb{E}_x\left[\int_K^\infty \mathrm{d}y \int_{T_y}^\infty \mathrm{d}T^* \exp\left(-\gamma(T^* - T_y) - \gamma T_y \right.\right. \\ &\qquad \left.\left. - \int_0^{T_y} X_s\,\mathrm{d}s - \int_{T_y}^{T^*} X_s\,\mathrm{d}s\right)\right] \\ &= \int_K^\infty \mathrm{d}y\, \mathbb{E}_x\left[\exp\left(-\gamma T_y - \int_0^{T_y} X_s\,\mathrm{d}s\right)\right] P_\gamma(y). \end{aligned}$$

To get the desired expression for the Laplace transform of the option price it remains for us to compute $\mathbb{E}_x[\exp(-\gamma T_y - \int_0^{T_y} X_s\,\mathrm{d}s)]$. From Theorem 9 and Remark 10, choosing $\theta = 1$, we obtain

$$\mathbb{E}_x\left[\exp\left(-\gamma T_y - \int_0^{T_y} X_s\,\mathrm{d}s\right)\right] = \mathrm{e}^{\frac{1}{\lambda}(y-x)} \frac{H_{\frac{\gamma}{\lambda},-\frac{1}{\lambda}}(x)}{H_{\frac{\gamma}{\lambda},-\frac{1}{\lambda}}(y)}.$$

The identity (5.3) follows. □

Finally, we conclude this section by investigating the possibility that the interest rates in the mean reverting stable Vasicek model, become negative. In this case, we have $\psi(u) = bu + c\delta^\alpha u^\alpha$, $\delta > 0$. The Laplace transform of the limiting distribution of the process X is given by

$$\begin{aligned} E[\exp(u\rho^X)] &= \exp\left(c\delta^\alpha u^\alpha \int_0^\infty \mathrm{e}^{-\lambda\alpha r}\,\mathrm{d}r + bu\int_0^\infty \mathrm{e}^{-\lambda r}\,\mathrm{d}r\right), \\ &= \exp\left(\frac{c\delta^\alpha}{\lambda\alpha} u^\alpha + \frac{b}{\lambda} u\right). \end{aligned}$$

We recognize the Laplace transform of a α-stable random variable with $\delta^\alpha/\lambda\alpha$, $\beta = -1$ and b/λ. In Table 1, we show the probability of a negative long-term interest rate

Table 1
Probabilities of negative long-term interest rate p^n and mean value $\bar{r}$ for different values of the stable index ($b = 0.01$, $\lambda = 0.1$, $\delta = 0.00025$)

α	$p^n(\approx)$	y
2	0	1
1.8	1.1×10^{-7}	0.0996
1.5	6.4×10^{-5}	0.099
1.2	0.015	0.086

p^n and the mean value $\bar{r}$ for different values of the index but with the other parameter being constant ($b = 0.01$, $\lambda = 0.1$ and $\delta = 0.00025$). These results are the outcomes of Monte Carlo simulation. We recall that for $\alpha = 2$ the mean value is simply given by the coefficient of the drift term b/λ, whereas for $1<\alpha<2$ the stable random variables without drift are not centered. We observe that the probability of negative interest rate decreases with the index α, but remains very small for moderate values of α. Moreover, the mean value of X stays almost unchanged for the same value of the index and equals the ratio $b/\lambda = 0.1$, which is a realistic level, for instance, for an annual interest rate. It is worth noting that it is possible to get both very small values for p^n and reasonable values for long-term interest rates y for any α by playing with the family of the parameters (λ, b, δ).

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