On some First Passage Time Problems
Motivated by Financial Applications

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A ma famille

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Abstract

From both theoretical and applied perspectives, first passage time problems for random processes are challenging and of great interest. In this thesis, our contribution consists on providing explicit or quasi-explicit solutions for these problems in two different settings.

In the first one, we deal with problems related to the distribution of the first passage time (FPT) of a Brownian motion over a continuous curve. We provide several representations for the density of the FPT of a fixed level by an Ornstein-Uhlenbeck process. This problem is known to be closely connected to the one of the FPT of a Brownian motion over the square root boundary. Then, we compute the joint Laplace transform of the L^1 and L^2 norms of the 3-dimensional Bessel bridges. This result is used to illustrate a relationship which we establish between the laws of the FPT of a Brownian motion over a twice continuously differentiable curve and the quadratic and linear ones. Finally, we introduce a transformation which maps a continuous function into a family of continuous functions and we establish its analytical and algebraic properties. We deduce a simple and explicit relationship between the densities of the FPT over each element of this family by a selfsimilar diffusion.

In the second setting, we are concerned with the study of exit problems associated to Generalized Ornstein-Uhlenbeck processes. These are constructed from the classical Ornstein-Uhlenbeck process by simply replacing the driving Brownian motion by a Lévy process. They are diffusions with possible jumps. We consider two cases: The spectrally negative case, that is when the process has only downward jumps and the case when the Lévy process is a compound Poisson process with exponentially distributed jumps. We derive an expression, in terms of

new special functions, for the joint Laplace transform of the FPT of a fixed level and the primitives of theses processes taken at this stopping time. This result allows to compute the Laplace transform of the price of a European call option on the maximum on the yield in the generalized Vasicek model. Finally, we study the resolvent density of these processes when the Lévy process is α-stable ($1 < \alpha \leq 2$). In particular, we construct their q-scale function which generalizes the Mittag-Leffler function.

Zusammenfassung

Grenzüberschreitungsprobleme in stochastischen Prozessen sind herausfordernd und sehr interessant, sowohl vom theoretischen als auch vom angewandten Standpunkt betrachtet. Der Beitrag dieser Dissertation besteht aus (quasi-)expliziten Lösungen für solche Probleme in zwei verschiedenen Fällen.

Im ersten Fall behandeln wir Probleme im Zusammenhang mit der Verteilung der ersten berschreitungszeit (First Passage Time, FPT) einer Brownschen Bewegung über eine stetige Kurve. Wir zeigen mehrere Darstellungen für die Dichte der FPT eines Ornstein-Uhlenbeck-Prozesses über einen konstanten Schwellwert. Dieses Problem ist bekanntermassen eng verbunden mit jenem der FPT einer Brownschen Bewegung über die Quadratwurzelfunktion. Wir berechnen dann die gemeinsame Laplace-Transformierte der L^1- und L^2-Normen der dreidimensionalen Bessel-Brücken. Dieses Resultat wird verwendet zur Illustration einer von uns hergestellten Beziehung zwischen der Verteilung der FPT einer Brownschen Bewegung über eine zweimal stetig differenzierbare Funktion und der Verteilung im quadratischen und im linearen Fall. Schliesslich führen wir eine Transformation ein, die eine stetige Funktion auf eine Familie von stetigen Funktionen abbildet, und wir zeigen die analytischen und algebraischen Eigenschaften dieser Transformation. Mit Hilfe einer selbstähnlichen Diffusion leiten wir eine einfache und explizite Beziehung her zwischen den Dichten der FPT über jedes Element der Familie.

Im zweiten Fall befassen wir uns mit dem Studium von Austrittsproblemen im Zusammenhang mit verallgemeinerten Ornstein -Uhlenbeck-Prozessen. Diese werden aus klassischen Ornstein-Uhlenbeck-Prozessen

konstruiert, indem man die treibende Brownsche Bewegung durch einen Lévy-Prozess ersetzt. Es sind dies Diffusionen mit mglichen Sprüngen. Wir betrachten zwei mgliche Fälle: Erstens den spektral negativen Fall, in dem der Prozess nur Abwärtssprünge aufweist, und zweitens den Fall, in dem der Lévy-Prozess ein verbundener Poisson-Prozess mit exponentialverteilten Sprüngen ist. Wir leiten eine Darstellung her basierend auf neuen, speziellen Funktionen für die gemeinsame Laplace-Transformierte der FPT über einen konstanten Schwellwert und den Primitiven dieser Prozesse betrachtet an der so definierten Stoppzeit. Dieses Resultat ermglicht die Berechnung der Laplace-Transformierten einer Europäischen Call-Option auf dem maximalen Zins im verallgemeinerten Vasicek-Modell. Schliesslich studieren wir die Dichte des Resolventen solcher Prozesse für den Fall, in dem der Lévy-Prozess α-stabil ist mit $1 < \alpha \leq 2$. Insbesondere konstruieren wir deren Skalenfunktion, welche die Mittag-Leffler-Funktion verallgemeinert.

Contents

Introduction

The motivation for studying first passage time problems is two-fold. On the one hand, they are of great theoretical interest since they are connected to many fields of mathematics such as probability theory, functional analysis, number theory and numerical analysis. On the other hand, its theory has drawn tremendous amount of attention in many scientific disciplines. Indeed, first passage time distributions are required in many phenomena in chemistry, physics, biology and neurology. For example, an understanding of such stopping times is important for those studying the theories of chemical reaction rates, neuron dynamics, and escape-rates in diffusion processes with absorbing boundaries. These systems typically depend on a random variable (or one that can be adequately approximated as random) reaching some threshold value.
In quantitative finance, such questions arise in many different practical issues such as the pricing of path-dependent options and credit risk. The former options, as tailor-made contingent claims, have become increasingly popular hedging and speculation tools in recent years. In particular, path-dependent options, most of them comprise barrier options, are successful to reduce the cost of hedging. These barrier options embed digital options. If the relative position of the underlying and the boundary matters at the date of maturity, these binary derivatives are of the European type and their valuation is simpler than European call or put options. On the other hand, if this relative position matters during the entire time to maturity, pricing these digital derivatives is more involved as they are path-dependent. In the latter case, they are dubbed one-touch digital options and their valuation boils down to computing *first passage time distributions*. Another important family of path-dependent options are the lookback ones. Their payoff depends

on the maximum or minimum underlying asset price attained during the option's life. Furthermore, other common assets can be showed to involve digital options when properly modelled. For example, corporate bondholders have a short position on digital positions, where the boundary corresponds to the default threshold. By the same token, Longstaff and Schwartz [79] price digital credit-spread options with the logarithm of the credit spread assumed to follow a mean-reverting process.

Despite the importance and wide applications of first passage times, explicit analytic solutions to such problems are not known except for very few instances. Among them, we mention that for one-dimensional time-homogeneous diffusions, the Laplace transform of the first passage time is given as a solution of a second order differential equation subject to some boundary conditions, see for instance the book of Borodin and Salminen [17] for a collection of explicit results. The Laplace transform of the first passage time, above and below, of a spectrally negative Lévy process is also known in terms of the Laplace exponent and the scale function associated to the process, see e.g. the thorough survey of Bingham [14]. In these two cases, several numerical methods have been developed to inverse these Laplace transforms, see for instance Abate and Whitt [1], Linetsky [78] and Rogers [106]. More generally, the mainstream of the research of the problem with general boundary crossings for Markov processes, is based on the Kolmogorov partial differential equations for the transition probability density function, and focuses on finding solutions of certain integral or differential equations for the first passage time densities. Our contribution in this thesis, is to find explicit solutions to some problems related to first passage times by using merely martingales techniques. More precisely, we shall consider the following two issues.

The first one deals essentially with the distribution of the first crossing time of the Brownian motion over some deterministic continuous functions. This is an old and still open problem. For instance, the formula which states that the density, denoted by p, of the first passage time of a Brownian motion over the linear boundary $c + bt$ is given by

$$p(t) = \frac{c}{t^{3/2}} \Phi\left(\frac{c + bt}{\sqrt{t}}\right)$$

with $\Phi(y) = \frac{1}{2\pi} e^{-y^2/2t}$, is called the Bachelier-Lévy formula, see e.g. Lerche [76]. Indeed, Lévy [77] refers in *Processus stochastiques et mouvement*

Brownien to Bachelier who has already treated first-passage densities in 1900 in his thesis *Théorie de la Spéculation*. In statistics the problem of determining the time of first passage of a Brownian motion to certain moving barriers arises asymptotically in sequential analysis, see Darling and Siegert [26], in computing the power of statistical test, see Durbin [36] and the iterated logarithm law, see Robbins and Siegmund [103], Novikov [84] and the references therein. We also mention that a review of applications in engineering can be found in Blake and Lindsey [15].

In the second part, we consider some exit problems associated to generalized Ornstein-Uhlenbeck processes, for short GOU. The classical Ornstein-Uhlenbeck process was first derived by Ornstein and Uhlenbeck [121] as the solution of the Langevin equation

$$\frac{du}{dt} = -\lambda u(t) + A(t)$$

where the first term on the right is due to the frictional resistance which is supposed proportional to the velocity. The second term represents the random forces (Maxwell's law or Gaussian distribution in this case). They were interested in computing the transition probabilities. Then, Doob [34] studied their path properties derived from the ones of the Brownian motion by using a deterministic time change. In this paper, he also studied the path properties of the solution of the Langevin equation with random forces given as symmetric stable distributions. Finally, Hadjiev [51] introduced the GOU process as the solution of the linear stochastic differential equation

$$dX_t = -\lambda X_t dt + dZ_t$$

where Z is a spectrally negative Lévy process, that is a process with stationary and independent increments and continuous in probability having only negative jumps. He gives an explicit form for the Laplace transform of their first passage times above. We mention that in the literature several terminologies can be found for this class of random processes: Ornstein-Uhlenbeck type processes, shot noise processes, filtered Poisson process, etc.. From a theoretical viewpoint, the interest of studying exit problems associated to this class of processes relies on understanding better the fluctuation of more general time-homogeneous Markov processes beyond the Lévy processes. Although the increments of the GOU processes are not independent neither stationary, it is still

possible to get explicit results thanks to the relationship with their underlying Lévy process. Moreover, GOU processes have found many applications in several fields. Recently, they have been used intensively in finance, for modelling the stochastic volatility of a stock price process, see e.g. Barndorff and Shephard [9], and for describing the dynamics of the instantaneous interest rate. The latter application, as a generalization of the Vasicek model, deserves a particular attention as these processes belong to the class of one factor affine term structure model. These are well known to be tractable, in the sense that it is easy to fit the entire yield curve by basically solving Riccati equations, see Duffie et al. [35] for a survey on affine processes.

Organisation and Outline of this Thesis

This thesis consists of five self-contained chapters, each with its own introduction. They are all devoted to the treatment of examples of first passage time problems and related objects. The settings differ from each other either by changing the process or the boundary. The processes considered in the thesis are Brownian motion, Gauss-Markov processes of Ornstein-Uhlenbeck type (continuous) and generalized Ornstein-Uhlenbeck processes (with jumps), while the boundaries are taken to be either constant or continuous deterministic functions. In the following lines, we discuss the content of each chapter and we quote in parenthesis the paper(s) related to each chapter.

Chapter 1. ([7]) In this Chapter, different expressions for the density of the first passage time of a fixed level by the classical Ornstein-Uhlenbeck process are gathered. This problem has attracted attention for a long time but the interest was renewed recently due to some general importance in many fields of applied mathematics. For instance, in finance, this density is used for the pricing of lookback options on yields in the Vasicek model.
The expressions consist of a series expansion involving parabolic functions and their zeros, the representation using Bessel bridges and an integral representation. Detailed algorithm for implementing each approach is provided and some numerical simulations are performed. This Chapter can be considered as a survey, for instance, the known series expansion for the density has never been rigorously proved in the litera-

ture. It can also be seen as a guiding case for computing the distribution of first passage time of more general diffusions.

Chapter 2. ([4]) The second Chapter is devoted to the study of the joint distribution of the couple $\left(\int_0^1 r_s\, ds, \int_0^1 r_s^2\, ds\right)$ where r is a 3-dimensional Bessel bridge between x and $y \geq 0$. The motivation for studying the law of this bivariate random variable comes from its intimate connection with the law of the first passage time of the Brownian motion over the square root and quadratic boundaries. It is also challenging to develop a probabilistic methodology to compute explicitly this joint Laplace transform. An instructive probabilistic construction of the parabolic cylinder is provided. Next, for the case $y = 0$ the distribution of the above couple is obtained by using merely stochastic tools. The case $y \neq 0$ is then studied by making use of the Feynman-Kac formula. Then, it is shown that the distribution of the first passage time of a continuous time random process over the linear and quadratic boundaries can be obtained as the limit of the distribution of a familly of first passage times of this process over any "smooth" (in a neighborood of 0) boundaries. This device is illustrated with the example of the first passage time of a Brownian motion over the square root boundary. In this case, it yields to an easy computation of some limits of ratio of parabolic cylinder functions. These limits appeared already in the literature but relied on complicated analytical arguments. Finally, the joint Laplace tranform of the bivariate random variable is used to derive some new explicit formulas concerning some functional of the 3- and 1- dimensional Bessel process and the radial part of the δ-dimensional Ornstein-Uhlenbeck processes.

Chapter 3. ([6],[3],[5]) The third chapter consists on the study of some functional transformations and their application to the boundary crossing problem for selfsimilar diffusions, which are either Brownian motion, Bessel processes or their natural powers.
Let B be a standard Brownian motion and f a continuous function on $\mathbb{R}^+$ with $f(0) \neq 0$. Introduce $T^{(f)} = \inf\{s \geq 0;\ B_s = f(s)\}$. The explicit determination of the distribution of $T^{(f)}$, even for elementary functions, is an old and difficult task which has been initiated by Bachelier (constant level) and Lévy (linear curve) and has attracted the attention of many researchers. The main result of this Chapter is an explicit relationship between the law of $T^{(f)}$ and the one of the first passage time of the Brownian motion to a family of curves obtained from f via

the following transform

$$\begin{array}{rccl} S^{(\alpha,\beta)}: & C(\mathbb{R}_0^+,\mathbb{R}^+) & \longrightarrow & C([0,\zeta^{(\beta)}),\mathbb{R}^+) \\ & f & \longmapsto & \left(\dfrac{1+\alpha\beta.}{\alpha}\right) f\left(\dfrac{\alpha^2.}{1+\alpha\beta.}\right) \end{array}$$

where $\alpha > 0, \beta \in \mathbb{R}$ and $\zeta^{(\beta)} = -\beta^{-1}$ when $\beta < 0$, and equals to $+\infty$ otherwise. We shall develop two different methodologies to establish this connection.
In order to describe the main steps of the first one, we need to introduce the Gauss-Markov process of Ornstein-Uhlenbeck type with parameter $\phi \in C([0,a),\mathbb{R}^+)$ (for short $GMOU$), denoted by $U^{(\phi)}$, defined by

$$U_t^{(\phi)} = \phi(t)\left(U_0^{(\phi)} + \int_0^t \phi^{-1}(s)\, dB_s\right), \quad 0 \le t < a,$$

where $U_0^{(\phi)} \in \mathbb{R}$. The first step consists on showing that the law of U is connected via a time-space harmonic function to the law of a family of GMOU processes whose parameters are obtained from ϕ as follows. For $\alpha > 0$ and β reals, we define the mapping $\Pi^{(\alpha,\beta)}$ by

$$\begin{array}{rccl} \Pi^{(\alpha,\beta)}: & \mathcal{C}_\infty(\mathbb{R}^+) & \longrightarrow & \mathcal{C}_\infty(\mathbb{R}^+) \\ & \phi & \longmapsto & \phi(.)\left(\alpha + \beta\displaystyle\int_0^{\cdot} \phi^{-2}(s)ds\right) \end{array}$$

where $\mathcal{C}_\infty(\mathbb{R}^+) := \bigcup_{b>0} C([0,b),\mathbb{R}^+)$. As a second step, we show that the law of the level crossing to a fixed boundary of a GMOU process is linked to the law of the first passage time of the Brownian motion to a specific curve via a deterministic time change. We now describe the transform Σ which connects the parameter of the GMOU and the curve. To a function $\phi \in \mathcal{C}_\infty(\mathbb{R}^+)$ we associate the increasing function $\tau(.) = \int_0^{\cdot} \frac{ds}{\phi^2(s)}$ and denote by A its inverse. We define the mapping Σ by

$$\begin{array}{rccl} \Sigma: & \mathcal{C}_\infty(\mathbb{R}^+) & \longrightarrow & \mathcal{C}_\infty(\mathbb{R}^+) \\ & \phi & \longmapsto & 1/\phi \circ A. \end{array}$$

The mapping Σ is called *Doob's transform*. Finally, the original transform is constructed by combining the two previous ones in the following way

$$S^{(\alpha,\beta)} = \Sigma \circ \Pi^{(\alpha,-\beta)} \circ \Sigma, \quad (\alpha,\beta) \in \mathbb{R}^+ \times \mathbb{R}_0^+.$$

We study in details these transformations by providing some algebraic and analytical properties.
For the second approach, we fix $\alpha = 1$ and set $S^{(1,\beta)} = S^{(\beta)}$. We observe that, when $\beta < 0$, the image by $S^{(\beta)}$ of a standard Brownian motion is a Brownian bridge of length $-\beta^{-1}$. Algebraic and analytic properties of $S^{(\beta)}$ are also studied.
As new classes of explicit examples, the family $a\sqrt{(1+\lambda_1 t)\,(1+\lambda_2 t)}$, where a, $\lambda_1 \neq \lambda_2$ are real numbers is considered for the Brownian motion. It is shown that this methodology also applies to the Bessel processes and in this case we investigate the first passage time over a straight line.
This Chapter ends up with a survey of the usual methods for the treatment of the first crossing time over a single curve by a Brownian motion.

Chapter 4. ([91]) Some first passage time problems are considered therein for spectrally negative generalized Ornstein-Uhlenbeck processes. For a fixed real $a > x$ we define the stopping time $T_a = \inf\{s \geq 0; X_s \geq a\}$ and introduce the primitive $I_t = \int_0^t X_s\,ds$ defined for $t \geq 0$. We recall the Laplace transform of T_a which has been computed by Hadjiev [51] and Novikov [86]. Note that in this case T_a is actually a hitting time since these processes do not have positive jumps and therefore they hit levels above continuously. Then, the attention is focussed on the joint distribution of the couple (T_a, I_{T_a}). The associated double-Laplace transform is provided in terms of new functions. The case when the underlying Lévy process is given as a sum of a spectrally negative Lévy process and an independent Compound Poisson process with exponential jumps is also considered. The explicit form for the joint Laplace transform is computed explicitly. The Chapter closes with an analytical formula for the Laplace transform with respect to time to maturity of the price of a European call option on maximum on yields in the framework of generalized Vasicek models. These models belong to the attractive class of affine term structure models.

Chapter 5. ([90]) The last Chapter is an attempt to find the law of the first passage time of a level below the starting point, the associated overshoot and the exit from an interval of spectrally negative α-stable $(1 < \alpha \leq 2)$ Ornstein-Uhlenbeck processes, denoted by X. These are particular instances of the processes studied in Chapter 4, that is the Lévy process is in this case an α-stable process. The solution of these problems for processes with jumps are only known explicitly in the sim-

pler case of spectrally negative Lévy process: The Laplace transform are obtained in terms of the so-called scale function and the Laplace exponent of these processes. All the techniques developed for the Lévy processes rely on their space homogeneity property. Since X does not have this property, a new approach using tools borrowed from the potential theory (q-potential of local time, q-resolvent density, dual process, switch identity between Markov processes) is suggested to solve these problems. The nice feature of this approach is to reduce the problems described above in the one of computing the Laplace transform of the hitting times (to any level) of X which can be obtained by finding an appropriate family of martingales. For the downward hitting at the level 0, this is done by introducing a new function which generalizes the Mittag-Leffler one. It is shown that this function is the so-called scale function of X. However, for any other levels y ($y \neq 0$) below, the problem remains open. The Laplace transforms of first passage times are provided for some related processes such as the process killed when it enters the negative half line and the process conditioned to stay positive. The law of the maximum of the associated bridge is characterized in terms of the q-resolvent density. By letting λ tend to zero in the definition of X (see the SDE in Chapter 4), so that λ-parameterized family converges to the driving Lévy process Z, some results are recovered for spectrally negative α-stable Lévy processes.

Chapter 1

Representations of the First Passage Time Density of an Ornstein-Uhlenbeck Process

Det er svært at spå, specielt om fremtiden.
Storm P.
(It is difficult to predict, especially about the future.)

1.1 Introduction

In this Chapter, we gather different expressions for the density function of the first passage time (or first hitting time) to a fixed level by an Ornstein-Uhlenbeck process, abbreviated as OU process. This density

is used in different areas of mathematical finance. Indeed, it is connected to some pricing formulas of interest rate path-dependent options when the dynamics of the underlying asset is assumed to be a mean reverting OU process. For this, we refer to [73] and the references therein. The knowledge of the sought density is also relevant in credit risk modelling, see e.g. Jeanblanc and Rutkowski [58]. It is also required in other fields of applied mathematics. For instance in biology, see [116], this is used for modelling the time between firings of a nerve cell. Recently, Leblanc et al. [73] and [74] showed that the density can be expressed as the Laplace transform of a functional of a 3-dimensional Bessel bridge. However, the authors used therein an erroneous spatial homogeneity property for the 3-dimensional Bessel bridge, a mistake that has been noticed by several authors, including [47]. The feature of this representation is of probabilistic nature and the details are given in Section 1.5. We provide two other explicit expressions obtained by different techniques. The feature of these two representations is of analytic nature. The first expression is a series expansion involving the eigenvalues of a Sturm-Liouville boundary value problem associated with the Laplace transform of the first passage time (see e.g. Keilson and Ross [63]). An analytic continuation argument is used to compute the cosine transform of the first passage time which gives an integral representation of the density. As discussed above – in specific contexts in mathematical finance – there is a need to perform numerical computations. The three representations suggest ways to approximate the density function. We point out that the OU process is considered here as a case study since it is possible to adapt readily the methodologies described below for a large class of one-dimensional diffusions. The remainder of this Chapter is organized as follows. In the next Section the OU process is reviewed and basic properties of the first passage time are presented. In Section 1.3, 1.4 and 1.5 the series, the integral, and the Bessel bridge representations of the density are respectively derived. Section 1.6 is devoted to numerical computations. Finally, some properties of Hermite and parabolic cylinder functions are recalled in Section 1.7.

1.2 Preliminaries on OU Processes

Let $B := (B_t, t \geq 0)$ be a standard Brownian motion. The associated OU process $U := (U_t, t \geq 0)$, with parameter $\lambda \in \mathbb{R}$, is defined to be

the unique solution of the equation

$$dU_t = dB_t - \lambda U_t\, dt, \quad U_0 = x \in \mathbb{R}. \tag{1.1}$$

This linear equation when integrated yields the realization

$$U_t = e^{-\lambda t}\Big(x + \int_0^t e^{\lambda s}\, dB_s\Big), \quad t \geq 0.$$

By the Dambis, Dubins-Schwarz Theorem, see [100, p.181], there is a Brownian motion $W := (W_t, t \geq 0)$, defined on the same probability space, such that

$$\int_0^t e^{\lambda s} dB_s = W_{\tau(t)}, \quad t \geq 0,$$

where $\tau(t) = (2\lambda)^{-1}(e^{2\lambda t} - 1)$. Hence, the representation $U_t = e^{-\lambda t}\big(x + W_{\tau(t)}\big)$ holds. We mention that this latter relation was first introduced by Doob [34]. He exploited this fact to derive some path properties of U. In particular, he showed that this process has almost surely continuous paths which are nowhere differentiable. In what follows, we suppose that $\lambda > 0$. In this case, U is positively recurrent and its semigroup has a unique invariant measure which is the law of a centered Gaussian random variable with variance $1/2\lambda$.

Remark 1.2.1 *Note that if U_0 is chosen to be distributed as the invariant measure and independent of B, we get the only stationary Gaussian Markov process.*

The process U is a Feller one. Its infinitesimal generator, denoted by $\mathcal{G}$, is given, on $\mathcal{C}_b^2(\mathbb{R})$, by

$$\mathcal{G}f(x) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x) - \lambda x \frac{\partial f}{\partial x}(x), \quad x \in \mathbb{R}.$$

Next, denote by $\mathbb{P}_x^{(\lambda)}$ the law of U when $U_0 = x \in \mathbb{R}$. Then, thanks to Girsanov's Theorem, for any fixed $t > 0$, the following absolute continuity relationship holds

$$d\mathbb{P}^{(\lambda)}_{x|\mathcal{F}_t} = \exp\Big(-\frac{\lambda}{2}(B_t^2 - x^2 - t) - \frac{\lambda^2}{2}\int_0^t B_s^2 ds\Big)\, d\mathbb{P}_{x|\mathcal{F}_t} \tag{1.2}$$

where $\mathbb{P}_x = \mathbb{P}_x^{(0)}$ stands for the law of B started at x.

Remark 1.2.2 *We point out that the Radon-Nikodym derivative (1.2) is a true martingale. Indeed, in the case of absolute continuity between solutions of stochastic differential equations, it is not necessary to use Novikov or Kazamaki criteria for checking the martingale property. We refer to Mckean [81, p.66-67] for the description of the conditions for this result to hold in a more general setting.*

Let $a \in \mathbb{R}$ be given and fixed, and introduce the first passage times

$$H_a = \inf\{s \geq 0;\, U_s = a\} \text{ and } T_a = \inf\{s \geq 0;\, B_s = a\}.$$

The law of H_a (resp. T_a) is absolutely continuous with respect to the Lebesgue measure and its density is denoted by $p^{(\lambda)}_{x\to a}(\cdot)$ (resp. $p_{x\to a}(\cdot)$) i.e. $\mathbb{P}^{(\lambda)}_x(H_a \in dt) = p^{(\lambda)}_{x\to a}(t)\, dt$, $t > 0$. The focus, in this Chapter, will be on the situation that U starts below the hitting barrier, that is $x < a$. The other case can be recovered by replacing a and x with $-a$ and $-x$ in the density (since $-U$ is also an OU process). For the Laplace transform of H_a, we recall the following well-known result, see Siegert [112] or Breiman [19].

Proposition 1.2.3 *For $x < a$, the Laplace transform of H_a is given by*

$$\mathbb{E}_x\left[e^{-\alpha H_a}\right] = \frac{\mathcal{H}_{-\alpha/\lambda}(-x\sqrt{\lambda})}{\mathcal{H}_{-\alpha/\lambda}(-a\sqrt{\lambda})} = \frac{e^{\lambda x^2/2}\mathcal{D}_{-\alpha/\lambda}(-x\sqrt{2\lambda})}{e^{\lambda a^2/2}\mathcal{D}_{-\alpha/\lambda}(-a\sqrt{2\lambda})} \tag{1.3}$$

where $\mathcal{H}_\nu(\cdot)$ and $\mathcal{D}_\nu(\cdot)$ stand for the Hermite and parabolic cylinder functions respectively, see Section 1.7 for a thorough study of these functions.

Remark 1.2.4 *In Proposition 2.2.3, we will show a new proof of this result which relies merely on probabilistic arguments.*

Proof. Thanks to the general theory of one-dimensional diffusion, we refer to Itô and McKean [57, p.150], the Laplace transform of the first passage time is the unique solution of the following Sturm-Liouville boundary value problem

$$\begin{aligned} \mathcal{G}u(x) &= \alpha u(x), \text{ for } x < a, \\ u(x)_{|x=a} &= 1 \text{ and } \lim_{x\to-\infty} u(x) = 0. \end{aligned} \tag{1.4}$$

This is a singular boundary value problem since the interval is not bounded. We refer to [57], where it is shown that the solution to the above problem takes the form

$$\mathbb{E}_x[e^{-\alpha H_a}] = \frac{\psi_\alpha(x)}{\psi_\alpha(a)}$$

where $\psi_\alpha(\cdot)$ is, up to some multiplicative constant, the unique increasing positive solution of the equation $\mathcal{G}\psi = \alpha\psi$. By the definition of Hermite functions, see Section 1.7, we get that $\psi_\alpha(x) = \mathcal{H}_{-\alpha/\lambda}(x\sqrt{\lambda})$ leading to (1.3). This completes the proof. □

Remark 1.2.5 *For $\lambda < 0$ the process U is transient. The study can be related to the recurrent case as follows. By the chain's rule, see e.g. Borodin and Salminen [17], we have for any fixed $t > 0$*

$$d\mathbb{P}^{(\lambda)}_{x|\mathcal{F}_t} = \exp\big(\lambda(U_t^2 - x^2 - t)\big)\, d\mathbb{P}^{(-\lambda)}_{x|\mathcal{F}_t}.$$

This combined with Doob's optional stopping Theorem yields

$$p^{(\lambda)}_{x\to a}(t) = \exp\big(\lambda(a^2 - x^2 - t)\big)p^{(-\lambda)}_{x\to a}(t), \quad t > 0.$$

Remark 1.2.6 *Note that thanks to the scaling property of B, we see that $\mathbb{E}_x\big[e^{-\alpha H_a}\big] = \mathbb{E}_{x\sqrt{\lambda}}\big[e^{-\alpha\lambda^{-1}H_{a\sqrt{\lambda}}}\big]$ and hence*

$$p^{(\lambda)}_{x\to a}(t) = \lambda p^{(1)}_{x\sqrt{\lambda}\to a\sqrt{\lambda}}(\lambda t), \quad t > 0. \tag{1.5}$$

Therefore, the study may be reduced to the case $\lambda = 1$.

Remark 1.2.7 *For the special case $a = 0$ there is a simple expression for $p^{(\lambda)}_{x\to 0}(\cdot)$. Indeed, we shall first recall that for the Brownian motion, recovered by letting $\lambda \to 0$, we have*

$$p_{x\to a}(t) = \frac{|a-x|}{\sqrt{2\pi t^3}} \exp\left(-\frac{(a-x)^2}{2t}\right), \quad t > 0. \tag{1.6}$$

Now, with $T_a^{(\sqrt{})} = \inf\{s \geq 0;\, W_s + x = a\sqrt{1+2\lambda s}\}$, Doob's transform implies the identity $T_a^{(\sqrt{})} = \tau(H_a)$ a.s., as noticed by [19]. We deduce that $p^{(\lambda)}_{x\to 0}(t) = \tau'(t)p_{x\to 0}(\tau(t))$, and recover thus

$$p^{(\lambda)}_{x\to 0}(t) = \frac{|x|}{\sqrt{2\pi}} \exp\left(-\frac{\lambda x^2 e^{-\lambda t}}{2\sinh(\lambda t)} + \frac{\lambda t}{2}\right)\left(\frac{\lambda}{\sinh(\lambda t)}\right)^{3/2}, \quad t > 0, \tag{1.7}$$

which appeared in Pitman and Yor [96].

Remark 1.2.8 *When, in (1.1), B is replaced by $(B_t + \mu t, t \geq 0)$, for some $\mu \in \mathbb{R}$, the resulting process is a mean reverting one. This is given by*

$$^{(\mu)}U_t = \frac{\mu}{\lambda} + e^{-\lambda t}\left(x - \frac{\mu}{\lambda} + \int_0^t e^{\lambda s}\, dB_s\right), \quad t > 0.$$

The corresponding first passage time density, denoted by $^{(\mu)}p^{(\lambda)}_{x\to a}(t)$, is easily seen to be related to that with $\mu = 0$ via

$$^{(\mu)}p^{(\lambda)}_{x\to a}(t) = p^{(\lambda)}_{x-\frac{\mu}{\lambda}\to a-\frac{\mu}{\lambda}}(t), \quad t > 0.$$

1.3 The Series Representation

This Section is devoted to inverting the Laplace transform of the distribution of H_a by means of the Cauchy Residue Theorem. Let $\mathcal{D}_\nu(\cdot)$ be the parabolic cylinder function with index $\nu \in \mathbb{R}$. For a fixed b, denote by $(\nu_{j,b})_{j\geq 1}$ the ordered sequence of positive zeros of the function $\nu \mapsto \mathcal{D}_\nu(b)$. We are now ready to state the following result which appeared without a rigorous justification in many references. For instance, we found it in [63] and also in [101] where the authors study the zeros of the parabolic cylinder functions. A similar expression is given in [87] for the density of the first passage time of the Brownian motion to the square root boundary, connected to the distribution we are focusing on by Doob's transform.

Theorem 1.3.1 *Fix $x < a$, then the density of H_a is given by the following series expansion*

$$p^{(\lambda)}_{x\to a}(t) = -\lambda e^{\lambda(x^2-a^2)/2} \sum_{j=1}^{\infty} \frac{\mathcal{D}_{\nu_{j,-a\sqrt{2\lambda}}}(-x\sqrt{2\lambda})}{\mathcal{D}'_{\nu_{j,-a\sqrt{2\lambda}}}(-a\sqrt{2\lambda})}\, e^{-\lambda \nu_{j,-a\sqrt{2\lambda}} t} \tag{1.8}$$

where $\mathcal{D}'_{\nu_{j,b}}(b) = \frac{\partial \mathcal{D}_\nu(b)}{\partial \nu}|_{\nu=\nu_{j,b}}$. For any $t_0 > 0$, the series converges uniformly for $t > t_0$.

Proof. The substitution $v(x) = e^{-x^2/4}\, u(x/\sqrt{2\lambda})$ transforms (1.4) into the Weber equation $v'' - \left(\frac{\alpha}{\lambda} - \frac{1}{2} + q(x)\right)v = 0$ where $q(x) := x^2/4$. A

fundamental solution of the latter equation is given by $x \mapsto \mathcal{D}_{-\alpha/\lambda}(-x)$. Since $x \mapsto q(x)$ is real-valued, continuous and $q(x) \to \infty$ as $x \to \infty$, the Weber operator has a pure point spectrum, we refer to Hille [54, Theorem 10.3.4]. Moreover, the eigenvalues are simple, positive and bounded from below, see [101] and [120] for more details about the distribution of the spectrum. As a consequence, the Laplace transform (1.3) is meromorphic as a function of the parameter α, whose poles are simple, negative and are given by the sequence $\{\alpha_j = -\lambda\nu_{j,-a\sqrt{2\lambda}}\}_{j\geq 1}$. The residue of the Laplace transform at $\alpha_j, j > 0$, is easily computed to be

$$\mathrm{Res}_{\alpha=\alpha_j}\, \mathbb{E}_x[e^{-\alpha\, H_a}] = -\lambda e^{\lambda(x^2-a^2)/2} \frac{\mathcal{D}_{-\alpha_j/\lambda}(-x\sqrt{2\lambda})}{\mathcal{D}'_{-\alpha_j/\lambda}(-a\sqrt{2\lambda})}.$$

To check that the conditions of [53] are satisfied, we make use of the asymptotic properties of parabolic cylinder functions recalled in Section 1.7. The Heaviside expansion Theorem in [53] gives the expression of the density where the parameters are given by the eigenvalues of the associated Sturm-Liouville equation. The uniform convergence of the series on $[t_0, \infty)$, for any $t_0 > 0$, follows from the asymptotic formulas (1.18) and (1.19). □

The following local limit result is essentially due to the fact that the series in formula (1.8) is uniformly convergent.

Corollary 1.3.2 *Let the situation be as in Theorem 1.3.1, then*

$$\lim_{T\to\infty} e^{\lambda\nu_{1,-a\sqrt{2\lambda}}T}\mathbb{P}_x^{(\lambda)}\left(H_a > T\right) = \frac{e^{\lambda(x^2-a^2)/2}}{\nu_{1,-a\sqrt{2\lambda}}} \frac{\mathcal{D}_{\nu_{1,-a\sqrt{2\lambda}}}(-x\sqrt{2\lambda})}{\mathcal{D}'_{\nu_{1,-a\sqrt{2\lambda}}}(-a\sqrt{2\lambda})}.$$

Remark 1.3.3 *The distribution of H_a is infinitely divisible and may be viewed as an infinite convolution of elementary mixtures of exponential distributions. Kent [66] establishes a link between the canonical measure of the first passage time of a fixed level by a one-dimensional diffusion and the spectral measure of its infinitesimal generator. When the left end point of the diffusion is not natural, the same author gives the series expansion based on the spectral decomposition, see [65]. However, in our case, the left-end point is natural therefore such methodology cannot be applied directly.*

1.4 The Integral Representation

In this Section, we compute the cosine transform of the distribution of H_a. Then the density $p^{(\lambda)}_{x\to a}(\cdot)$ can be computed out from the cosine transform, and its inverse, on $L^1(\mathbb{R}^+)$, via

$$p^{(\lambda)}_{x\to a}(t) = \frac{1}{2\pi}\int_0^\infty \cos(\alpha t)\mathbb{E}_x\left[\cos(\alpha H_a)\right]\, d\alpha, \quad t>0.$$

Theorem 1.4.1 *Fix $x<a$, then the density of H_a is given by*

$$p^{(\lambda)}_{x\to a}(t) = \frac{\lambda}{2\pi}\int_0^\infty \cos(\alpha\lambda t)\widehat{\mathcal{H}}_{-\alpha}\left(-\sqrt{\lambda}a, -\sqrt{\lambda}x\right)\, d\alpha \tag{1.9}$$

where

$$\widehat{\mathcal{H}}_\alpha(a,x) = \frac{\mathcal{H}r_\alpha(a)\mathcal{H}r_\alpha(x)+\mathcal{H}i_\alpha(x)\mathcal{H}i_\alpha(a)}{\mathcal{H}r^2_\alpha(a)+\mathcal{H}i^2_\alpha(a)}$$

and $\mathcal{H}r_\alpha(\cdot)$ and $\mathcal{H}i_\alpha(\cdot)$ are specified by formulas (1.15) and (1.16) respectively.

Proof. To simplify the notation in the proof we only consider the case $\lambda=1$, and the general case $\lambda>0$ can be recovered from (1.5). The Laplace transform (1.3) is analytic on the domain $\{\alpha\in\mathbb{C};\ \mathrm{Re}(\alpha)\geq 0\}$. Moreover, from the proof of Theorem 1.3.1, it is clear that the ratio of the parabolic cylinder function is analytic on the domain $\{\alpha\in\mathbb{C};\ \mathrm{Re}(\alpha)>-\nu_{1,-a\sqrt{2}}\}$, where we recall that $\nu_{1,-a\sqrt{2}}$ is the smallest positive zero of the function $\nu\mapsto\mathcal{D}_\nu(-a\sqrt{2})$. By analytical continuation, we deduce that the Laplace transform is analytical on the domain $\{\alpha\in\mathbb{C};\ \mathrm{Re}(\alpha)>-\nu_{1,-a\sqrt{2}}\}$. It follows that

$$\begin{aligned}\mathbb{E}_x[\cos(\alpha H_a)] &= \mathrm{Re}\left(\frac{\mathcal{H}_{i\alpha}(-x)}{\mathcal{H}_{i\alpha}(-a)}\right)\\ &= \frac{\mathcal{H}r_{-\alpha}(-a)\mathcal{H}r_{-\alpha}(-x)+\mathcal{H}i_{-\alpha}(-x)\mathcal{H}i_{-\alpha}(-a)}{\mathcal{H}r^2_{-\alpha}(-a)+\mathcal{H}i^2_{-\alpha}(-a)}.\end{aligned}$$

The statement follows from the injectivity of the cosine transform. □

1.5 The Bessel Bridge Representation

As mentioned in the Introduction, computing explicitly $p^{(\lambda)}_{x\mapsto a}(t)$ amounts to characterizing the distribution of a quadratic functional of the 3-dimensional Bessel bridge. In order to recall the connection we need to provide some properties of the 3-dimensional Bessel process R . This process might be defined as the radial part of a 3-dimensional Brownian motion. In what follows, we just give some results which we shall use in this Section and we postpone to Chapter 2 a more detailed study of these processes. First, set for $y \in \mathbb{R}^+$, $L_y = \sup\{s \geq 0;\ R_s = y\}$, then Williams' time reversal result, see e.g. Revuz and Yor [100, p.498], says that the processes $\big(y - B_{T_y - u},\ u \leq T_y\big)$ and $(R_u,\ u \leq L_y)$ are equivalent. A second time reversal result which we call the switching identity, states that, for $y \in \mathbb{R}^+$, the processes $(R_s, s \leq t)$ conditionally on $R_0 = x$ and $R_t = y$ and $(R_{t-s}, s \leq t)$ conditionally on $R_0 = y$ and $R_t = x$ have the same distribution, see [100, p.468, Exercise 3.7]. The process $(R_s, s \leq t)$ conditionally on $R_0 = x$ and $R_t = y$, which we simply denote by r, is the so-called 3-dimensional Bessel bridge over the interval $[0, t]$ between x and y. It is the unique strong solution of the stochastic differential equation, for $s < t$,

$$dr_s = \Big(\frac{y - r_s}{t - s} + \frac{1}{r_s}\Big)ds + dB_s, \quad r_0 = x,\ r_t = y.$$

Now, we quote the following result from [47] and provide its detailed proof for the sake of completeness.

Theorem 1.5.1 *Fix $x < a$, then the density of H_a is given by*

$$p^{(\lambda)}_{x\to a}(t) = e^{-\lambda(a^2 - x^2 - t)/2}\mathbb{E}_{0\to a-x}\left[e^{-\frac{\lambda^2}{2}\int_0^t (r_u - a)^2\,du}\right]p_{x\to a}(t) \qquad (1.10)$$

where r is a 3-dimensional Bessel bridge over the interval $[0, t]$ between 0 and $a - x$ and $p_{x\to a}(\cdot)$ is given in (1.6).

Remark 1.5.2 *From this result, we shall derive, in Chapter 2, the joint Laplace transform of the L^1 and L^2 norms of the 3-dimensional Bessel bridges.*

Proof. Combining relation (1.2) and Doob's optional stopping Theorem, we get

$$p^{(\lambda)}_{x\to a}(t) = e^{-\frac{\lambda}{2}(a^2-x^2-t)} b(t) p_{x\to a}(t),$$

where we set

$$b(t) = \mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}\int_0^t B_u^2\,du} \,\middle|\, T_a = t \right].$$

Next, we use successively the spatial homogeneity, the symmetry of B, Williams' time reversal identity, the transience property of R and the commuting identity for R, in order to write

$$\begin{aligned} b(t) &= \mathbb{E}_{a-x}\left[e^{-\frac{\lambda^2}{2}\int_0^t (B_u-a)^2\,du} \,\middle|\, T_0 = t \right] \\ &= \mathbb{E}_0\left[e^{-\frac{\lambda^2}{2}\int_0^t (R_u-a)^2\,du} \,\middle|\, L_{a-x} = t \right] \\ &= \mathbb{E}_0\left[e^{-\frac{\lambda^2}{2}\int_0^t (R_u-a)^2\,du} \,\middle|\, R_t = a-x \right] \\ &= \mathbb{E}_{a-x}\left[e^{-\frac{\lambda^2}{2}\int_0^t (R_{t-u}-a)^2 du} \,\middle|\, R_t = 0 \right], \end{aligned}$$

which completes the proof.

□

1.6 Numerical Illustrations

Two standard techniques for approximating the density of the first passage time of diffusions are: the numerical approach to the solution of the partial differential equation associated to the density (analytic method) and direct Monte Carlo simulation (probabilistic method). The three representations of the density suggest alternative ways to perform numerical computations in the OU process case. Below, we provide a short description of these approaches. We illustrate them, in the last subsection, with two examples.

Series representation. The first approximation is to use the series expansion (1.8). The infinite series is truncated after the first N terms,

that is,

$$f_S(t) = -\lambda e^{\lambda(x^2-a^2)/2} \sum_{j=1}^{N} b_j \exp\big(-\lambda a_j t\big), \quad t > 0,$$

where $a_j = \nu_{j,-a\sqrt{2\lambda}}$ and $b_j = D_{\nu_{j,-a\sqrt{2\lambda}}}(-x\sqrt{2\lambda})/D'_{\nu_{j,-a\sqrt{2\lambda}}}(-a\sqrt{2\lambda})$. For t small, $f_S(t)$ is negative or decreasing. Let t_0 be the point where $f_S(t_0) = 0$ or $f'_S(t_0) = 0$. Hence, the approximation of $p^{(\lambda)}_{x\to a}(\cdot)$ is given by $f_S(t)$ for $t \geq t_0$ and 0 for $0 < t < t_0$. The parabolic cylinder function $\mathcal{D}_\nu(x)$ can be approximated by the series expansion given by (1.17) and (1.12). From this, numerical values of $\nu_{j,-a\sqrt{2\lambda}}$, $\mathcal{D}_{\nu_{j,-a\sqrt{2\lambda}}}(-x\sqrt{2\lambda})$ and $\mathcal{D}'_{\nu_{j,-a\sqrt{2\lambda}}}(-a\sqrt{2\lambda})$ can be estimated where the last term is computed by the differential quotient. A problem is to choose suitable N for a prescribed truncation error. Since we are approximating a density, there are many ways to measure the quality of the chosen truncation parameter N. We give an average error $\bar{e}$ based on large-n asymptotics that is independent of the argument t and is easy to compute. Integrating the absolute value of the Nth term of the series and using the asymptotic formulas (1.18) and (1.19) yield

$$\begin{aligned} \int_0^\infty |\lambda e^{\lambda(x^2-a^2)/2}\, b_N e^{-\lambda a_N t}|\, dt &= e^{\lambda(x^2-a^2)/2}\, |b_N|/a_N \\ &\sim \pi^{-1} e^{\lambda(x^2-a^2)/2}\, N^{-1}. \end{aligned}$$

The average error is defined to by $\bar{e} = \pi^{-1}e^{\lambda(x^2-a^2)/2}N^{-1}$. When $\nu_{j,-a\sqrt{2\lambda}}$, $j = 1, \ldots, N$, are estimated, it is easy to numerically compute the expectation of a bounded function of the first passage time (e.g. prices of interest rate options presented in [73]) using the approximation. Then $\bar{e}$ gives a measure of how precise the expectation is estimated. In the examples below we chose $N = 100$ and in Example 1 $\bar{e}$ is equal to 0.005.

Integral representation. It is not a good method to approximate the integral in (1.9) by the corresponding Riemann sum. Instead, we make use of the trapezoidal rule. The approximation formula for $p^{(\lambda)}_{x\to a}(\cdot)$ by the integral representation is then given by

$$f_I(t) = \frac{e^{A/2}}{2t} h^x_a\left(\frac{A}{2t}\right) + \frac{e^{A/2}}{t} \sum_{k=1}^{N} (-1)^k \operatorname{Re}\left(h^x_a\left(\frac{A+2ik\pi}{2t}\right)\right) \quad (1.11)$$

where $h_a^x(\alpha) = \mathcal{H}_{\alpha/\lambda}(-x)/\mathcal{H}_{\alpha/\lambda}(-a)$ and $A > 0$ is a constant. It follows from Section 1.4 that the Laplace transform is given by $\mathbb{E}_x^{(\lambda)}[e^{-\nu\sigma_a}] = \mathcal{H}_{-\nu/\lambda}(-x\sqrt{\lambda})/\mathcal{H}_{-\nu/\lambda}(-a\sqrt{\lambda})$. Also, for this approach, the question remains about a good choice for A and N. The numerical computation of the integral leads to the discretisation error and the truncation error (both depends on the argument t). A bound for the discretisation error is Ce^{-A} where C is constant that dominates the density. In the examples below $A = 18.1$ so the discretisation error is of order 10^{-7}. There is no simple bound for the truncation error. One can choose N when the value of the last term is small. We set $N = 500$ in the examples which is a conservative choice. In practice, one can determine A and N based on trial and error. We refer to Abate and Whitt [1], for precise statements and more details on this approximation method.

Bessel bridge representation. For the Bessel bridge approach, it is needed to resort to some simulation techniques to compute the functional of the 3-dimensional Bessel bridges in the expression (1.10). With the notation $\mathbb{E}\left[G\left(\int_0^t g(r_s)\,ds\right)\right]$ where G is some measurable and bounded function and g is a regular function, the three steps to follow are

1. First, we compute the integral by considering the corresponding Riemann sum
$$\mathbb{E}\left[G\left(\int_0^t g(r_s)\,ds\right)\right] \simeq \mathbb{E}\left[G\left(\sum_{k=1}^n g\left(r_{k\frac{t}{n}}\right)\right)\right].$$

2. We approach r with another process $\bar{r}$ by means of the Euler scheme.
$$\mathbb{E}\left[G\left(\sum_{k=1}^n g\left(r_{k\frac{T}{n}}\right)\right)\right] \simeq \mathbb{E}\left[G\left(\sum_{k=1}^n g\left(\bar{r}_{k\frac{T}{n}}\right)\right)\right].$$
The same step of discretisation is chosen for the Euler scheme and the Riemann sum.

3. Finally, to estimate the expectation we use Monte Carlo method by simulating a large number M of independent paths of the process $\bar{r}$
$$\mathbb{E}\left[G\left(\sum_{k=1}^n g\left(\bar{r}_{k\frac{T}{n}}\right)\right)\right] \simeq \frac{1}{M}\sum_{i=1}^M G\left(\sum_{k=1}^n g\left(\bar{r}^{(i)}_{k\frac{T}{n}}\right)\right).$$

Putting these steps together, at the end, the approximation formula for (1.10) is given by

$$f_B(t) = e^{-\lambda(a^2-x^2-t)/2} \frac{1}{M} \sum_{i=1}^{M} G\left(\sum_{k=1}^{n} g\left(\bar{r}^{(i)}_{k\frac{T}{n}} \right) \right) p_{x\to a}(t).$$

Implementation and results. The two first approaches are analytic methods and very easy to implement using programs like Maple or Mathematica, where it is possible to use built-in functions. However, these require the knowledge of the Laplace transform of the first passage time which can be computed only for some specific continuous Markov processes. The Bessel bridge approach is a probabilistic method. Its main advantage compared to the direct Monte Carlo one is that it overcomes the problem of detecting the time at which the approximated process crosses the boundary. We refer to [46] for an explanation of the difficulties encountered with the direct Monte Carlo method. We also emphasize that this algorithm estimates directly the density whereas the direct Monte Carlo provides an approximation of the distribution function. This method can readily be used to treat similar problems for continuous Markov processes which laws are absolutely continuous with respect to the Wiener measure.

In order to test the performance of the three methodologies, we carried out two numerical examples. In both examples we have used the following approximation parameters. For the series representation we used $N = 100$ in the truncated series $f_S(\cdot)$. For the integral method, we have chosen $A = 18.1$ and took $N = 500$ terms in the series of $f_I(\cdot)$. In the approximation $f_B(\cdot)$, the Bessel bridge method, we have simulated $M = 10^5$ paths of the Bessel bridge with $n = 1000$ time steps on the interval $[0, 4]$. In both examples we took the parameter of the OU process to be $\lambda = 1$, which is sufficient by (1.5).

Example 1: We examine the example $a = 0$, which is the only case where the density is known in closed form, indeed given by (2.9). The OU process is starting from $x = -1$. The numerical approximations of the density $p^{(1)}_{-1\to 0}(t)$ are collected in Table 1. The table shows that all the analytical approaches are accurate up to 10^{-5} digits whereas the simulation approach is accurate up to 10^{-3} digits. Note that for the series method $t_0 = 0.044$ and hence for $t = 0.04$ the approximated value for the density is set to be 0 as described above. In fact, $f_S(0.04) =$

-0.0019.

Example 2: In this example the OU process starts from $x = 0$. We computed the density $p^{(1)}_{0\to a}(t)$ for a equals $0.50, 0.75$ and 1.00.

In Figure 1, the results of the three densities are presented. In this example there is no check of the numerical values since there is no closed form formulas. But from Figure 1, one sees that the three methods give numerical values which are very close and can hardly be distinguished.

t	0.04	0.08	0.1	0.25
Explicit	0.000310	0.057549	0.144538	0.762172
Series	–0.001865	0.057540	0.144538	0.762172
Integral	0.000310	0.005754	0.144538	0.762172
Bessel Bridge	0.000310	0.057538	0.144534	0.762074
t	0.5	0.75	1	1.5
Explicit	0.760954	0.584084	0.441483	0.257945
Series	0.760954	0.584084	0.441483	0.257945
Integral	0.760954	0.584084	0.441483	0.257945
Bessel Bridge	0.760946	0.584362	0.441648	0.258012
t	2	2.5	3	4
Explicit	0.154101	0.092934	0.056248	0.020670
Series	0.154101	0.092934	0.056248	0.020670
integral	0.154122	0.092612	0.055968	0.020670
Bessel Bridge	0.154107	0.092841	0.056203	0.020596

Table 1.1: *Different values of the density $p^{(1)}_{-1\to 0}(t)$ of the first passage time to the level $a = 0$ for an OU process starting from $x = -1$ with parameter $\lambda = 1$.*

1.7 Hermite Functions and their Complex Decomposition

The special functions used in previous Sections are recalled below and most the results can be found in Lebedev [72, Chapter 10]. The Hermite

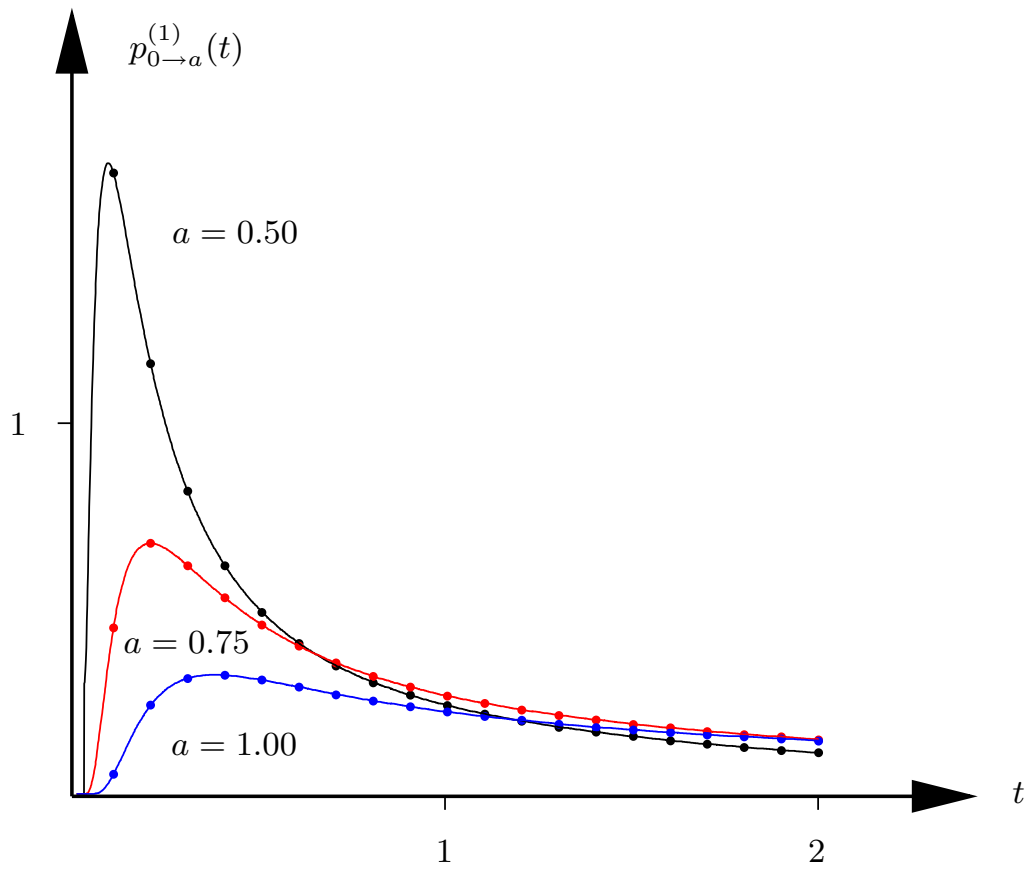

Figure 1.1: *A drawing of the density $t \mapsto p^{(\lambda)}_{x\to a}(t)$ for three values of a when $\lambda = 1$ and $x = 0$. Solid line: Series representation. Dashed line = Bessel Bridge approach.*

function $\mathcal{H}_\nu(z)$ is defined by

$$\mathcal{H}_\nu(z) = \frac{2^\nu \Gamma(\frac{1}{2})}{\Gamma(\frac{1-\nu}{2})} \Phi\Big(-\frac{\nu}{2}, \frac{1}{2}; z^2\Big) + \frac{2^{\nu+\frac{1}{2}}\Gamma(-\frac{1}{2})}{\Gamma(-\frac{\nu}{2})} z\Phi\Big(\frac{1-\nu}{2}, \frac{3}{2}; z^2\Big)$$

where Φ denotes the confluent hypergeometric function and Γ the gamma function. The Hermite function has the following series representation

$$\mathcal{H}_\nu(z) = \frac{1}{2\Gamma(-\nu)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \Gamma\Big(\frac{m-\nu}{2}\Big)(2z)^m, \quad |z| < \infty, \qquad (1.12)$$

and satisfies the recurrence relations

$$\frac{\partial \mathcal{H}_\nu(z)}{\partial z}|_{z=a} = 2\nu \mathcal{H}_{\nu-1}(a), \quad \mathcal{H}_{\nu+1}(z) = 2z\mathcal{H}_\nu(z) - 2\nu\mathcal{H}_{\nu-1}(z).$$

$\mathcal{H}_\nu(z)$ is an entire function in both the variable z and parameter ν. The couple $\mathcal{H}_\nu(\pm\cdot)$ forms a fundamental solution to the ordinary Hermite equation

$$v'' - 2zv' + 2\nu v = 0. \tag{1.13}$$

The Hermite function, see [72, p.297], has the integral representation

$$\mathcal{H}_\nu(z) = \frac{2^{\nu+1}}{\Gamma((1-\nu)/2)} \int_0^\infty e^{-u^2} u^{-\nu} (u^2+z^2)^{\nu/2}\, du, \tag{1.14}$$

for $\mathrm{Re}(\nu) < 1$ and $|\arg z| < \pi/2$. In particular, we have

$$\mathcal{H}_\nu(0) = 2^\nu \frac{\Gamma(1/2)}{\Gamma((1-\nu)/2)}.$$

With the notation

$$\frac{\mathcal{H}_{i\nu}(z)}{\mathcal{H}_{i\nu}(0)} = \mathcal{H}r_\nu(z) + i\mathcal{H}i_\nu(z),$$

we get, from the representation (1.14),

$$\mathcal{H}r_\nu(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \cos\left(\frac{\nu}{2}\log\left(1+\left(\frac{z}{u}\right)^2\right)\right) du \tag{1.15}$$

$$\mathcal{H}i_\nu(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \sin\left(\frac{\nu}{2}\log\left(1+\left(\frac{z}{u}\right)^2\right)\right) du. \tag{1.16}$$

Replacing ν by $i\nu$ in (1.13) and equalizing the real and imaginary parts yield the system

$$\mathcal{G}\mathcal{H}r - 2\nu\mathcal{H}i = 0 \text{ and } \mathcal{G}\mathcal{H}i + 2\nu\mathcal{H}r = 0,$$

with boundary conditions

$$\mathcal{H}r(0) = 1, \quad \mathcal{H}i(0) = 0.$$

The Weber equation

$$v'' + \left(\nu + \frac{1}{2} - \frac{z^2}{4}\right)v = 0$$

has as a particular solution the parabolic cylinder function

$$\mathcal{D}_\nu(z) = 2^{-\nu/2} e^{-z^2/4} \mathcal{H}_\nu(z/\sqrt{2}), \quad z \in \mathbb{R}. \tag{1.17}$$

We have the asymptotic formulas

$$\mathcal{D}_\nu(z) \sim \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi\nu} z^{-\nu-1} e^{z^2/4} \text{ for } |z| \to +\infty,\ \frac{\pi}{4} < |\arg z| < \frac{5\pi}{4},$$

$$\mathcal{D}_\nu(z) \sim z^\nu e^{-z^2/4} \text{ for } |z| \to +\infty, \quad |\arg z| < \frac{3\pi}{4},$$

$$\mathcal{D}_\nu(z) \sim \sqrt{2} e^{-\frac{1}{4}(2\nu+1)} \cos\left(z\sqrt{\nu+\frac{1}{2}} - \frac{\pi}{2}\nu\right)\left(1 + O(\nu^{-\frac{1}{4}})\right)$$

$$\text{for } \nu \to +\infty, z \in \mathbb{R}.$$

We deduce from the later formula the following large-n asymptotics

$$\nu_{n,a} \sim 2n - 1 + \frac{2\lambda a}{\pi^2} + \frac{4a}{\pi}\sqrt{n - \frac{1}{4} + \frac{\lambda a}{\pi^2}} \tag{1.18}$$

and

$$\frac{\mathcal{D}_{\nu_n}(-x\sqrt{2\lambda})}{\mathcal{D}'_{\nu_n}(-a\sqrt{2\lambda})} \sim \frac{(-1)^{n+1} 2\sqrt{\nu_n + \frac{1}{2}}}{\pi\sqrt{\nu_n + \frac{1}{2}} - \sqrt{2\lambda} a} \cos\left(x\sqrt{\lambda(2\nu_n+1)} - \frac{\pi}{2}\nu_n\right) \tag{1.19}$$

where $\nu_n = \nu_{n,-a\sqrt{2\lambda}}$ and for a fixed a, $\nu_{n,a}$ denotes the nth positive zero of the function $\nu \mapsto \mathcal{D}_\nu(a)$. We point out that the above representations for the Hermite function might obviously be fit to the parabolic cylinder one.

Chapter 2

On the Joint Law of the L^1 and L^2 Norms of the 3-Dimensional Bessel Bridge

Agdud mebla idles d'arggaz mebla iles.
Berber proverb.
(A people without culture, it is like a man without words.)

2.1 Introduction

Let $r := (r_s, s \leq t)$ be a 3-dimensional Bessel bridge over the interval $[0, t]$ between x and y, where x, y are some positive real numbers and t is a fixed time horizon. Introduce the couple of random variables

$$\left(N_t^{(1)}(r), N_t^{(2)}(r)\right) = \left(\int_0^t r_s ds, \int_0^t r_s^2 ds\right). \tag{2.1}$$

In this Chapter, we aim to compute explicitly its joint Laplace transform. Let W be a standard real-valued Brownian motion started at

$x \in \mathbb{R}$ and recall that $T_a^{(\sqrt{})} = \inf\{s \geq 0;\ W_s = a\sqrt{1+2\lambda s}\}$, where $\lambda > 0$ and $a \in \mathbb{R}$. As mentioned in Remark 1.2.7, Doob's transform allows to relate $T_a^{(\sqrt{})}$ to the first passage time of the level a by an Ornstein-Uhlenbeck process with parameter λ. That is with $H_a = \inf\{s \geq 0;\ U_s = a\}$ and

$$U_t = e^{-\lambda t}\left(x + \int_0^t e^{\lambda s} dB_s\right), \quad t \geq 0, \tag{2.2}$$

where B is another real-valued Brownian motion defined on the same probability space, we have $T_a^{(\sqrt{})} = \frac{1}{2\lambda}\log(1+2\lambda H_a)$ $a.s.$. We shall see that the determination of the distribution of H_a, or equivalently that of $T_a^{(\sqrt{})}$, amounts to studying the joint distribution of the L^1 and L^2 norms of a 3-dimensional Bessel bridge. While we are interested in the joint law, we mention that there is a substantial literature devoted to the study of the law of the L^1 norm of the Brownian excursion, that is when $x = y = 0$, see e.g. [80], [49], [92] and [59]. The L^2 norm of the Bessel bridge, which is closely related to the Lévy stochastic area formula, has been also intensively studied by many authors including for instance [97], [45] and the references therein.

Then, motivated by recovering the results for the L^1 and L^2 norms of a 3-dimensional Bessel bridge from the joint law, we develop a stochastic device which allows to get the limit when one of the parameters of the Laplace transform tends to 0. To this end, we establish a relationship between the first passage times of the Brownian motion to a large class of (smooth) curves to the linear or quadratic ones. As a by-product, we show some connections between certain stochastic objects and some special functions. We will show that this device applies to continuous time stochastic processes.

The Chapter is organized as follows. In the next Section, after some preliminaries on the 3-dimensional Bessel process, we derive the sought joint law in terms of transforms via stochastic techniques for the case $y = 0$. In particular, we give a probabilistic construction of the parabolic cylinder function which characterizes the Laplace transform of the first passage time of a fixed level by an Ornstein-Uhlenbeck process. For any $y > 0$, we resort to the Feynman-Kac formula. Then, in Section 2.3 we show some relationships between first passage times over some moving boundaries for general stochastic processes which we apply to

the first passage time of the Brownian motion over the square root boundary. This link allows to get some asymptotic results for ratio of parabolic cylinder functions. We end up this Chapter by making some connections between the studied law and the one of some other functionals.

2.2 On the Law of $\left(N_t^{(1)}(r), N_t^{(2)}(r)\right)$

The 3-dimensional Bessel process, denoted by R, is defined to be the unique strong solution of

$$dR_t = dB_t + \frac{1}{R_t}dt, \quad R_0 = x \geq 0.$$

This is a strong Markov process with speed measure given by $m(dy) = 2y^2dy$. Its semigroup is absolutely continuous with respect to m with density

$$q_t(x,y) = \frac{1}{2\sqrt{2\pi t}}\frac{1}{yx}\left(e^{-\frac{1}{2t}(x-y)^2} - e^{-\frac{1}{2t}(x+y)^2}\right), \quad x,y,t > 0,$$

and by passage to the limit as y tends to zero we obtain

$$q_t(x,0) = \frac{1}{\sqrt{2\pi t^3}}e^{-\frac{x^2}{2t}}, \quad x,t > 0.$$

We shall denote by $\mathbb{Q}_x$ the law of R when it is started at x and we simply write $\mathbb{Q}$ for $x = 0$. Next, for y and $t \geq 0$, the conditional measure $\mathbb{Q}^t_{x,y} = \mathbb{Q}_x[\,.\mid R_t = y]$, viewed as a probability measure on $\mathcal{C}([0,t],[0,\infty))$, stands for the law of the 3-dimensional Bessel bridge starting at x and ending at y at time t. Since R is transient, we have $\mathbb{Q}^t_{x,y} = \mathbb{Q}_x[\,.\mid L_y = t]$ where $L_y = \sup\{s \geq 0;\ R_s = y\}$. Williams' time reversal relationship states that, for $R_0 = 0$, $B_0 = x > 0$, the processes $(R_{L_x - s}, s \leq L_x)$ and $(B_s, s \leq T_0)$ are equivalent.

Next, introduce, for $x, y, \beta \geq 0$. the resolvent kernel, or the Green's function, G given by

$$G_\beta(x,y)dy = \int_0^\infty e^{-\beta t}\mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}\int_0^t R_s^2\,ds - \alpha\int_0^t R_s\,ds}, R_t \in dy\right]\,dt.$$

As we shall see below we have $G_\beta(x,y) = w_\beta^{-1} m(y)\varphi_\beta(x\wedge y)\psi_\beta(x\vee y)$ where φ_β (resp. ψ_β) is the unique, up to some multiplicative positive constant, decreasing, positive and bounded at $+\infty$ solution (resp. increasing, positive and bounded at 0 solution) of the Sturm-Liouville equation

$$2^{-1}\phi''(x) + x^{-1}\phi'(x) - \left(2^{-1}\lambda^2 x^2 + \alpha x + \beta\right)\phi(x) = 0, \quad x > 0. \quad (2.3)$$

For a fixed $t \geq 0$, let us introduce the notation

$$\Upsilon^{\lambda,\alpha}_{x\to y}(t) = \mathbb{E}_{x\to y}\left[e^{-\frac{\lambda^2}{2}N_t^{(2)}(r) - \alpha N_t^{(1)}(r)}\right], \quad \lambda, x \text{ and } \alpha \geq 0.$$

We denote simply $\Upsilon^{\lambda,\alpha}_x(t)$ (resp. $\Upsilon^{\lambda,\alpha}(t)$) for $\Upsilon^{\lambda,\alpha}_{x\to 0}(t)$ (resp. $\Upsilon^{\lambda,\alpha}_{0\to 0}(t)$).

Remark 2.2.1 *We point out that, thanks to the scaling property of Bessel processes, we have the identity* $\Upsilon^{\lambda,\alpha}_{x\to y}(t) = \Upsilon^{\lambda t^2, \alpha t^{3/2}}_{\frac{x}{\sqrt{t}}\to\frac{y}{\sqrt{t}}}(1)$.

2.2.1 Stochastic Approach for the Case $y = 0$

In here we show how to solve the Sturm-Liouville boundary value problem (2.3) by using stochastic devices.

Theorem 2.2.2 *For $x > 0$ and β, α and $\lambda \geq 0$, we have*

$$\int_0^\infty e^{-\beta t} q_t(x,0)\Upsilon^{\lambda,\alpha}_x(t)\,dt = \frac{1}{x}\frac{\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\frac{\alpha}{\lambda^2})\right)}{\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}.$$

Consequently, We have

$$\int_0^\infty \left(e^{-\beta t} - 1\right)\Upsilon^{\lambda,\alpha}(t)\frac{dt}{\sqrt{2\pi t^3}} = \sqrt{2\lambda}$$
$$\times\left(\frac{\mathcal{D}^{(x)}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}{\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)} - \frac{\mathcal{D}^{(x)}_{\frac{\alpha^2}{2\lambda^3}-\frac{1}{2}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}{\mathcal{D}_{\frac{\alpha^2}{2\lambda^3}-\frac{1}{2}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}\right)$$

where $\mathcal{D}^{(x)}_\nu(y) = \frac{\partial \mathcal{D}_\nu(x)}{\partial x}|_{x=y}$.

Proof. We fix $a = \alpha/\lambda^2$, observe that

$$\Upsilon_x^{\lambda, a\lambda^2}(t) = e^{a^2\lambda^2 t/2}\, \mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}\int_0^t (R_u+a)^2\, du} \,\middle|\, R_t = 0\right]. \tag{2.4}$$

Following a line of reasoning similar to the proof of Theorem 1.5.1, we get

$$\mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}\int_0^t (R_u+a)^2\, du} \,\middle|\, R_t = 0\right] = \mathbb{E}_{x+a}\left[e^{-\frac{\lambda^2}{2}\int_0^t B_u^2\, du} \,\middle|\, T_a = t\right]. \tag{2.5}$$

Now, thanks to the absolute continuity relationship (1.2) and Doob's optional stopping Theorem, we can write

$$p^{(\lambda)}_{x+a\to a}(t) = e^{\frac{\lambda}{2}(x^2+2ax+t)}\mathbb{E}_{x+a}\left[e^{-\frac{\lambda^2}{2}\int_0^t B_u^2\, du} \,\middle|\, T_a = t\right] p_{x\to 0}(t). \tag{2.6}$$

A combination of (2.4), (2.5) and (2.6) leads to

$$e^{(\frac{1}{2}a^2\lambda^2 - \frac{\lambda}{2})t} p^{(\lambda)}_{x+a\to a}(t) = e^{\frac{\lambda}{2}x^2 + a\lambda x} p_{x\to 0}(t)\, \Upsilon_t^{\lambda,\alpha}(x).$$

By taking the Laplace transform with respect to the variable t on both sides, we get

$$\int_0^\infty e^{-\beta t} q_t(x,0)\Upsilon_x^{\lambda,\alpha}(t)\, dt = \frac{1}{x}\mathbb{E}_{x+a}\left[e^{-(\beta+\frac{\lambda}{2}-\frac{\alpha^2}{2\lambda^2})H_a}\right].$$

We now derive the expression of the Laplace transform of H_a. Although this is a well-known result, see Proposition 1.2.3, below we give a new proof which relies on probabilistic arguments.

Proposition 2.2.3 *For any $x, a \in \mathbb{R}$ and $\beta \geq 0$, we have*

$$\mathbb{E}_x\left[e^{-\beta H_a}\right] = \frac{e^{\lambda x^2/2}\mathcal{D}_{-\beta/\lambda}(\varepsilon x\sqrt{2\lambda})}{e^{\lambda a^2/2}\mathcal{D}_{-\beta/\lambda}(\varepsilon a\sqrt{2\lambda})} \tag{2.7}$$

where $\varepsilon = sgn(x-a)$ and $\mathcal{D}_\nu$ stands for the parabolic cylinder function which admits the following integral representation

$$\mathcal{D}_\nu(z) \quad = \quad \frac{2^{1/2}e^{-z^2/4}}{\Gamma\left(\frac{1-\nu}{2}\right)}\int_0^\infty \left(t^2+z^2\right)^{\nu/2} t^{-\nu} e^{-t^2/2}\, dt \tag{2.8}$$

where $Re(\nu) < 1, |arg(z)| < \frac{\pi}{2}$.

Proof. Recall that Doob's transform implies the identity $T_a^{(\sqrt{\ })} = \tau(H_a)$ a.s., where $\tau(t) = (2\lambda)^{-1}(e^{2\lambda t} - 1)$. Specializing on $a = 0$ we deduce that $p^{(\lambda)}_{x\to 0}(t) = \tau'(t)p_{x\to 0}(\tau(t))$. Hence, the expression

$$p^{(\lambda)}_{x\to 0}(t) = \frac{|\,x\,|}{\sqrt{2\pi}} \exp\Big(- \frac{\lambda x^2 e^{-\lambda t}}{2\sinh(\lambda t)} + \frac{\lambda t}{2}\Big) \Big(\frac{\lambda}{\sinh(\lambda t)}\Big)^{3/2}. \tag{2.9}$$

It follows that

$$\begin{aligned}
\mathbb{E}_x\left[e^{-\beta H_0}\right] &= \int_0^\infty e^{-\beta t}\tau'(t)p_{x\to 0}(\tau(t))\,dt \\
&= \frac{|x|}{\sqrt{2\pi}}\int_0^\infty (1+2\lambda t)^{-\beta/2\lambda}\, t^{-3/2}e^{-x^2/2t}\,dt \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty \left(t^2+\lambda x^2\right)^{-\beta/2\lambda}\, t^{\beta/\lambda}e^{-t^2}\,dt.
\end{aligned}$$

Next, the continuity of the paths of U yields the following identity

$$H_{x\to 0} \overset{(d)}{=} H_{x\to a} + \hat{H}_{a\to 0}, \quad x \le a \le 0,$$

where thanks to the strong Markov property $\hat{H}_{a\to 0}$ is independent of $H_{x\to a}$. It follows that

$$\mathbb{E}_x\left[e^{-\beta H_a}\right] = \frac{\int_0^\infty \left(t^2+\lambda x^2\right)^{-\beta/2\lambda} t^{\beta/\lambda}e^{-t^2}\,dt}{\int_0^\infty \left(t^2+\lambda a^2\right)^{-\beta/2\lambda} t^{\beta/\lambda}e^{-t^2}\,dt}.$$

By using the integral representation of the parabolic cylinder function (2.8), we get

$$\mathbb{E}_x\left[e^{-\beta H_a}\right] = \frac{e^{\lambda x^2/2}\mathcal{D}_{-\beta/\lambda}(x\sqrt{2\lambda})}{e^{\lambda a^2/2}\mathcal{D}_{-\beta/\lambda}(a\sqrt{2\lambda})}, \quad x \le a.$$

We complete the proof of the Proposition by observing that the symmetry of B in (2.2) allows to recover the case $x \ge a$. □

The proof of the first assertion of Theorem 2.2.2 is then completed by putting pieces together. To prove the second one, it is enough to let x

tend to 0 in the following formula

$$\int_0^\infty \left(e^{-\beta t}-1\right) e^{-x^2/2t}\Upsilon_x^{\lambda,\alpha}(t)\,\frac{dt}{\sqrt{2\pi t^3}} =$$
$$\frac{1}{x}\left(\frac{\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\frac{\alpha}{\lambda^2})\right)}{\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)} - \frac{\mathcal{D}_{-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\frac{\alpha}{\lambda^2})\right)}{\mathcal{D}_{-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}\right).$$

□

Below, we give a straightforward reformulation of the previous result, which is based on the Laplace transform inversion formula. To this end, we recall the expression of the density of H_a as a series expansion which can be found for instance in Theorem 1.3.1. That is, for x and a real numbers, we have

$$p_{x\to a}^{(\lambda)}(t) = -\lambda e^{\lambda(x^2-a^2)/2}\sum_{n=1}^{\infty}\frac{\mathcal{D}_{\nu_{n,\varepsilon\sqrt{2\lambda}a}}(\varepsilon\sqrt{2\lambda}x)}{\mathcal{D}'_{\nu_{n,\varepsilon\sqrt{2\lambda}a}}(\varepsilon\sqrt{2\lambda}a)}\,e^{-\lambda\nu_{n,\varepsilon\sqrt{2\lambda}a}t}$$

where we set $\varepsilon = \mathrm{sgn}(x-a)$, $\mathcal{D}'_{\nu_{n,b}}(b) = \frac{\partial\mathcal{D}_\nu(b)}{\partial\nu}|_{\nu=\nu_{n,b}}$ and the sequence $(\nu_{j,b})_{j\geq 0}$ stands for the ordered positive zeros of the function $\nu \to \mathcal{D}_\nu(b)$.

Corollary 2.2.4 *For λ, α, x and $t > 0$, we have*

$$\Upsilon_x^{\lambda,\alpha}(t) = -\frac{\lambda\sqrt{2\pi t^3}}{x}e^{\frac{1}{2}\left((\frac{\alpha^2}{\lambda^2}-\lambda)(t-1)+\frac{x^2}{t}\right)}\sum_{n=1}^{\infty}\frac{\mathcal{D}_{\nu_{n,c}}\left(\sqrt{2\lambda}x+c\right)}{\mathcal{D}'_{\nu_{n,c}}(c)}\,e^{-t\lambda\nu_{n,c}} \tag{2.10}$$

where we set $c = \sqrt{2}\alpha\lambda^{-3/2}$.

Remark 2.2.5 *It would be interesting to find a probabilistic methodology to extend the result for the case $y > 0$.*

2.2.2 Extension to $y > 0$ Using the Feynman-Kac Formula

Our aim here is to provide an extension of the previous result to any positive real numbers y by using the Feynman-Kac formula.

Theorem 2.2.6 *For $y, x \geq 0$ and $\beta > 0$, we have*

$$\int_0^\infty e^{-\beta t} q_t(x,y)\, \Upsilon^{\lambda,\alpha}_{x\to y}(t)\, dt = \frac{\Gamma(\frac{\beta}{\lambda} + \frac{1}{2} - \frac{\alpha^2}{2\lambda^3})y}{\sqrt{\lambda\pi}x}$$
$$\frac{\mathcal{S}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x\wedge y + \frac{\alpha}{\lambda^2}), c\right)\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x\vee y + \frac{\alpha}{\lambda^2})\right)}{\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}(c)}$$

where $\mathcal{S}_\alpha(x,y) = \mathcal{D}_\alpha(-x)\mathcal{D}_\alpha(y) - \mathcal{D}_\alpha(x)\mathcal{D}_\alpha(-y)$, $x\wedge y = inf(x,y)$ and $x \vee y = sup(x,y)$.

Proof. We shall prove our statement by following a method which is similar to that used by Shepp [110] for computing the double Laplace transform of the integral of a Brownian bridge. Set $F^y_\epsilon(x) = \frac{1}{2\epsilon}\mathbb{I}_{\{|x-y|<\epsilon\}}$ and $a(x) = \left(\frac{\lambda^2}{2}x^2 + \alpha x\right)$. First, note that

$$\lim_{\epsilon\to 0}\mathbb{E}_x\left[\int_0^\infty e^{-\beta t}e^{-\int_0^t a(R_s)\,ds}F^y_\epsilon(R_t)\,dt\right] = \int_0^\infty e^{-\beta t}q_t(x,y)\,\Upsilon^{\lambda,\alpha}_{x\to y}(t)\,dt.$$

Then, the Feynman-Kac formula states that

$$\phi_\epsilon(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\beta t}e^{-\int_0^t a(R_s)\,ds}F^y_\epsilon(R_t)\,dt\right]$$

is the bounded solution of

$$\frac{1}{2}\phi''_\epsilon(x) + \frac{1}{x}\phi'_\epsilon(x) - (a(x)+\beta)\phi_\epsilon(x) = F^y_\epsilon(x), \quad x > 0. \tag{2.11}$$

In order to solve this equation, we first consider the following homogeneous one

$$\frac{1}{2}\phi''(x) + \frac{1}{x}\phi'(x) - (a(x)+\beta)\phi(x) = 0, \quad x > 0.$$

Setting $\phi(x) = x^{-1}v(x)$, we get that v satisfies the Weber equation

$$\frac{1}{2}v''(x) = \left(\frac{\lambda^2}{2}\bar{x}^2 - \frac{\alpha^2}{2\lambda^2} + \beta\right)v(x), \quad x > 0, \tag{2.12}$$

where $\bar{x} = x + \frac{\alpha}{\lambda^2}$. A fundamental solution of (2.12) is expressed in terms of the parabolic cylinder function $\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}\bar{x}\right)$, see e.g. [48].

Thus, the solution of (2.12) which is positive, decreasing and bounded at ∞ is given by

$$\varphi(x) \quad = \quad x^{-1}\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}\bar{x}\right), \quad x > 0.$$

The solution of (2.12) which is positive, increasing and bounded at 0 has the form

$$\psi(x) \quad = \quad x^{-1}\left(c_1\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(-\sqrt{2\lambda}\bar{x}\right) + c_2\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}\bar{x}\right)\right)$$

where c_1 and c_2 are constants. With $c_1 = \mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-\frac{3}{2}}\right)$ and $c_2 = -\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(-\sqrt{2}\alpha\lambda^{-\frac{3}{2}}\right)$, we check that $\psi(x)$ is bounded at 0. The two solutions are linearly independent and their Wronskian, normalized by the derivative of the scale function $s'(x) = x^{-2}$, is given by

$$w_\beta \quad = \quad \mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-\frac{3}{2}}\right) w_\beta^D$$

where $w_\alpha^D = \frac{2\sqrt{\lambda\pi}}{\Gamma(\frac{\beta}{\lambda}+\frac{1}{2}-\frac{\alpha^2}{2\lambda^3})}$ is the Wronskian of the parabolic cylinder functions. Next, we recall the Green formula for the solution of the nonhomogeneous ODE (2.11), that is with second member given by F_ϵ^y

$$\phi_\epsilon(x) = \frac{1}{w_\beta}\left(\varphi(x)\int_0^x \psi(r)F_\epsilon^y(r)\, m(dr) + \psi(x)\int_x^\infty \varphi(r)F_\epsilon^y(r)\, m(dr)\right)$$

where we recall that the speed measure m of the 3-dimensional Bessel process is $m(dr) = 2r^2\, dr$. The proof is then completed by passing to the limit as ϵ tends to 0. □

Remark 2.2.7 *Observing that* $\lim_{x\to 0} x^{-1}\mathcal{S}_\alpha(x,y) = w_\alpha^D$, *we recover the result of Proposition 2.2.2.*

Remark 2.2.8 *In the same vein than Corollary 2.2.4, it is possible to derive an expression of the joint Laplace transform* $\Upsilon_{x\to y}^{\lambda,\alpha}(t)$ *as a series expansion.*

2.3 Connection Between the Laws of First Passage Times

Let $\lambda > 0$ and consider a function f which is twice continuously differentiable on a neighborhood of 0. Let Z be a continuous time stochastic process starting at $x \in \mathbb{R}$, $x \neq f(0)$. Introduce the stopping times

$$\begin{aligned} T_\delta^{(f,\mu)}(\lambda) &= \inf\{s \geq 0;\ Z_s = \delta f(\lambda s) - f(0) - \mu\lambda s\}, \\ T_\alpha^{(1)} &= \inf\{s \geq 0;\ Z_s = \alpha s\}, \\ T_\alpha^{(2)} &= \inf\{s \geq 0;\ Z_s = -\frac{\alpha}{2}s^2\}. \end{aligned}$$

We simply write $T_\delta^{(f)}(\lambda) = T_\delta^{(f,0)}(\lambda)$. We shall describe a device which allows to connect the laws of these first passage times. As an application, we shall apply this technique to the first passage time of a Brownian motion over the square root boundary and derive some limit results of ratios of the parabolic cylinder functions. These limit results have already been shown with analytical techniques such as the Laplace's method or the method of steepest descent, see [89]. However our approach is new, straightforward and relies only on probabilistic arguments and could readily be extended to other examples.

Proposition 2.3.1 *Let $\delta^{(1)} = \alpha/\lambda$. Assume $f'(0) \neq 0$, then*

$$\lim_{\lambda \to 0} T_{\delta^{(1)}}^{(f)}(\lambda) = L^{\alpha f'(0)} \quad a.s.. \tag{2.13}$$

Next, let $\delta^{(2)} = \alpha/\lambda^2$. Assume $f''(0) \neq 0$, then

$$\lim_{\lambda \to 0} T_{\delta^{(2)}}^{(f,\delta^{(2)})}(\lambda) = S^{-\alpha f''(0)} \quad a.s.. \tag{2.14}$$

Proof. Using the following Taylor expansion

$$f(\lambda t) = f(0) + \lambda f'(0)t + \frac{\lambda^2}{2} f''(0)t^2 + o(\lambda^2),$$

we get that

$$\lim_{\lambda \to 0} T_\delta^{(f,\delta)}(\lambda) = \inf\{s \geq 0;\ Z_s = \delta\lambda(f'(0) - \mu)s + \delta\frac{\lambda^2 f''(0)}{2}s^2 + o(\lambda^2)\}.$$

The first (resp. second) assertion is then obtained by choosing $\delta = \delta^{(1)}$ (resp. $\delta^{(2)}$). □

2.3.1 Brownian Motion and the Square Root Boundary

We apply the previous technique to the first passage time of the Brownian motion over the curve $f(t) = \sqrt{1+2t}$. It allows to evaluate some known limits of the ratio of parabolic cylinder functions by a stochastic approach.

Linear case

Set $\mu = 0$.

Corollary 2.3.2 *Let $\beta > 0$, $x, \alpha \in \mathbb{R}$ then we have*

$$\lim_{\lambda\to 0} \frac{\mathcal{D}_{-\frac{\beta}{2\lambda}}\left(\sqrt{2\lambda}(x+\frac{\alpha}{\lambda})\right)}{\mathcal{D}_{-\frac{\beta}{2\lambda}}\left(\sqrt{2}\alpha\lambda^{-1/2}\right)} = e^{-|x|\sqrt{\alpha^2+2\beta}}.$$

As a consequence, we also have

$$\lim_{\lambda\to 0} \quad \lambda e^{\lambda(x^2-2\frac{\alpha}{\lambda}x)/2} \sum_{n=1}^{\infty} \frac{\mathcal{D}_{\nu_{n,\frac{\alpha}{\lambda}\sqrt{2\lambda}}}(\sqrt{2\lambda}x)}{\mathcal{D}'_{\nu_{n,\frac{\alpha}{\lambda}\sqrt{2\lambda}}}(\sqrt{2}\alpha\lambda^{-1/2})} (1+2\lambda t)^{-\nu_{n,\varepsilon a\sqrt{2\lambda}/2}t}$$
$$= \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{1}{2t}(x-\alpha t)^2}.$$

Proof. First, set $\delta = -\alpha/\lambda$. Then, by combining Doob's transform with Proposition 2.2.3, we recover the result of Breiman [19] about the Mellin transform of $T_\delta^{(\sqrt{})}$

$$\mathbb{E}_{x+\frac{\alpha}{\lambda}}\left[(1+2\lambda T_\delta^{(\sqrt{})})^{-\beta/2\lambda}\right] = e^{\alpha x} \frac{\mathcal{D}_{-\frac{\beta}{2\lambda}}\left(\sqrt{2\lambda}(x+\frac{\alpha}{\lambda})\right)}{\mathcal{D}_{-\frac{\beta}{2\lambda}}\left(\sqrt{2}\alpha\lambda^{-1/2}\right)}.$$

Next, recall that the Laplace transform of $T_\alpha^{(1)}$ is specified by, see e.g. [62, p.197],

$$\mathbb{E}_x\left[e^{-\beta T_\alpha^{(1)}}\right] = e^{\alpha x - |x|\sqrt{\alpha^2+2\beta}}.$$

The first statement follows readily from the first assertion of Proposition 2.3.1. The second one is an immediate reformulation of the previous one in terms of density functions. □

Quadratic case

In what follows, we investigate the second order expansion. We start by computing the law of $T_\alpha^{(2)}$, the first passage time of the Brownian motion over the second order boundary. We denote by $q_{x\to\alpha}$ its density. In the case $x\alpha < 0$, its law has been computed by Groeneboom [49] and Salminen [107] in terms of the Airy function, denoted by Ai, see e.g. [72]. For the sake of completeness we recall their approach.

Lemma 2.3.3 *For β and $\alpha, x > 0$, hold the relations*

$$\mathbb{E}_x\left[e^{-\beta T_\alpha^{(2)}} G(T_\alpha^{(2)})\right] = \frac{Ai\left(2^{1/3}\frac{\beta+\alpha x}{\alpha^{2/3}}\right)}{Ai\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)}$$

where $G(t) = e^{\frac{1}{6}\alpha^2 t^3}$ and

$$\mathbb{P}_x(T_\alpha^{(2)} \in dt) = (2\alpha^2)^{1/3} e^{-\frac{1}{6}\alpha^2 t^3} \sum_{k=0}^{\infty} \frac{Ai\left(\upsilon_k - (2\alpha)^{1/3}x\right)}{Ai'(\upsilon_k)} e^{2^{-1/3}\alpha^{2/3}\upsilon_k t}\, dt$$

where $(\upsilon_k)_{k\geq 0}$ is the decreasing sequence of negative zeros of the Airy function.

Proof. Denote by $\mathbb{P}^\alpha$ the law of the process $\left(B_t + \frac{\alpha}{2}t^2,\ t \geq 0\right)$. We have the following absolute continuity relationship

$$\begin{aligned} d\mathbb{P}^\alpha_{x|\mathcal{F}_t} &= e^{\alpha\int_0^t s\, dB_s - \frac{\alpha^2}{6}t^3}\, d\mathbb{P}_{x|\mathcal{F}_t} \\ &= e^{\alpha t B_t - \alpha\int_0^t B_s\, ds - \frac{\alpha^2}{6}t^3}\, d\mathbb{P}_{x|\mathcal{F}_t}, \quad t > 0, \end{aligned}$$

where the last line follows from Itô's formula. An application of the Doob's optional stopping Theorem yields

$$\mathbb{E}_x\left[e^{-\beta T_\alpha^{(2)}} G(T_\alpha^{(2)})\right] = \mathbb{E}_x\left[e^{-\beta T_0 - \alpha\int_0^{T_0} B_s ds}\right].$$

As in the previous Section, the expectation on the right-hand side can be estimated via the Feynman-Kac formula. It is the solution to the boundary value problem

$$\begin{aligned} &\frac{1}{2}\varphi''(x) - (\alpha x + \beta)\varphi(x) = 0, \\ &\varphi(0) = 1, \quad \lim_{x\to\infty}\varphi(x) = 0, \end{aligned}$$

which is given in terms of the Airy function, see e.g. [59]. The expression of the density is a consequence of the Laplace transform inversion formula and the residues Theorem, see [49] or [107] for more details. □

Next, we define the process $U^{(\mu)} := (U_t^{(\mu)},\ t \geq 0)$ as the solution to the stochastic differential equation

$$dU_t^{(\mu)} = \left(-\lambda U_t^{(\mu)} + \mu e^{\lambda t}\right) dt + dB_t, \quad U_0^{(\mu)} = x \in \mathbb{R}.$$

Note that $U^{(\mu)}$ can also be expressed as follows

$$U_t^{(\mu)} = e^{-\lambda t}\left(x - \frac{\mu}{2\lambda} + \frac{\mu}{2\lambda}e^{2\lambda t} + \int_0^t e^{\lambda s} dB_s\right), \qquad t \geq 0.$$

For x, a real numbers, we introduce the stopping time $H_a^{(\mu)} = \inf\{s \geq 0;\ U_s^{(\mu)} = a\}$ and denote by $p_{x\to a}^{(\lambda,\mu)}(t)$ its density. Let us also introduce the function $G_\lambda(t) = e^{\frac{\mu^2}{2}\tau_t - \mu e^{\lambda t} a}$, $t \geq 0$. The law of $H_a^{(\mu)}$ is characterized in the following.

Proposition 2.3.4 *For $\beta > 0$, we have*

$$\mathbb{E}_x\left[e^{-\beta H_a^{(\mu)}} G_\lambda(H_a^{(\mu)})\right] = \frac{e^{\lambda x^2/2 - \mu x}\mathcal{D}_{-\frac{\beta}{\lambda}}\left(\varepsilon x\sqrt{2\lambda}\right)}{e^{\lambda a^2/2}\mathcal{D}_{-\frac{\beta}{\lambda}}\left(\varepsilon a\sqrt{2\lambda}\right)} \tag{2.15}$$

where we set $\varepsilon = sgn(x-a)$. In particular,

$$p_{x\to 0}^{(\lambda,\mu)}(t) = \frac{\mid x \mid}{\sqrt{2\pi}}\, e^{-\mu e^{\lambda t}(\frac{\mu}{2}\sinh(\lambda t) - a) - \mu x - \frac{\lambda x^2 e^{-\lambda t}}{2\sinh(\lambda t)} + \frac{\lambda t}{2}} \left(\frac{\lambda}{\sinh(\lambda t)}\right)^{3/2}. \tag{2.16}$$

Proof. The first assertion follows from the following absolutely continuity relationship

$$d\mathbb{P}_{x|\mathcal{F}_t}^{(\lambda,\mu)} = e^{\mu e^{\lambda t} X_t - \mu x - \frac{\mu^2}{2}\tau_t}\, d\mathbb{P}_{x|\mathcal{F}_t}^{(\lambda)}, \quad t > 0, \tag{2.17}$$

and the application of Doob's optional stopping Theorem. We point out that the exponential martingale is the one associated with the Gaussian martingale $\left(B_{\tau(t)}, t \geq 0\right)$. The expression of the density in the case $a = 0$ is obtained from the Laplace inversion formula of the parabolic cylinder function, see formula (2.9). □

Remark 2.3.5 *An expression of the density $p^{(\lambda,\mu)}_{x\to a}(t)$ is given in Daniels [22] as a contour integral. The author used a technique suggested by Shepp [109].*

Let us recall the notation $T_a^{(\sqrt{;}\mu)} = \inf\{s \geq 0; B_s + \mu s = a\sqrt{1+2\lambda s} - a\}$. We recall that from Doob's transform, we have

$$H_a^{(\mu)} = \tau(T_a^{(\sqrt{;}\mu)}) \quad a.s.. \tag{2.18}$$

We are now ready to state the following limit result.

Corollary 2.3.6 *For β, α and $x > 0$, we have*

$$\lim_{\lambda\to 0} \frac{\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\frac{\alpha}{\lambda^2})\right)}{\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)} = \frac{Ai\left(2^{1/3}\frac{\beta+\alpha x}{\alpha^{2/3}}\right)}{Ai\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)}.$$

Proof. Substituting β by $\beta - \frac{\alpha^2}{2\lambda^2}$, x by $x + \frac{\alpha}{\lambda^2}$ and setting $a = \frac{\alpha}{\lambda^2}$ and $\mu = \frac{\alpha}{\lambda}$ in (2.15), we get

$$\begin{aligned}
&\frac{\mathcal{D}_{-\frac{\beta}{\lambda}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\frac{\alpha}{\lambda^2})\right)}{\mathcal{D}_{-\frac{\beta}{\lambda}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)} = e^{-\frac{\lambda}{2}x^2+\frac{\alpha^2}{\lambda^3}} \\
&\times \int_0^\infty e^{-(\beta-\frac{\alpha^2}{2\lambda^2})t+\frac{\alpha^2}{2\lambda^2}\tau_t-\frac{\alpha^2}{\lambda^3}e^{\lambda t}} p^{(\lambda,\frac{\alpha}{\lambda})}_{x+\frac{\alpha}{\lambda^2}\to\frac{2\alpha}{\lambda^2}}(t)\,dt.
\end{aligned}$$

Note that $\tau\left(H^{(\alpha/\lambda)}_{\frac{\alpha}{\lambda^2}}\right) \to T^{(2)}_\alpha$ a.s., as $\lambda \to 0$. Thus, we have

$$\begin{aligned}
&\lim_{\lambda\to 0} e^{-\frac{\lambda}{2}x^2+\frac{\alpha^2}{\lambda^3}} \int_0^\infty e^{-(\beta-\frac{\alpha^2}{2\lambda^2})t+\frac{\alpha^2}{2\lambda^2}\tau_t-\frac{\alpha^2}{\lambda^3}e^{\lambda t}} p^{(\lambda,\frac{\alpha}{\lambda})}_{x+\frac{\alpha}{\lambda^2}\to\frac{\alpha}{\lambda^2}}(t)\,dt \\
&= \int_0^\infty e^{-\beta t+\frac{1}{6}\alpha^2 t^3} q_{x\to\alpha}(t)\,dt \\
&= \frac{Ai\left(2^{1/3}\frac{\beta+\alpha x}{\alpha^{2/3}}\right)}{Ai\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)}
\end{aligned}$$

where the last line follows from Lemma 2.3.3. □

Remark 2.3.7 *By analogy to the results of Section 2.2, we have*

$$\lim_{\lambda\to 0}\int_0^\infty e^{-\beta t}q_t(x,0)\Upsilon_x^{\lambda,\alpha}(t)\,dt=\frac{1}{x}\frac{Ai\left(2^{1/3}\frac{\beta+\alpha x}{\alpha^{2/3}}\right)}{Ai\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)},$$

$$\int_0^\infty\left(e^{-\beta t}-1\right)\Upsilon^{0,\alpha}(t)\,\frac{dt}{\sqrt{2\pi t^3}}=(2\alpha)^{1/3}\left(\frac{Ai'\left(\frac{2^{1/3}\beta}{\alpha^{2/3}}\right)}{Ai\left(\frac{2^{1/3}\beta}{\alpha^{2/3}}\right)}-\frac{Ai'(0)}{Ai(0)}\right)$$

and

$$\Upsilon_x^{0,\alpha}(t)=\frac{1}{x}\sqrt{2\pi t^3}e^{\frac{x^2}{2t}}(2\alpha^2)^{1/3}\sum_{k=0}^{\infty}\frac{Ai\left(\upsilon_k-(2\alpha)^{1/3}x\right)}{Ai'(\upsilon_k)}e^{(\frac{\alpha^2}{2})^{1/3}\upsilon_k t}\,dt.$$

Remark 2.3.8 *We mention that Lachal [70] get the following identity*

$$\mathbb{E}_x\left[e^{-\beta H_0-\alpha\int_0^{H_0}U_s\,ds}\right]\quad=\quad e^{\frac{\lambda}{2}x^2}\frac{\mathcal{D}_{-\frac{\beta}{\lambda}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\frac{\alpha}{\lambda^2})\right)}{\mathcal{D}_{-\frac{\beta}{\lambda}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}$$

which gives the following relationship

$$\int_0^\infty e^{-\beta t}q_t(x,0)\Upsilon_x^{\lambda,\alpha}(t)\,dt\quad=\quad\frac{1}{x}e^{-\frac{\lambda}{2}x^2}\mathbb{E}_x\left[e^{-(\beta+\frac{\lambda}{2})H_0-\alpha\int_0^{H_0}U_s\,ds}\right].$$

We also indicate that the author computed the limit as $\lambda\to 0$ to recover the result of Biane and Yor [13], Lefebvre [75] stating that

$$\mathbb{E}_x\left[e^{-\beta T_0-\alpha\int_0^{T_0}B_s\,ds}\right]\quad=\quad\frac{Ai\left(2^{1/3}\frac{\beta+\alpha x}{\alpha^{2/3}}\right)}{Ai\left(2^{1/3}\frac{\beta}{\alpha^{2/3}}\right)}.$$

In order to compute the expression of the limit of the Laplace transform, Lachal used an asymptotic result of the parabolic cylinder function which has been derived by the method of steepest descent in [31].

2.3.2 Another Limit

From Proposition 2.2.2, we readily derive

$$\lim_{\alpha\to 0}\int_0^\infty e^{-\beta t}xe^{-x^2/2t}\Upsilon_x^{\lambda,\alpha}(t)\,\frac{dt}{\sqrt{2\pi t^3}}=\frac{\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}}\left(\sqrt{2\lambda}x)\right)}{\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}}(0)}.$$

We recall the following well known results regarding the Laplace transform of the L^2 norm of Bessel bridges. In conjunction with (2.9), for the special case $\alpha = 0$, we extract the relationship

$$\Upsilon_x^\lambda(t) = \frac{1}{x}\sqrt{2\pi t^3}\left(\frac{\lambda t}{\sinh(\lambda t)}\right)^{\frac{3}{2}} e^{-\frac{x^2}{2t}(\lambda t \coth(\lambda t)-1)}. \tag{2.19}$$

Since in this case the zeros of the function $\nu \mapsto \mathcal{D}_\nu(0) = 2^\nu \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1-\nu}{2})}$ correspond to the odd poles of the Γ function, we also have

$$\Upsilon_x^\lambda(t) = -\frac{\lambda}{x}\sqrt{2\pi t^3} e^{\frac{x^2}{2t}} \sum_{n=1}^{\infty} \frac{\mathcal{D}_{2n+1}(x\sqrt{2\lambda})}{\mathcal{D}_{2n+1}^{(\nu)}(0)} e^{-2(n+1)\lambda t}.$$

We precise that from the expression (2.19), it is easy to extend the result to δ-dimensional Bessel bridges, for any $\delta > 0$, which is closely related to the Generalized Lévy stochastic area formula, see e.g. [97]. Indeed, denoting by $\Upsilon_x^{\lambda,(\delta)}$ its Laplace transform, thanks to the additivity property of squared Bessel processes, we have

$$\Upsilon_x^{\lambda,(\delta)}(t) = \frac{1}{x}\sqrt{2\pi t^3}\left(\frac{\lambda t}{\sinh(\lambda t)}\right)^{\frac{\delta}{2}} e^{-\frac{x^2}{2t}(\lambda t \coth(\lambda t)-1)}. \tag{2.20}$$

In [45] the inverse of the Laplace transform $\Upsilon_x^{\lambda,(\delta)}(t)$ is given in terms of the parabolic cylinder functions.

2.4 Comments and some Applications

Our aim here is first to examine the law of the studied functional when the fixed time t is replaced by some interesting stopping times and when we consider both the 3-dimensional Bessel process and the reflected Brownian motion (i.e. the 1-dimensional Bessel process). To a stopping time S we associate the following notation, with $\delta = 1, 3$,

$$\Upsilon_x^{\lambda,\alpha,(\delta)}(S) = \mathbb{E}_x\left[e^{-\beta S - \frac{\lambda^2}{2}\int_0^S R_u^2\,du - \alpha\int_0^S R_u\,du}\right]$$

where, for this Section, R stands for a δ-dimensional Bessel process starting from $x \geq 0$. We denote by $\mathbb{Q}_x^\delta$ its law. Next, with $K_y = \inf\{s \geq 0;\ R_s = y\}$ and $S = K_y$, we state the following result.

Proposition 2.4.1 *Let $x \geq y$,*

$$\Upsilon_x^{\lambda,\alpha,(3)}(K_y) = \frac{y}{x} \frac{\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\alpha\lambda^{-2})\right)}{\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(y+\alpha\lambda^{-2})\right)}. \tag{2.21}$$

Proof. First, we recall the following absolute continuity relationship

$$d\mathbb{Q}^{(3)}_{x|\mathcal{F}_t} = (R_t/x)\, d\mathbb{P}_{x|\mathcal{F}_t}\,, \quad \text{on } \{K_0 > t\}.$$

Then, observe that $K_y < K_0$ a.s. since $x \geq y$. Next, denote by ${}^{(\mu)}H_x$ the first passage time to a fixed level $x \in \mathbb{R}$ of the mean reverting OU process with parameter $\mu \in \mathbb{R}$. As mentioned in Remark 1.2.8, the determination of its density, denoted by ${}^{(\mu)}p^{(\lambda)}_{x\to a}(t)$, can be reduced to the case $\mu = 0$ as follows

$${}^{(\mu)}p^{(\lambda)}_{x\to a}(t) = p^{(\lambda)}_{x-\frac{\mu}{\lambda}\to a-\frac{\mu}{\lambda}}(t), \quad t > 0.$$

Thus, we have

$$\begin{aligned}
\Upsilon_x^{\lambda,\alpha,(3)}(K_y) &= \mathbb{E}_x\left[e^{-\beta K_y - \frac{\lambda^2}{2}\int_0^{K_y} R_s^2\, ds - \alpha\int_0^{K_y} R_s\, ds}\right] \\
&= \frac{y}{x}\mathbb{E}_x\left[e^{-\beta T_y - \frac{\lambda^2}{2}\int_0^{T_y} B_s^2\, ds - \alpha\int_0^{T_y} B_s\, ds}\right] \\
&= \frac{y}{x}e^{\frac{\lambda}{2}(y^2-x^2)}\mathbb{E}_x\left[e^{-(\beta+\frac{\lambda}{2})H_y - \alpha\int_0^{H_y} U_s\, ds}\right] \\
&= \frac{y}{x}e^{\frac{\lambda}{2}(y^2-x^2)+\frac{\alpha}{\lambda}(y-x)}\,\mathbb{E}_x\left[e^{-(\beta+\frac{\lambda}{2}-\frac{\alpha^2}{2\lambda^2})^{(\frac{\alpha}{\lambda})}H_y}\right] \\
&= \frac{y}{x}\frac{\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\alpha\lambda^{-2})\right)}{\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(y+\alpha\lambda^{-2})\right)}.
\end{aligned}$$

□

Corollary 2.4.2 *Let $\alpha, \beta, \lambda \geq 0$. Then, for any $x \geq y \geq 0$, we have*

$$\Upsilon_x^{\lambda,\alpha,(1)}(K_y) = \frac{\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(x+\alpha\lambda^{-2})\right)}{\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2\lambda}(y+\alpha\lambda^{-2})\right)}. \tag{2.22}$$

Proof. The result follows from the absolute continuity relationship

$$d\mathbb{Q}^{(3)}_{x|\mathcal{F}_t} = (R_t/x)\, d\mathbb{Q}^{(1)}_{x|\mathcal{F}_t}, \quad \text{on } \{K_0 > t\},$$

where $\mathbb{Q}^1$ stands for the law of the reflected Brownian motion and K_0 is the first time when the canonical process hits 0. □

Next, let $(\sigma_l, l \geq 0)$ be defined as the right continuous inverse process of the local time $(l_t, t \geq 0)$ at 0 of the reflected Brownian motion. It is a $\frac{1}{2}$-stable subordinator, its Laplace exponent is given by

$$\mathbb{E}\left[e^{-\beta\sigma_l}\right] = e^{-l\sqrt{2\beta}}.$$

We denote by n and $(e_u, 0 \leq u \leq V)$ Itô's measure associated with the reflected Brownian motion and the generic excursion process under n respectively. We recall that with the choice of the normalization of the local time via the occupation formula with respect to the speed measure, we have $n(V \in dt) = \frac{dt}{\sqrt{2\pi t^3}}$, see e.g. [56].

Proposition 2.4.3 *Let $\alpha, \beta, \lambda \geq 0$.*

$$-\log\left(\Upsilon^{\lambda,\alpha,(1)}(\sigma_1)\right) = \sqrt{2\lambda}\frac{\mathcal{D}^{(x)}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}{\mathcal{D}_{-\frac{\beta}{\lambda}-\frac{1}{2}+\frac{\alpha^2}{2\lambda^3}}\left(\sqrt{2}\alpha\lambda^{-3/2}\right)}. \tag{2.23}$$

Proof. From the exponential formula of excursions theory, see e.g. [100] and the fact that conditionally on $V = t$ the process $(e_u, u \leq V)$ is a 3-dimensional Bessel bridge over $[0, t]$ between 0 and 0. We get

$$\begin{aligned} -\log\left(\Upsilon^{\lambda,\alpha,(1)}(\sigma_1)\right) &= \int n(de)\left(1 - e^{-\beta V - \frac{\lambda^2}{2}\int_0^V e_u^2\, du - \alpha\int_0^V e_u\, du}\right) \\ &= \int_0^\infty \left(1 - e^{-\beta t}\Upsilon^{\lambda,\alpha}(t)\right)\frac{dt}{\sqrt{2\pi t^3}}. \end{aligned}$$

Next, set $J(\beta) = \int_0^\infty \left(1 - e^{-\beta t}\Upsilon^{\lambda,\alpha}(t)\right)\frac{dt}{\sqrt{2\pi t^3}}$. Thus, we have

$$J(\beta) - J(0) = \int_0^\infty \left(1 - e^{-\beta t}\right)\Upsilon^{\lambda,\alpha}(t)\frac{dt}{\sqrt{2\pi t^3}}.$$

The statement follows from Proposition 2.2.2. □

Finally, we shall extend the above computations to the radial part of a δ-dimensional Ornstein-Uhlenbeck process, with $\delta = 1, 3$, denoted by O, with parameter $\theta \in \mathbb{R}^+$. The law of this process, when started at $x > 0$, is denoted by $\mathbb{P}_x^{(\theta),\delta}$. Girsanov's Theorem gives

$$d\mathbb{P}^{(\theta),\delta}_{x|\mathcal{F}_t} = e^{-\frac{\theta}{2}(R_t^2 - x^2 - \delta t) - \frac{\theta^2}{2}\int_0^t R_u^2\, du}\, d\mathbb{Q}^{\delta}_{x|\mathcal{F}_t}, \quad t > 0. \tag{2.24}$$

We also shall need the densities of its semigroup which are given, for $x, y, t > 0$, by

$$p_t^{(\theta),\delta}(x,y) = \frac{y}{x}\frac{\lambda\sqrt{2}e^{\frac{3\lambda t}{2}}}{\sqrt{\pi\sinh(\lambda t)}} e^{-\frac{\lambda e^{-\lambda t}}{2\sinh(\lambda t)}(x^2+y^2)} \sinh\left(\frac{\lambda x y}{2\sinh(\lambda t)}\right).$$

We simply write $p_t^{(\theta),1}(x) = p_t^{(\theta),1}(0,x)$ For a fixed $t \geq 0$, we set

$${}^{(\theta)}\Upsilon^{\lambda,\alpha,(\delta)}_{x\to y}(t) = \mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}\int_0^t O_u^2\, du - \alpha\int_0^t O_u\, du} \,\middle|\, O_t = y\right], \quad \lambda, x, \alpha \geq 0.$$

Proposition 2.4.4 *Set $\kappa = \lambda^2 + \theta^2$, $\omega_1 = \beta + \frac{\theta}{2}$ and $\omega_3 = \beta + \frac{3\theta}{2}$. For x and $\beta > 0$, we have*

$$\int_0^\infty e^{-\beta t} p_t^{(\theta),1}(x)\, {}^{(\theta)}\Upsilon^{\lambda,\alpha,(1)}_x(t)\, dt = e^{-\frac{\theta}{2}x^2}\frac{\mathcal{D}_{-\frac{\omega_1}{\kappa}-\frac{1}{2}+\frac{\alpha^2}{2\kappa^3}}\left(\sqrt{2\kappa}(x+\frac{\alpha}{\kappa^2})\right)}{\mathcal{D}_{-\frac{\beta}{\kappa}-\frac{1}{2}+\frac{\alpha^2}{2\kappa^3}}\left(\sqrt{2}\alpha\kappa^{-3/2}\right)}$$

and

$$\int_0^\infty e^{-\beta t} p_t^{(\theta),3}(x)\, {}^{(\theta)}\Upsilon^{\lambda,\alpha,(3)}_x(t)\, dt = e^{-\frac{\theta}{2}x^2} x\frac{\mathcal{D}_{-\frac{\omega_3}{\kappa}-\frac{1}{2}+\frac{\alpha^2}{2\kappa^3}}\left(\sqrt{2\kappa}(x+\frac{\alpha}{\kappa^2})\right)}{\mathcal{D}_{-\frac{\beta}{\kappa}-\frac{1}{2}+\frac{\alpha^2}{2\kappa^3}}\left(\sqrt{2}\alpha\kappa^{-3/2}\right)}.$$

Proof. From the absolute continuity relationship (2.24), we have

$$\begin{aligned}&\mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}\int_0^t O_s^2\, ds - \alpha\int_0^t O_s\, ds}\right]\\ &= \mathbb{E}_x\left[e^{-\frac{\theta}{2}(R_t^2 - x^2 - \delta t)} e^{-(\frac{\lambda^2+\theta^2}{2})\int_0^t R_s^2\, ds - \alpha\int_0^t R_s\, ds}\right].\end{aligned}$$

The results follow by the same reasoning as for the proof of Proposition 2.2.2. □

Chapter 3

Study of some Functional Transformations with an Application to some First Crossing Problems for Selfsimilar Diffusions

Though this be madness, yet there is method in't.
W. Shakespeare (Hamlet, II,i,206)

3.1 Introduction and Preliminaries on some Nonlinear Spaces

Let B be a standard Brownian motion and f a continuous function on $\mathbb{R}^+$ such that $f(0) \neq 0$. We consider the first passage time problem consisting on the determination of the distribution of the stopping time $T^{(f)} = \inf\{s \geq 0;\ B_s = f(s)\}$. Following Strassen [117], we know that

if f is continuously differentiable then the law of $T^{(f)}$ is absolutely continuous with respect to the Lebesgue measure with a continuous density. This problem, which has been studied since the early 1900's, originally attracted researchers because of its connections to sequential analysis, non-parametric tests and iterated logarithm law, see [84], [103] and the references therein. From the explicit viewpoint, some elaborated fine methods have proven efficiency each for specific elementary examples. For instance, the Bachelier-Lévy formula for the straight lines, Doob's transform for the square root boundaries [19], [83], and the direct application of Girsanov's Theorem for the quadratic functions [49]. In the general setting, the celebrated method of images allows, at least theoretically, to solve the problem for a class of curves which are solutions, in the unknown x for a fixed t, of implicit equations of the type

$$h(x,t) \overset{(def)}{=} \int_0^\infty e^{ux-\frac{u^2}{2}t} F(du) = a \tag{3.1}$$

where a is some fixed positive constant and F is a positive σ-finite measure. Another method which is worth to be mentioned, discovered by Durbin in [36] and [37], transforms the problem into the calculation of a conditional expectation. Further reading about asymptotic studies, numerical techniques and other recent applications can be found in Borovkov and Novikov [18], Darling and al. [26], Daniels [21], [23], [24] [25], Di Nardo et al. [30], Durbin [39], Ferebee [40], [41], Novikov et al. [85], [86], Peskir [94], [93], Pötzelberger and Wang [99], Ricciardi et al. [102], Roberts [104], Roberts and Shortland [105], Siegmund [113] and the references therein.

We proceed by giving some notation. Let $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$ and $\mathbb{R}_\infty^+ = \mathbb{R}^+ \cup \{\infty\}$. We recall that for a set $I \subset \mathbb{R}^+$, $C(I, \mathbb{R}^+)$ denotes the space of positive continuous function defined on I. We also define, for a fixed couple $(a,b) \in \mathbb{R}_\infty^+ \times \mathbb{R}_\infty^+$, the nonlinear functional space

$$A_a^{-2,b} \quad = \quad \left\{ h \in C\left([0,a), \mathbb{R}^+\right); \int_0^a h^{-2}(s)\, ds = b \right\}.$$

Thus, we have the following identities $C\left([0,a), \mathbb{R}^+\right) = \bigcup_{b \geq 0} A_a^{-2,b}$ and we set $C_\infty\left(\mathbb{R}^+\right) := \bigcup_{b>0} C\left([0,b), \mathbb{R}^+\right)$.

In this Chapter, we aim to provide an explicit relationship between the law of $T^{(f)}$ and the one of the first passage time of the Brownian motion

over an element of a family of curves obtained from f via the following transform

$$\begin{aligned} S^{(\alpha,\beta)}: \quad C(\mathbb{R}_0^+,\mathbb{R}^+) &\longrightarrow C([0,\zeta^{(\beta)}),\mathbb{R}^+) \\ f &\longmapsto \left(\frac{1+\alpha\beta.}{\alpha}\right) f\left(\frac{\alpha^2.}{1+\alpha\beta.}\right) \end{aligned} \tag{3.2}$$

where $\beta \in \mathbb{R}, \alpha \in \mathbb{R}^+$ and $\zeta^{(\beta)} = -\beta^{-1}$ when $\beta < 0$, and equals $+\infty$ otherwise. Note that to simplify notation , we write $f^{(\alpha,\beta)} = S^{(\alpha,\beta)} f$. For $\alpha = 1$, we shall refer to $S^{(\beta)} = S^{(1,\beta)}$ as the family of elementary transformations. We take two different routes to establish this connection. First, we shall show it directly by studying the analytical transformations allowing to construct Brownian bridges from a Brownian motion. In order to describe the second approach, we need to introduce the Gauss-Markov process of Ornstein-Uhlenbeck type with parameter $\phi \in A_a^{-2,b}$ (for short $GMOU^{\phi}$), denoted by $U^{(\phi)}$, defined by

$$U_t^{(\phi)} = \phi(t)\left(U_0^{(\phi)} + \int_0^t \phi^{-1}(s)\, dB_s\right), \quad 0 \le t < a, \tag{3.3}$$

where B is a standard Brownian motion and $U_0^{(\phi)} \in \mathbb{R}$. We shall drop the exponent ϕ when it is not ambiguous. The first step consists on showing that the law of $U^{(\phi)}$ is connected via a time-space harmonic function to the laws of a family of GMOU processes whose parameters are obtained from ϕ as follows. For $\alpha > 0$ and β real numbers, we define the mapping $\Pi^{(\alpha,\beta)}$ by

$$\begin{aligned} \Pi^{(\alpha,\beta)}: \quad C_\infty\left(\mathbb{R}^+\right) &\longrightarrow C_\infty\left(\mathbb{R}^+\right) \\ \phi &\longmapsto \phi(.)\left(\alpha + \beta\int_0^{\cdot} \phi^{-2}(s)ds\right). \end{aligned} \tag{3.4}$$

Thus, we shall show that there exists a Doob's h-transform between the law of $U^{(\phi)}$ and $U^{(\theta)}$ where $\theta = \Pi^{(\alpha,\beta)}\phi$. As a second step, we show that the law of the level crossing to a fixed boundary of a GMOU process is linked to the law of the first passage time of the Brownian motion to a specific curve via a deterministic time change. We now describe the transform Σ which connects the parameter of the GMOU and the curve. To a function $\phi \in C_\infty\left(\mathbb{R}^+\right)$ we associate the increasing function $\tau^{(\phi)}(.) = \int_0^{\cdot} \frac{ds}{\phi^2(s)}$ and denote by $\varrho^{(\phi)}$ its inverse. To simplify notation, when there will be no confusion, these will be simply denoted by τ and

ϱ . We define the mapping Σ by

$$\begin{array}{rccc} \Sigma: & C_\infty\left(\mathbb{R}^+\right) & \longrightarrow & C_\infty\left(\mathbb{R}^+\right) \\ & \phi & \longmapsto & 1/\phi \circ \varrho. \end{array} \tag{3.5}$$

We call the mapping Σ *Doob's transform.* Finally, we introduce the last transform which is obtained by combining the two previous ones in the following way

$$S^{(\alpha,\beta)} = \Sigma \circ \Pi^{(\alpha,-\beta)} \circ \Sigma, \quad (\alpha,\beta) \in \mathbb{R}^+ \times \mathbb{R}_0^+. \tag{3.6}$$

In the diagram below, we show how these transformations are connected.

$$\begin{array}{ccc} f & \xrightarrow{S^{(\alpha,\beta)}} & f^{(\alpha,\beta)} \\ \Sigma \Big\downarrow & & \Big\uparrow \Sigma \\ \phi & \xrightarrow[\Pi^{(\alpha,-\beta)}]{} & \theta \end{array}$$

We shall note that for $\alpha = 1$, we have the identity $S^{(1,\beta)} = S^{(\beta)}$, which explains the notation . It will turn out that the methodology also applies to the Bessel processes which is in agreement with the title. Indeed, the Bessel processes (or their powers) together with the Brownian motion, form the class of selfsimilar diffusions with continuous paths, see Lamperti [71].

In what follows, we introduce some spaces which will be the basis of our study. Let us denote by Mr^+ the space of positive Radon measures defined on $\mathbb{R}_0^+$. Fix $\mu \in Mr^+$, and introduce the associated Sturm-Liouville equation

$$\phi'' = \mu\phi, \quad \text{on} \quad \mathbb{R}_0^+, \tag{3.7}$$

defined in the sense of distributions. The solutions are gathered in the set

$$SL^{(\mu,+)}(\mathbb{R}_0^+) \quad = \quad \left\{\phi \in C(\mathbb{R}_0^+, \mathbb{R}^+); \phi'' = \mu\phi\right\}.$$

Finally, we introduce the set of positive convex functions, that is

$$V^+(\mathbb{R}_0^+) \quad = \quad \left\{\phi \in C(\mathbb{R}_0^+, \mathbb{R}^+);\ \phi \,\text{convex}\right\}.$$

We now explain the organization of the Chapter. In Section 3.2, we start by providing some properties of the family of transformations

$\{\Pi^{(\alpha,\beta)}; \alpha \in \mathbb{R}^+, \beta \in \mathbb{R}\}$. Then, we recall elementary properties of Gauss-Markov processes of type (3.3) and study the action of Π on their parameters. Section 3.3 is devoted to the study of Doob's mapping Σ and its switching role in the context of the first passage time problem. Results on boundary crossing for the Brownian motion and their analogues for Bessel processes, obtained through both the family $\{S^{(\alpha,\beta)}; \alpha, \beta \in \mathbb{R}^+ \times \mathbb{R}\}$ and $\{S^{(\beta)}; \beta \in \mathbb{R}\}$, are collected in Section 3.4. We close the Chapter by providing a survey on the most known fine methods for the study of the distribution of first passage time of a Brownian motion over a given smooth function.

3.2 Sturm-Liouville Equation and Gauss-Markov Processes

Consider the equation (3.7) for some $\mu \in Mr^+$. It is easy to check that if ϕ is a solution then so is $\vartheta(.) = \phi \int_0^{\cdot} \frac{ds}{\phi^2(s)}$ and the set of solutions to this equation is given by the vectorial space spanned by ϕ and ϑ. Furthermore, all positive solutions of (3.7) are convex. Moreover, we know that there exists a unique positive decreasing solution, denoted by φ, such that $\varphi(0) = 1$. It satisfies $\lim_{t\to\infty} \varphi(t) \in [0, 1]$ and the strict inequality $\varphi(\infty) < 1$, except in the trivial case $\mu = 0$. Moreover, under the condition $\int (1 + s)\mu(ds) < \infty$, we have $\varphi(\infty) > 0$. See, for instance, the Appendix 8 of Revuz and Yor [100] for a detailed discussion about this topic. Writing $\psi = \varphi \int_0^{\cdot} \frac{ds}{\varphi^2(s)}$, we have the following characterization of the space of positive convex function.

Lemma 3.2.1

$$V^+(\mathbb{R}_0^+) = \bigcup_{\mu\in Mr^+} \bigcup_{\alpha>0} \bigcup_{\beta\geq 0} \{\Pi^{(\alpha,\beta)}\varphi;\ \varphi'' = \mu\varphi\}.$$

Proof. The result follows after these identities

$$\begin{aligned} V^+(\mathbb{R}_0^+) &= \bigcup_{\mu\in Mr^+} SL^{(\mu,+)}(\mathbb{R}_0^+) \\ &= \bigcup_{\mu\in Mr^+} \bigcup_{\alpha>0} \bigcup_{\beta\geq 0} \{\alpha\varphi + \beta\psi;\ \varphi'' = \mu\varphi\} \end{aligned}$$

$$= \bigcup_{\mu \in Mr^+} \bigcup_{\alpha>0} \bigcup_{\beta \geq 0} \{\Pi^{(\alpha,\beta)}\varphi;\ \varphi'' = \mu\varphi\}.$$

□

Next, we state some elementary properties of the family $\Pi^{(\alpha,\beta)}$.

Proposition 3.2.2 *1. Fix $\alpha, \alpha' \in \mathbb{R}^+$ and $\beta, \beta' \in \mathbb{R}$. Then,*

$$\Pi^{(\alpha,\beta)} \circ \Pi^{(\alpha',\beta')} = \Pi^{(\alpha\alpha',\alpha'\beta+\beta'/\alpha)}.$$

In particular $\Pi^{(\alpha,\beta)} \circ \Pi^{(1/\alpha,-\beta)} = Id$.

2. $(\Pi^{(1,\beta)})_{\beta\geq 0}$ is a semigroup.

3. Fix $a, b \in \mathbb{R}^+_\infty$, and $\alpha, \beta \in \mathbb{R}^+$. If $\phi \in A_a^{-2,b}$ then $\Pi^{(\alpha,\beta)}\phi \in A_a^{-2,c}$ with $c = b/(\alpha(\alpha+\beta b))$ and $\Pi^{(\alpha,-\beta)}\phi \in A_{a'}^{-2,b'}$ with $a' = a$, $b' = \frac{b}{\alpha(\alpha-\beta b)}$ if $b < \alpha/\beta$ and $a' = \varrho(\frac{\alpha}{\beta})$, $b' = \infty$ otherwise.

4. For α and β real numbers, $\Pi^{(\alpha,\beta)}$ preserves the convexity and concavity.

Proof. The proof of the first two items follows from some easy algebra. For (3), fix $\phi \in A_a^{-2,b}$ and recall that $\Pi^{(\alpha,\beta)}\phi(t) = \phi(t)(\alpha+\beta\tau(t))$. Then, by integration we get

$$\begin{aligned} \tau^{(\Pi^{(\alpha,\beta)}\phi)}(t) &= \int_0^t \frac{ds}{\left(\Pi^{(\alpha,\beta)}\phi(s)\right)^2} \\ &= \frac{\tau(t)}{\alpha(\alpha+\beta\tau(t))}. \end{aligned}$$

Next, if $\alpha, \beta > 0$, then $\Pi^{(\alpha,\beta)}\phi > 0$ on $[0, a)$ and $\tau^{(\Pi^{(\alpha,\beta)}\phi)}(a) = \frac{b}{\alpha(\alpha+\beta b)}$. On the other hand, $\Pi^{(\alpha,-\beta)}\phi > 0$ on $[0, a \wedge \varrho(\frac{\alpha}{\beta}))$. Finally, we have $\tau^{(\Pi^{(\alpha,-\beta)}\phi)}(a) = \frac{b}{\alpha(\alpha-\beta b)}$ and $\tau^{(\Pi^{(\alpha,-\beta)}\phi)}\left(\varrho(\frac{\alpha}{\beta})\right) = \infty$. The proof of the last item is obtained by differentiating twice in the sense of distributions. □

Remark 3.2.3 *We point out that $\Pi^{(1,\beta)}$ is related to the transformation T_β introduced by Donati et al. in [32] as follows $\Pi^{(1,\beta)} = \exp \circ T_\beta \circ \log$.*

Next, to a function $\phi \in A^{-2,b}_{\zeta^{(\phi)}}$ we associate the Gauss-Markov process of Ornstein-Uhlenbeck type $U^{(\phi)}$ starting from $U_0^{(\phi)} \in \mathbb{R}$ which is defined by (3.3). Next, we denote by $\mathbb{P}_x^{(\phi)}$ the law of $U^{(\phi)}$ when started at $U_0^{(\phi)} = x \in \mathbb{R}$ and write simply $\mathbb{P}^{(\phi)}$ when $x = 0$. Similarly, $\mathbb{P}_x$ stands for the law of $(x+B_t, t \geq 0)$. To a fixed $y \in \mathbb{R}$ we associate the first passage time $H_y^{(\phi)} = \inf\left\{s \geq 0;\ U_s^{(\phi)} = y\right\}$. Without loss of generality, we choose the normalization $\phi(0) = 1$ and we emphasize that the study also applies to negative-valued function thanks to the symmetry property of B.

$U^{(\phi)}$ is a continuous Gaussian process with mean $m(t) = U_0^{(\phi)}\phi(t)$ and covariance

$$\begin{aligned} v(s,t) &= \phi(t \vee s)\Pi^{0,1}\phi(s \wedge t), \\ &= \phi(t \vee s)\vartheta(s \wedge t), \quad s,t \leq \zeta^{(\phi)}. \end{aligned}$$

Remark 3.2.4 *First note that with the choice $\phi \equiv 1$, $U^{(1)}$ is simply a Brownian motion. Also, by choosing $\phi(t) = e^{-\lambda t}$, with $\lambda \in \mathbb{R}$, $U^{(\phi)}$ boils down to the classical Ornstein-Uhlenbeck process. Moreover, in this case, by taking $\lambda > 0$ and $U_0^{(\phi)}$ to be centered, normally distributed with variance $1/2\lambda$ and independent of B, we get the only stationary Gaussian Markov process.*

The laws of associated hitting times of constant levels of the family $\left\{U^{(\phi)}, \phi \in SL^{(\mu,+)}(\mathbb{R}_0^+)\right\}$ are all related. Next, to a couple $(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}$ we associate $\theta = \Pi^{(\alpha,\beta)}\phi$. Then, $\theta \in A^{-2,b'}_{\zeta^{(\theta)}}$ where $\zeta^{(\theta)}$ and b' are given in Proposition 3.2.2. Denote by $\zeta = \zeta^{(\phi)} \wedge \zeta^{(\theta)}$ and introduce the function, for $t < \zeta$ and $x \in \mathbb{R}$,

$$\mathcal{M}(t,x) = \left(\frac{\phi(t)}{\theta(t)}\right)^{\frac{1}{2}} e^{\frac{\beta}{2}\frac{x^2}{\phi(t)\theta(t)}}. \tag{3.8}$$

In particular, we have $\mathcal{M}(0,x) = \alpha^{-1/2}e^{\frac{\beta x^2}{2\alpha}}$. We are now ready to state the following.

Lemma 3.2.5 *The process $\left(\mathcal{M}(t, U_t^{(\phi)}),\ 0 \leq t < \zeta\right)$ is a $\mathbb{P}^{(\phi)}$-martingale.*

PROOF. In the special case $\phi \equiv 1$, observe that

$$\begin{aligned}\mathcal{M}(t, B_t) &= \frac{1}{\sqrt{\alpha+\beta t}} e^{\frac{\beta}{2}\frac{B_t^2}{\alpha+\beta t}} \\ &= e^{\beta\int_0^t \frac{B_s dB_s}{\alpha+\beta s} - \frac{\beta^2}{2}\int_0^t \frac{B_s^2}{(\alpha+\beta s)^2} ds}\end{aligned}$$

which is a bounded $\mathbb{P}$-local martingale and hence a true martingale. The martingale property follows from the fact that $\mathcal{M}(t, U_t^{(\phi)})$ has the same distribution as $\mathcal{M}(\tau(t), B_t)$. □

We are now ready to state the main result of this Section.

Theorem 3.2.6 *For $x, y \in \mathbb{R}$, we have*

$$\mathbb{P}_x^{(\theta)}\left(H_y^{(\theta)} \in dt\right) = \frac{\mathcal{M}(t,y)}{\mathcal{M}(0,x)} \mathbb{P}_x^{(\phi)}\left(H_y^{(\phi)} \in dt\right), \quad t < \zeta. \tag{3.9}$$

PROOF. From the previous Lemma, we deduce by using Girsanov's Theorem that

$$d\mathbb{P}_{x|\mathcal{F}_t}^{(\theta)} = \frac{\mathcal{M}(t, U_t^{(\phi)})}{\mathcal{M}(0,x)} d\mathbb{P}_{x|\mathcal{F}_t}^{(\phi)}, \quad t < \zeta. \tag{3.10}$$

Next, on the set $\{H_y^{(\phi)} \le t\} \in \mathcal{F}_{t\wedge H_y}$, we have $\mathcal{M}(t \wedge H_y, U_{t\wedge H_y}^{(\phi)}) = \mathcal{M}(H_y, y)$. So Doob's optional stopping Theorem implies

$$\begin{aligned}\mathbb{P}_x^{(\theta)}(H_y \le t) &= \mathbb{E}_x\left[1_{\{H_y \le t\}} \frac{\mathcal{M}(t, U_t^{(\phi)})}{\mathcal{M}(0,x)}\right] \\ &= \mathbb{E}_x\left[1_{\{H_y \le t\}} \mathbb{E}_x\left[\frac{\mathcal{M}(t, U_t^{(\phi)})}{\mathcal{M}(0,x)} \Big| \mathcal{F}_{t\wedge H_y}\right]\right] \\ &= \mathbb{E}_x\left[1_{\{H_y \le t\}} \frac{\mathcal{M}(H_y, y)}{\mathcal{M}(0,x)}\right].\end{aligned}$$

Our claim follows then by differentiation. □

Remark 3.2.7 *To a process X and a function ϕ, we associate the process $M^{(\phi)}(X)$ defined for any fixed $t < \zeta^{(\phi)}$ by*

$$M_t^{(\phi)}(X) = \frac{1}{\sqrt{\phi(t)}} e^{\frac{1}{2}\frac{\phi'(t)}{\phi(t)} X_t^2 - \frac{1}{2}\int_0^t X_s^2 \mu(ds)}$$

where $\phi'' = \mu\phi$ in the sense of distributions. By checking that the jumps of the various involved processes cancel, we see that $M_t^{(\phi)}(U^{(\phi)})$ is continuous. Because, for any fixed $t < \zeta^{(\phi)}$, the random variable $U_t^{(\phi)}$ is proper, that is $\mathbb{P}_x^{(\phi)}(U_t^{(\phi)} < \infty) = 1$, we conclude that $M_t^{(\phi)}(U^{(\phi)}))$ is a true $\mathbb{P}_x$-martingale. In other words, $\mathbb{E}[M_t^{(\phi)}(U^{(\phi)})] = 1$ for any $t < \zeta^{(\phi)}$.

Remark 3.2.8 *Relation (3.10) is also obtained by the chain rule as follows. For $t < \zeta^{(\phi)}$, we have*

$$\begin{aligned} d\mathbb{P}^{(\phi)}_{x|\mathcal{F}_t} &= \frac{M_t^{(\phi)}(B)}{M_0^{(\phi)}(x)}\, d\mathbb{P}_{x|\mathcal{F}_t} \\ &= \frac{M_0^{(\theta)}(x) M_t^{(\theta)}(B)}{M_t^{(\theta)}(B) M_0^{(\theta)}(x)} \frac{M_t^{(\phi)}(B)}{M_0^{(\phi)}(x)}\, d\mathbb{P}_{x|\mathcal{F}_t} \\ &= \frac{\mathcal{M}(t, U_t^{(\phi)})}{\mathcal{M}(0,x)} d\mathbb{P}^{(\theta)}_{x|\mathcal{F}_t}. \end{aligned}$$

3.3 Doob's Transform and Switching of First Passage Time Problem

Recall that Doob's transform Σ is defined by formula (3.5) of Section 3.1. We start by providing some elementary properties of Σ.

Proposition 3.3.1 *1. Σ is an involution, i.e. $\Sigma^2 = Id$.*

2. Fix $a, b \in \mathbb{R}_\infty^+$. $\Sigma\left(A_a^{-2,b}\right) = A_b^{-2,a}$.

3. Let $\phi, \theta \in C(\mathbb{R}_0^+, \mathbb{R}^+)$ with ϕ non increasing such that $\phi \leq \theta$ then we have $\Sigma\phi \geq \Sigma\theta$.

4. Σ preserves and reverses the monotonicity.

5. Σ transforms a convex function into a concave one.

6. If $f \in C^{(n)}\left(\mathbb{R}_0^+, \mathbb{R}^+\right)$ for some $n \in \mathbb{N}$ then $sgn\left(f^{(n)}(\Sigma f)^{(n)}\right) = -1$.

7. *Recall that for* $(\alpha,\beta)\in\mathbb{R}^+\times\mathbb{R}$, $S^{(\alpha,\beta)}=\Sigma\circ\Pi^{(\alpha,-\beta)}\circ\Sigma$. *Let* $\phi\in A_a^{-2,b}$, *then* $S^{(\alpha,\beta)}\phi\in A_{a'}^{-2,b'}$ *where* $a'=a$, $b'=b/(\alpha(\alpha+\beta b))$ *if* $b<-\alpha/\beta$ *and* $a'=\varrho(\alpha/\beta)$, $b'=\infty$ *otherwise. Moreover, we have*
$$S^{\alpha,\beta}\phi(\cdot)=\left(\frac{1+\alpha\beta\cdot}{\alpha}\right)\phi\left(\frac{\alpha^2\cdot}{1+\alpha\beta\cdot}\right).$$

Proof. (1) Let $\phi\in C_\infty(\mathbb{R}^+)$. From the identity $\phi\circ\varrho=1/\Sigma\phi$, we deduce that $\varrho(.)=\int_0^\cdot\phi^2(\varrho(s))ds$. Hence, $\Sigma^2\phi=\Sigma\left(1/\phi\circ\varrho\right)=\phi\circ\varrho\circ\tau=\phi$. (2) Let $\phi\in A_a^{-2,b}$ and denote $f=\Sigma\phi$. First, note that τ is an homeomorphism from $[0,a)$ into $[0,b)$. Hence, $f\in C\left([0,b),\mathbb{R}^+\right)$. We conclude by observing that $\varrho(b)=a$. (3) follows from the fact that $\varrho^{(\phi)}\le\varrho^{(\theta)}$ which implies that $\phi\circ\varrho^{(\phi)}\le\phi\circ\varrho^{(\theta)}\le\theta\circ\varrho^{(\theta)}$. Items (4) and (5) are immediate consequences of the fact that ϱ is increasing. (6) is shown by induction. Note that (4) and (5) give $n=1,2$. Since $\Sigma\phi\circ\tau=1/\phi$, it follows that $(\Sigma\phi)'=-\phi'\circ\varrho$. Furthermore, we see that if φ is decreasing (resp. increasing) then f is increasing (resp. decreasing). (7) From Propositions 3.2.2 and 3.3.1, we deduce readily the first part of the assertion. For the identity, we set $\vartheta=\phi\tau$. By integration, we see that
$$\int_0^\cdot\frac{ds}{(\alpha\phi(s)-\beta\vartheta(s))^2}=\frac{1}{\alpha}\frac{\tau(\cdot)}{\alpha-\beta\tau(\cdot)}.$$
Inverting and using the fact that $\tau\circ\varrho=Id$, yields
$$\int_0^\cdot\frac{ds}{\left(\Sigma\circ\Pi^{\alpha,-\beta}\phi\right)^2(s)}=\varrho\left(\frac{\alpha^2\cdot}{1+\alpha\beta\cdot}\right).$$
The item follows by differentiation. □

We show that Σ transforms the space of positive solutions of the Sturm-Liouville equation (3.7) to the space of solutions to a non-linear second order differential equation. As we pointed out, we have
$$\Sigma\{SL^{(\mu)}(\mathbb{R}_0^+)\}=\bigcup_{\alpha>0}\bigcup_{\beta\ge0}\{\left(S^{(\alpha,\beta)}\circ\Sigma\right)\varphi\}. \tag{3.11}$$

Theorem 3.3.2 *Let* $\mu\in Mr^+$ *and let* φ *be the positive decreasing solution of (3.7). Consider on* $\mathbb{R}_0^+$ *the nonlinear differential equation* $f^3f''d.=-\mu(d\varrho.)$ *defined in the sense of distributions. Then, this*

equation has a unique positive, increasing and concave solution such that $f(0) = 1$ *and* $f'(0) = -\varphi'(0)$. *Furthermore, its positive solutions on* $\mathbb{R}^+$ *are given by the set* $\Sigma\{SL^{(\mu)}(\mathbb{R}_0^+)\}$.

Proof. From the identity $\phi(\cdot)(\Sigma\phi)(\tau(\cdot)) = 1$, we deduce that $\phi'(\cdot) = -(\Sigma\phi)'(\tau(\cdot))$ and $\phi''(\cdot) = -(\Sigma\phi)''(\tau(\cdot))/\phi^2(\cdot)$ in the sense of distributions. The proof is then completed by respectively putting pieces together and performing the change of variable $s = \tau(t)$, keeping in mind that we can integrate the other way around. □

Lemma 3.3.3 $\phi \in A_\infty^{-2,\infty} \cap V^+(\mathbb{R}_0^+)$ *if and only if* ϕ *is decreasing and* $\int (1+s)\dfrac{\phi''(s)}{\phi(s)}\,ds = \infty$. *Consequently* $f \in A_\infty^{-2,\infty}$ *and is concave if and only if* $\Sigma f \in A_\infty^{-2,\infty} \cap V^+(\mathbb{R}_0^+)$.

Proof. As discussed in Section 2, we have $\varphi(\infty) = 0$ if $\int(1+s)\frac{\varphi''(s)}{\varphi(s)}ds = \infty$. The second assertion is an immediate consequence of the first one. □

The following result is required later and is important in the derivation of our classification of concave boundaries with respect to the behavior of the tail of the distribution of the corresponding boundary crossing random times at $+\infty$.

Proposition 3.3.4 *Let* $g \in A_{\zeta^{(g)}}^{-2,b}$. *Moreover, we assume that* g *is concave and write* $g'_+(.)$ *for its right derivative. Then, there exists a unique increasing and concave function* f *with* $f(0) = 1$ *such that, for any* $t < \zeta^{(g)}$, *we have the following identity*

$$g(t) = \left(\frac{1+\alpha\beta t}{\alpha}\right) f\left(\frac{\alpha^2 t}{1+\alpha\beta t}\right) \tag{3.12}$$

where $\alpha = 1/g(0)$ *and* $\beta = g'_+(0) - \alpha f'(0)$. *Furthermore, we have* $f \in A_{\zeta^{(f)}}^{-2,b'}$ *where* $\zeta^{(f)} = \zeta^{(g)}, b' = \alpha^2 b/(1-\alpha\beta b)$ *if* $b > 1/(\alpha\beta)$ *or* $\zeta^{(f)} = \varrho^{(g)}(1/(\alpha\beta)), b' = \infty$ *otherwise. Consequently, if* $\zeta^{(g)} = \infty$ *and* $\beta > 0$, *we have*

$$g(t) \sim \beta f\left(\frac{\alpha}{\beta}\right) t, \quad \text{as } t \to \infty. \tag{3.13}$$

Proof. Items (5) and (2) of Proposition 3.3.1 implies that $\phi = \Sigma g$ is convex and $\phi \in A_b^{-2,\zeta^{(g)}}$. Define then μ by $\phi'' = \mu\phi$ in the sense of distributions. This is a positive Radon measure on $[0, b)$ and the associated equation (3.7) admits a unique couple of solutions (φ, ψ) satisfying the conditions on $[0, b)$ fixed in Section 3.2. In particular, φ is positive, convex and decreasing with $\varphi(0) = 1$. Thus, there exists a unique pair $(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}$ such that $\phi = \Pi^{(\alpha,-\beta)}\varphi$. By choosing $f = \Sigma\varphi$, we get a function which fulfills the required properties. Finally, we easily check that $\alpha = 1/g(0)$ and $\beta = -\phi'(0)+\alpha\varphi'(0)$. The result follows then by recalling that $g' = -(\Sigma\phi)'$ and the fact that $f = S^{(1/\alpha,-\beta)}g$. $\square$

Remark 3.3.5 *The quantity $\varphi'(0)$ already appeared as the Lévy exponent of a subordinator. Indeed, let $(l_t^a, a \in \mathbb{R}, t \geq 0)$ be a bi-continuous version of the local time of B. Write l_t for the local time of B at the level 0 and denote by σ its right inverse i.e. $\sigma_r = \inf\{s \geq 0;\ l_s \geq r\}$. If g is a C^1-function with compact support in $(0, \infty)$ then the stopped process $\left(\int_0^{\sigma_r} g(B_s)\, ds, r \geq 0\right)$ is a subordinator. Its Laplace-Lévy exponent is given by*

$$\mathbb{E}\left[e^{-\lambda \int_0^{\sigma_1} g(B_s)\, ds}\right] = e^{\varphi'(0)}, \quad \lambda \geq 0,$$

where φ is defined as above with $\mu(dx) = \lambda g(x)dx$, $x \geq 0$. This is nothing but a reformulation of the second Ray-Knight Theorem which states that $\left(l_{\sigma_r}^a, a \geq 0\right)$ is a squared Bessel process of dimension 0. More generally, the couple (φ, ψ) is involved in fine studies of other functionals of Bessel processes and, for interested readers, we refer to [100, Chap. XI and XII].

Now, we turn to the relationship between the first passage time to a constant level by a $GMOU^{(\phi)}$ with $\phi \in \mathcal{A}_a^{-2,b}$, denoted simply by U, and the first passage time to the curve $\Sigma\phi$ by the Brownian motion. By Dumbis, Dubins-Schwarz Theorem, see Revuz and Yor [100, p.181], there exists a unique standard Brownian motion W such that, for any $t \geq 0$, we have

$$U_t^{(\phi)} = \phi(t)\left(U_0^{(\phi)} + W_{\tau(t)}\right), \quad U_0^{(\phi)} \in \mathbb{R}, \tag{3.14}$$

where $\tau(t) = \int_0^t \phi^{-2}(s)\, ds$. We recall that the relation (3.14) was first introduced by Doob in [34] in the case of the stationary Ornstein-Uhlenbeck process, see Chapter 1 for more details. For the sake of simplicity, throughout the rest of this Section, we set $U_0^{(\phi)} = 0$. Recall the

notation $H_y^{(\phi)} = \inf\left\{s \geq 0;\ U_s^{(\phi)} = y\right\}$, we simply write $H^{(\phi)} = H_1^{(\phi)}$, and finally $T^{(f)} = \inf\{s \geq 0;\ B_s = f(s)\}$ where $f \in C(\mathbb{R}_0^+, \mathbb{R}^+)$. In the following lines we generalize the idea of Breiman [19] which consists on using Doob's transform to connect the law of the first passage time of the Brownian motion to the square root boundary and the one of the Ornstein-Uhlenbeck process to a fixed level. We have the following result.

Theorem 3.3.6 *For any $\phi \in A_a^{-2,b}$ we have the equality in law*

$$H^{(\phi)} = \int_0^{T^{(\Sigma\phi)}} \frac{ds}{(\Sigma\phi)^2(s)}, \quad \text{on } [0, a \wedge b).$$

As a consequence, we have, for $t < a \wedge b$,

$$\mathbb{P}^{(\phi)}\left(H^{(\phi)} \in dt\right) = \tau'(t)\mathbb{P}\left(T^{(\Sigma\phi)} \in d\tau(t)\right).$$

In particular, $\mathbb{P}^{(\phi)}\left(H^{(\phi)} < a\right) = \mathbb{P}\left(T^{(\Sigma\phi)} < b\right)$.

Proof. We can write

$$\begin{aligned} H^{(\phi)} &= \inf\left\{s \geq 0;\ \phi(s)\int_0^s \phi^{-1}(u)\, dB_u = 1\right\} \\ &= \inf\left\{s \geq 0;\ \phi\left(\varrho\left(\tau(s)\right)\right) W_{\tau(s)} = 1\right\} \\ &= \varrho(T^{(\Sigma\phi)}) \end{aligned}$$

which gives the first statement. The second one follows. □

Remark 3.3.7 *By observing that $H_0^{(\phi)} = \tau(T_0)$ a.s., we derive from Proposition 3.6.1 an expression for $h_x^{\phi''/\phi}$. Indeed, for $x > 0$, we have, for $t < \zeta^{(\phi)}$,*

$$h_x^{\phi''/\phi}(0,t) = \left(\frac{\tau(t)}{t}\right)^{3/2}\left(\frac{\phi(0)}{\phi(t)}\right)^{1/2} e^{-x^2\left(\frac{\phi'(0)}{\phi(0)} + \frac{1}{t} - \frac{1}{\tau(t)}\right)}. \tag{3.15}$$

3.4 First Passage Time and the Elementary Family of Mappings

In this Section, we present two different methodologies to derive the relationship between the density of first passage time of a curve to a parameterized family of curves by the Brownian motion.

3.4.1 The Composition Approach

Now, we are interested in the transform $S^{(\alpha,\beta)} = \Sigma \circ \Pi^{(\alpha,-\beta)} \circ \Sigma, \quad (\alpha,\beta) \in \mathbb{R}^+ \times \mathbb{R}$. Note that it enjoys the following property. For $\alpha, \alpha' \in \mathbb{R}^+$ and $\beta, \beta' \in \mathbb{R}$, we have

$$S^{(\alpha,\beta)} \circ S^{(\alpha',\beta')} = S^{(\alpha\alpha',\alpha'\beta+\beta'/\alpha)}.$$

In particular $S^{(\alpha,\beta)} \circ S^{(1/\alpha,-\beta)} = Id$. We leave to the next subsection the study of the case $\alpha = 1$. Next, to simplify notation we write $f^{(\alpha,\beta)} = S^{(\alpha,\beta)} f$ and fix $f \in \mathcal{A}_a^{-2,b}$. We recall that $f^{(\alpha,\beta)} \in A_{\zeta^{(f)}}^{-2,b'}$ where $\zeta^{(f)} = a$, $b' = b/(\alpha(\alpha+\beta b))$ if $b < -\alpha/\beta$ and $\zeta^{(f)} = \varrho(\alpha/\beta)$, $b' = \infty$ otherwise. We point out that we can extend the domain of action of the map $S^{(\alpha,\beta)}$ to the space of probability measures. That is, in the absolute continuous case, to the measure $\mu(dt) = h(t)dt$ we associate $S^{(\alpha,\beta)}(\mu)(dt) = S^{(\alpha,\beta)}(h(t))dt$. We are now ready to state the main result of this Chapter.

Theorem 3.4.1 *For any $t < \zeta^{(f)}$, we have the relationship*

$$\mathbb{P}\left(T^{(f^{(\alpha,\beta)})} \in dt\right) = \left(\frac{\alpha}{1+\alpha\beta t}\right)^{\frac{5}{2}} e^{-\frac{\alpha\beta f^{(\alpha,\beta)}(t)^2}{2(1+\alpha\beta t)}} S^{(\alpha,\beta)}\left(\mathbb{P}\left(T^{(f)} \in dt\right)\right). \tag{3.16}$$

Proof. Let $\theta = \Sigma f^{(\alpha,\beta)}$ and $\phi = \Pi^{(1/\alpha,\beta)}\theta$. Then, from Theorems 3.2.6 and 3.3.6, we get successively, with the obvious notation,

$$\begin{aligned}
\mathbb{P}\left(T^{(f^{(\beta)})} \in dt\right) &= \varrho^{(\theta)'}(t)\mathbb{P}^{(\theta)}\left(H \in d\varrho^{(\theta)}(t)\right) \\
&= \frac{\mathcal{M}(\varrho^{(\theta)}(t),1)}{\mathcal{M}(0,0)} \varrho^{(\theta)'}(t)\mathbb{P}^{(\phi)}\left(H \in d\varrho^{(\theta)}(t)\right)
\end{aligned}$$

$$= \frac{\mathcal{M}(\varrho^{(\theta)}(t),1)}{\mathcal{M}(0,0)}\varrho^{(\theta)'}(t)\tau^{(\phi)'}(\varrho^{(\theta)}(t)) \times\mathbb{P}\left(T^{(f)} \in d\tau^{(\phi)}(\varrho^{(\theta)}(t))\right).$$

Next, note that $\theta(\varrho^{(\theta)}(t)) = 1/f^{(\alpha,\beta)}(t)$. From Proposition 3.2.2 we observe that $\tau^{(\theta)}(\varrho^{(\phi)}(t)) = t/(\alpha(\alpha-\beta t))$, then we easily deduce that $\tau^{(\phi)}(\varrho^{\theta}(t)) = \alpha^2 t/(1+\alpha\beta t)$. Finally, we conclude by observing that $\phi(\varrho^{(\theta)}(t)) = (1+\alpha\beta t)/(\alpha f^{(\alpha,\beta))}(t))$.

□

We postpone to the next Section the investigations of some known examples.

3.4.2 The Family of Elementary Transformations

We shall now focus on the family $\left\{S^{(\beta)};\beta\in\mathbb{R}\right\}$, defined in (3.2), and study its elementary properties as well as its application to the boundary crossing problem for the Brownian motion and Bessel processes. A way to think about this family is, as we have seen, its realization as the composition $\Sigma\circ\Pi^{(1,-\beta)}\circ\Sigma$. We shall prove that it is possible to derive our relationship between crossing boundaries distribution directly without going through the study of GMOU processes. We proceed by providing some properties of the studied family.

Proposition 3.4.2 *The family $\left\{S^{(\beta)},\beta\in\mathbb{R}\right\}$ has the following properties.*

1. *For $\alpha,\beta\in\mathbb{R}$, we have $S^{(\alpha)}\circ S^{(\beta)} = S^{(\alpha+\beta)}$.*
2. *$(S^{(\beta)})_{\beta\geq 0}$ is a semigroup.*
3. *For a fixed $\beta\in\mathbb{R}$ the mapping $S^{(\beta)}$ is linear and its invariant subspace is the set of linear functions.*

Proof. The statements (1), (2) and the first part of (3) are obvious. We also easily check that the space of linear functions is invariant. Next, assume that there exists $\beta\in\mathbb{R}$ and a continuous mapping $f:\mathbb{R}^+\to$

$\mathbb{R}$ such that $S^{(\beta)}(f) = f$ on $[0, \zeta^{(\beta)}[$. Then, for all $n \in \mathbb{N}$ we have $S^{(n\beta)}(f) = f$ on $[0, \zeta^{(n\beta))}[$. If $\beta < 0$, then f and $S^{(\beta)}(f)$ are different since their are not defined on the same domain. In the other case, we observe that f is invariant by $S^{(\beta)}$ if and only if $\hat{f}(t) = f(t)/t$ is invariant by $(1+\beta.)^{-1}S^{(\beta)}$. Repeating this procedure, we obtain that $\hat{f}$ is invariant through the transformation $(1+n\beta.)^{-1}S^{(n\beta)}$ for any $n \in \mathbb{N}^*$. Then, letting $n \to \infty$ and using the right continuity of $\hat{f}$, we get that $\hat{f}(t) = \lim_{s\to 0} \hat{f}(s)$ for $t \in \mathbb{R}^+$. In other words, f is linear. $\square$

At a first stage, we shall show how a specific element of this family allows to realize a standard Brownian bridge $B^{(br)}$, of length $T > 0$, from a given Brownian motion B. Recall that $B^{(br)}$ is the unique solution, on $[0, T[$, of the linear equation

$$B_t^{(br)} = B_t - \int_0^t \frac{B_s^{(br)}}{T-s} ds, \quad t < T,$$

which, when integrated, yields the well-known expression

$$B_t^{(br)} = (T-t) \int_0^t \frac{dB_s}{T-s}, \quad t < T. \tag{3.17}$$

Note that the law of $B^{(br)}$ is also obtained as a Doob's h-transform of that of B. Indeed, denoting by $\mathbb{P}^{(br)}$ and $\mathbb{P}$, respectively, the laws of $B^{(br)}$ and B, then, for $0 \le t < T$, these probability measures are related as follows

$$\begin{aligned} d\mathbb{P}^{(br)}_{|\mathcal{F}_t} &= \left(\frac{T}{T-t}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\frac{B_t^2}{T-t}} d\mathbb{P}_{|\mathcal{F}_t} \\ &= \frac{1}{\sqrt{1+\beta t}} e^{\frac{\beta}{2}\frac{B_t^2}{1+\beta t}} d\mathbb{P}_{|\mathcal{F}_t} \end{aligned} \tag{3.18}$$

where $\beta = -T^{-1}$. In the sequel, we write simply $\zeta^{(\beta)}$ for $\zeta^{(1+\beta t)}$, that is $\zeta^{(\beta)} = -\beta^{-1}$ for $\beta < 0$. Observe that the above results remain true when $\beta > 0$. Consequently, as an extension of the family of standard Brownian bridges, we introduce the real-parameterized family of GMOU processes with parameters $\{(1+\beta t),\ \beta \in \mathbb{R}\}$, defined, for a fixed $\beta \in \mathbb{R}$, by

$$U_t^{(\beta)} = (1+\beta t) \int_0^t \frac{dB_s}{1+\beta s}, \quad t < \zeta^{(\beta)}.$$

This implies readily that

$$U_t^{(\beta)} = S^{(\beta)}(B^{(\beta)})_t,\ t < \zeta^{(\beta)}, \tag{3.19}$$

where $B^{(\beta)}$ is the martingale

$$B_t^{(\beta)} = \int_0^{\frac{t}{1-\beta t}} \frac{dB_s}{1+\beta s}, \quad t < \zeta^{(-\beta)}. \tag{3.20}$$

Next, we have

$$\langle \int_0^{\cdot} \frac{dB_s}{1+\beta s} \rangle_t = \frac{t}{1+\beta t} \underset{t\to\zeta\beta}{\longrightarrow} \begin{cases} \frac{1}{\beta}, & \beta > 0 \\ \infty, & \text{otherwise.} \end{cases}$$

Thus, if $\beta \leq 0$ then $B^{(\beta)}$ is a Brownian motion defined on $\mathbb{R}^+$. Otherwise, we can extend the definition of $B^{(\beta)}$ such that it becomes a Brownian motion on $\mathbb{R}^+$ as

$$B_t^{(\beta)} = \begin{cases} \int_0^{\frac{t}{1-\beta t}} \frac{dB_s}{1+\beta s}, & t \leq \beta^{-1} \\ \int_0^{\frac{1}{\beta}} \frac{dB_s}{1+\beta s} + \tilde{B}_{t-\frac{1}{\beta}}, & t > \beta^{-1} \end{cases}$$

where $\tilde{B}$ is an another Brownian motion, independent of B.

Next, we need to introduce $H_f^{(-\beta)} = \inf\{s \geq 0;\ U_s^{(-\beta)} = f(s)\}$ and, for convenience, write simply $f^{(\beta)} = S^{(\beta)}(f)$. The support of $H_f^{-\beta}$ is the interval $[0, \beta^{-1}]$ when β is positive. Similarly, we close the curve $f^{(\beta)}$ at $-\beta^{-1}$ when β is negative. Theorem 3.3.6 allows us to connect $H_f^{(-\beta)}$ and $T^{(f^{(\beta)})}$ as follows

$$H_f^{(-\beta)} \overset{(d)}{=} \frac{T^{(f^{(\beta)})}}{1+\beta T^{(f^{(\beta)})}} \quad \text{and} \quad T^{(f^{(\beta)})} \overset{(d)}{=} \frac{H_f^{(-\beta)}}{1-\beta H_f^{(-\beta)}}. \tag{3.21}$$

We carry on our discussion by observing that we can also extend the domain of the family of transformations $S^{(\beta)}$ to the space of probability measures in the same fashion than for $S^{(\alpha,\beta)}$. The main result of this Section is the following.

Theorem 3.4.3 *For any $t < \zeta^{(\beta)}$, we have the relationship*

$$\mathbb{P}\left(T^{(f^{(\beta)})} \in dt\right) = \frac{1}{(1+\beta t)^{5/2}} e^{-\frac{1}{2}\frac{\beta}{1+\beta t} f^{(\beta)}(t)^2} S^{(\beta)}\left(\mathbb{P}(T^{(f)} \in dt)\right). \tag{3.22}$$

Proof. Introduce the funtion $h(t,x) = \frac{1}{\sqrt{1+\beta t}} e^{\frac{\beta}{2}\frac{x^2}{1+\beta t}}$. From (3.18), it is clear that $h\left(t \wedge T^{(f)}, X_{t\wedge T^{(f)}}\right)$ is a uniformly integrable martingale. Next, thanks to Lemma 3.21 and using the dominated convergence, we can write, for any $\lambda \geq 0$,

$$\begin{aligned}
& \mathbb{E}_x\left[e^{-\lambda T^{f^{(\beta)}}} \mathbb{I}_{\{T^{(f^{(\beta)})}<\zeta^{(\beta)}\}}\right] \\
= \; & \mathbb{E}_x\left[e^{-\lambda\left(\frac{H_f^{(-\beta)}}{1-\beta H_f^{(-\beta)}}\right)} \mathbb{I}_{\{H_f^{(-\beta)}<\zeta^{(-\beta)}\}}\right] \\
= \; & \mathbb{E}\left[\frac{1}{\sqrt{1-\beta T^{(f)}}} \exp\left\{-\lambda \frac{T^{(f)}}{1-\beta T^{(f)}} - \frac{\beta}{2}\frac{f^2(T^{(f)})}{1-\beta T^{(f)}}\right\} \mathbb{I}_{\{T^{(f)}<\zeta^{(-\beta)}\}}\right] \\
= \; & \int_0^{\zeta^{(-\beta)}} \frac{1}{\sqrt{1-\beta t}} \exp\left\{-\lambda \frac{t}{1-\beta t} - \frac{\beta}{2}\frac{f^2(t)}{1-\beta t}\right\} \mathbb{P}\left(T^{(f)} \in dt\right) \\
= \; & \int_0^{\zeta^{(\beta)}} \frac{e^{-\lambda r}}{(1+\beta r)^{3/2}} e^{-\frac{\beta}{2}(1+\beta r) f^2\left(\frac{r}{1+\beta r}\right)} \mathbb{P}\left(T^{(f)} \in d\left(\frac{r}{1+\beta r}\right)\right)
\end{aligned}$$

We complete the proof by using the injectivity of the Laplace transform and make use of $S^{(\beta)}$ in the notation. □

Remark 3.4.4 *Note that in the proof of the previous Theorem, the condition of f being positive can be relaxed and one can consider real valued function instead.*

We shall now be concerned with some properties of the resulting curves and the distributions of the corresponding crossing times. As usual, the notation $f \sim g$ stands for $\lim_{t\to\infty} f(t)/g(t) = 1$. In the case $\beta < 0$ we shall split the discussion into two cases depending on whether the limit $\lim_{t\to+\infty} f(t)/t = \tilde{f}(\infty)$ is finite or not. We have the following local limit result.

Theorem 3.4.5 *In the case $\beta > 0$, we have*

$$\lim_{t\to\infty} \frac{t^{3/2}}{dt} e^{\frac{1}{2}\beta(1+\beta t) f^2\left(\frac{1}{\beta}\right)} \mathbb{P}\left(T^{(f^{(\beta)})} \in dt\right) = \frac{\beta^{-3/2}}{d\beta^{-1}} \mathbb{P}\left(T^{(f)} \in d\left(\beta^{-1}\right)\right).$$

Proof. It is an immediate consequence of the fact that when $\beta > 0$ we have $f^{(\beta)} \sim \beta f(1/\beta) t$. □

3.4.3 Some Examples

In the Table below, we collect the images by $S^{(\beta)}$ of the most studied curves and mention that some other curves for which the density is known explicitly can be found in Lerche [76, p.27]. For any real numbers a, b and b_1, we have the following correspondences

f	$f^{(\beta)}$
$a + bt$	$a + (b + a\beta)t$
$\sqrt{1 + 2bt}$	$\sqrt{(1 + \beta t)(1 + (\beta + 2b)t)}$
$(b + t)^2$	$\frac{(b+(1+\beta))t)^2}{1+\beta t}$
$\frac{a}{2} - \frac{t}{a} \ln \left(\frac{b+\sqrt{b^2+4b_1 e^{-\frac{a^2}{t}}}}{2} \right)$ $a > 0, b \geq 0, b_1 > -b^2/4$	$\frac{a(1+\beta t)}{2} - \frac{t}{a} \ln \left(\frac{b+\sqrt{b^2+4b_1 e^{-\frac{a^2(1+\beta t)}{t}}}}{2} \right)$

Remark 3.4.6 *We refer to Lemma 2.3.3 for the expression of the density of the quadratic curve. We also mention that the density of the first passage time to the last boundary has been derived by Daniels [22], by using the method of images, and is given by*

$$\mathbb{P}(T^{(f)} \in dt) = \frac{1}{\sqrt{2\pi t^3}} \left(e^{-\frac{f(t)^2}{2t}} - \frac{b_1}{2} e^{-\frac{(f(t)-a)^2}{2t}} \right) dt.$$

We proceed by studying the two first examples given in the Table.

1. In the first example, taking $b = 0$, we easily recover the Bachelier-Lévy formula which is the distribution of the first passage time of the Brownian motion to the linear curve $(\mu t, t \geq 0)$, denoted by $T_\mu^{(1)}$. Indeed, by choosing $f = a$ and $\beta = \mu/a$ then we have $f^{(\beta)}(t) = a + \mu t$. Recall that, for the hitting time of the level a by B, we have the well-known formula

$$\mathbb{P}_a \left(T_0 \in dt \right) = \frac{|a|}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}} \, dt. \tag{3.23}$$

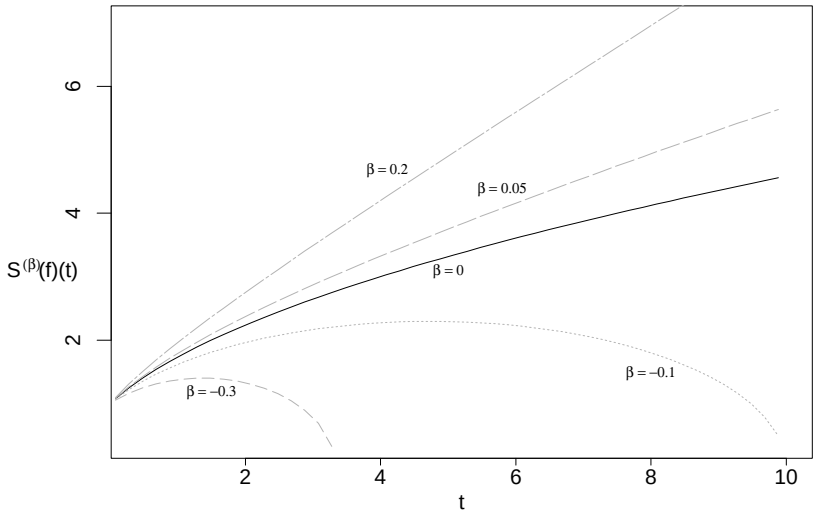

Figure 3.1: *Image of the function* $\sqrt{1+2t}$ *by* $S^{(\beta)}$ *for several values of* β.

Thus, a straightforward application of Theorem 3.4.1 yields

$$\mathbb{P}_a\left(T_\mu^{(1)} \in dt\right) = \frac{|a|}{\sqrt{2\pi t^3}} e^{-\mu a - \frac{\mu^2}{2}t - \frac{a^2}{2t}}\, dt$$

which is also easily checked by using Girsanov's Theorem.

2. Now, we suggest to compute the law of the first passage time of the Brownian motion to the square root of a quadratic function, see Figure 1. More precisely, we seek to determine the distribution of the stopping time

$$T_a^{(\lambda_1,\lambda_2)} = \inf\left\{s \geq 0;\ B_s = a\sqrt{(1+\lambda_1 s)\,(1+\lambda_2 s)}\right\}$$

where a and $\lambda_1 < \lambda_2$ are fixed real numbers. We do not treat the case $\lambda_1 = \lambda_2$ since it is elementary. First, we assume that $\lambda_2 = 0$ and, to simplify notation, we set $\lambda_1 = \lambda$ and $T_a^{(\lambda,0)} = T_a^{(\lambda)}$. It is the case studied by L. Breiman in [19], which is linked to the first passage time to a fixed level by an Ornstein-Uhlenbeck process. Indeed, with $U_t = e^{-\lambda t/2}\int_0^t e^{\lambda s/2} dB_s$, for any $t \geq 0$ and $H_a = \inf\{s \geq 0;\ U_s = a\}$, we have the equality in law

$$T_a^{(\lambda)} \overset{(d)}{=} \lambda^{-1}\left(e^{\lambda H_a} - 1\right) \tag{3.24}$$

which might be seen as a particular case of Theorem 3.3.6. Then, we observe that it is enough to consider only the case $a > 0$ since the other can be recovered from the symmetry of the Brownian motion. We complete the computation by using (3.24) to get, for $t < \zeta^{(\lambda)}$,

$$\mathbb{P}\left(T_a^{(\lambda)} \in dt\right) = \frac{1}{1+\lambda t}\mathbb{P}\left(H_a \in d\cdot\right)\big|_{\cdot=\frac{1}{\lambda}\log(1+\lambda t)} dt \qquad (3.25)$$

where several representations of the density of H_a can be found in Chapter 1. Now, if $\lambda_1 < \lambda_2$ then on $[0, \zeta^{(\lambda_1)}]$, which is the support of $T_a^{(\lambda_1,\lambda_2)}$ if λ_1 is positive and its support is finite otherwise, we have

$$S^{(\lambda_1)}\left(\sqrt{1+(\lambda_2-\lambda_1)\cdot}\right) = \sqrt{(1+\lambda_2\cdot)\,(1+\lambda_1\cdot)}.$$

By using Theorem 3.4.3, we obtain, for $t < \zeta^{(\lambda_1)}$,

$$\mathbb{P}\left(T_a^{(\lambda_1,\lambda_2)} \in dt\right) = \frac{e^{-\frac{1}{2}\lambda_1(1+\lambda_2 t)}}{(1+\lambda_1 t)^{5/2}} S^{(\lambda_1)}\left(\mathbb{P}(T_a^{(\lambda_2-\lambda_1)} \in dt)\right)$$

which ends by using (3.25).

3.5 Application to Bessel Processes

We start by recalling some well-known facts concerning Bessel processes. Let δ, $z \geq 0$ and set $\nu = \frac{\delta}{2} - 1$. It is plain that the stochastic differential equation

$$dQ_t^{(\nu)} = 2\sqrt{|Q_t^{(\nu)}|}dB_t + \delta dt, \quad Q_0^{(\nu)} = z,$$

admits a unique strong solution, see e.g. [100]. A realization of a Bessel process of dimension δ (or of index ν) is given by $R^{(\nu)} = \sqrt{Q^{(\nu)}}$. In particular, for $\delta > 1$, it is the unique solution of the equation

$$dR_t^{(\nu)} = dB_t + \frac{\delta - 1}{2R_t^{(\nu)}}\, dt, \quad R_0^{(\nu)} = x = \sqrt{z}.$$

The Laplace transform of the squared Bessel process $Q^{(\nu)}$ takes the following form

$$\mathbb{E}_z\left[e^{-\lambda Q_t^{(\nu)}}\right] = \frac{1}{(1+2\lambda t)^{\nu+1}\Gamma(\nu+1)} e^{-\frac{\lambda z}{1+2\lambda t}}, \quad \lambda \geq 0, \ (3.26)$$

and the density of the semigroup is given by

$$\begin{aligned} p_t^\nu(z,y) &= \frac{1}{2t}\left(\frac{y}{z}\right)^{\nu/2} e^{-\frac{z+y}{2t}} \mathcal{I}_\nu\left(\frac{\sqrt{zy}}{t}\right), \quad t>0,\ z\neq 0, \\ p_t^\nu(0,y) &= \frac{1}{(2t)^{\nu+1}\Gamma(\nu+1)} e^{-\frac{y}{2t}}, \quad t>0, \end{aligned}$$

where we recall that $\mathcal{I}_\nu$ is the modified Bessel function of the first kind of index ν. The case $z=0$ is obtained by passage to the limit and, for further results on Bessel processes, we refer to [100]. Observe that the law of $S^\beta(R^{(\nu)})$, denoted by $\mathbb{Q}_x^{(\nu,\beta)}$, is absolutely continuous with respect to that of $R^{(\nu)}$, denoted by $\mathbb{Q}_x^{(\nu)}$, and these are related via the mutual relation

$$d\mathbb{Q}_x^{(\nu,\beta)}{}_{|\mathcal{F}_t} = \frac{1}{(1+\beta t)^{\nu+1}} e^{-\frac{\beta}{2}\left(\frac{R_t^2}{1+\beta t}-x^2\right)} d\mathbb{Q}_x^{(\nu)}{}_{|\mathcal{F}_t} \tag{3.27}$$

for any $t<\zeta^{(\beta)}$. We also note that (3.19) remains true when $B^{(\beta)}$ and $U^{(\beta)}$ are replaced, respectively, by $R^{(\nu,\beta)}$ and $S^\beta(R)$, where these objects are defined following the same procedure. Next, we shall recall and provide an interpretation of Lamperti's relation [71] in terms of the mappings we introduced earlier. The latter states that, for any fixed $\nu>0$, there exists a Brownian motion B such that one has

$$e^{B_t^\nu} = R^{(\nu)}\left(\int_0^t e^{2B_s^\nu}\, ds\right) \tag{3.28}$$

where $B_t^\nu = B_t + \nu t$ for any $t\geq 0$. This reads $\Sigma\left(e^{-B^\nu}\right) = R^{(\nu)}$ and $\Sigma\left(e^{B^\nu}\right) = 1/R^{(\nu)}_{\cdot}$. We do not intend to go further in this direction however the following result is worth to be mentioned.

Corollary 3.5.1 *We have the equalities*

$$\begin{aligned} \Pi^{(\alpha,\beta)}\left(e^{-B^\nu}\right) &= e^{-B^\nu}\left(\alpha+\beta\int_0^{\cdot} e^{2B_s^\nu}\, ds\right) \\ &= \Pi^{(\alpha,\beta)}\circ\Sigma\left(R^{(\nu)}\right) \\ &= \Sigma\circ S^{(\alpha,\beta)}\left(R^{(\nu)}\right). \end{aligned}$$

Now, we are ready to state the analogue of Theorem 3.4.1 in the Bessel setting. We modify the notation by introducing $K^{(f)} = \inf\{s\geq 0; R_s^{(\nu)} =$

$f(s)\}$, for any $f \in C\left(\mathbb{R}_0^+, \mathbb{R}^+\right)$, for Bessel processes. The proof of the following result will be omitted since it is similar to the Brownian case.

Theorem 3.5.2 *For any $t < \zeta^{(\beta)}$, holds the relationship*

$$\mathbb{Q}_x^{(\nu)}\left(K^{(f^{(\beta)})} \in dt\right) = \frac{e^{\frac{-\beta}{2}\left(\frac{f^{(\beta)}(t)^2}{1+\beta t} - x^2\right)}}{(1+\beta t)^{\nu+3}} S^{(\beta)}\left(\mathbb{Q}_x^{(\nu)}\left(K^{(f)} \in dt\right)\right). \tag{3.29}$$

We shall now make explicit computations for the case where f is taken to be a straight line i.e. $f(t) = a + bt$, $t \geq 0$, where $a > 0$ and b is some fixed real numbers. Observe that with $\beta = b/a$, we have $S^{(\beta)}(a) = \{a + bt,\ t \geq 0\}$ when $b > 0$ and $S^{(\beta)}(a) = \{a + bt,\ t \leq -b/a\}$ otherwise. Next, set $K^{(a+b\cdot)} = \inf\{s \geq 0;\ R_s^{(\nu)} = a + bs\}$ and set $\hat{H}_a^{(\beta)} = \inf\{s \geq 0;\ S^{(\beta)}(R_s^{(\nu)}) = a\}$. Note that if $b < 0$ then the support of $K^{(a+b\cdot)}$ is $(0, -b/a)$ and recall that the distribution of $K^{(a)}$ is characterized by

$$\mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}K^{(a)}}\right] = \begin{cases} \frac{x^{-\nu}\mathcal{I}_\nu(x\lambda)}{a^{-\nu}\mathcal{I}_\nu(a\lambda)}, & x \leq a, \\ \\ \frac{x^{-\nu}\mathcal{K}_\nu(x\lambda)}{a^{-\nu}\mathcal{K}_\nu(a\lambda)}, & x \geq a. \end{cases}$$

for $\lambda > 0$, where $\mathcal{K}_\nu$ is the modified Bessel function of the second kind. In particular, for $x < a$, we have

$$\mathbb{Q}_x^{(\nu)}(K^{(a)} \in dt) \quad = \quad \sum_{k=1}^{\infty} \frac{x^{-\nu} j_{\nu,k}\mathcal{J}_\nu(j_{\nu,k}\frac{x}{a})}{a^{2-\nu}\mathcal{J}_{\nu+1}(j_{\nu,k})} e^{-j_{\nu,k}^2 t/2a^2}\, dt \tag{3.30}$$

where $(j_{\nu,k})_{k\geq 1}$ is the ordered increasing sequence of the zeros of Bessel function of the first kind $\mathcal{J}_\nu$, see e.g. [17]. We shall now characterize the distribution of $K^{(a+b\cdot)}$ in terms of its Laplace transform and compute its density in the case $x < a$.

Theorem 3.5.3 *For $\lambda > 0$, we have*

$$\mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}K^{(a+b\cdot)}}\mathbb{I}_{\{K^{(a+b\cdot)}<\infty\}}\right] = \begin{cases} C\int_0^\infty \frac{\mathcal{I}_\nu(x\sqrt{2u})}{\mathcal{I}_\nu(a\sqrt{2u})} p_{b/2a}^\nu(\bar{x}, u)\, du, x \leq a, \\ C\int_0^\infty \frac{\mathcal{K}_\nu(x\sqrt{2u})}{\mathcal{K}_\nu(a\sqrt{2u})} p_{b/2a}^\nu(\bar{x}, u)\, du, x \geq a, \end{cases}$$

where $C = e^{-\frac{b}{2a}(a^2-x^2)}\frac{x^{-\nu}}{a^{-\nu}}$ *and* $\bar{x} = (\lambda^2+b^2)/2$. *In particular, for* $x < a$, *we have*

$$\frac{\mathbb{Q}_x^{(\nu)}(K^{(a+b\cdot)} \in dt)}{dt} = \frac{e^{\frac{b}{2a}(a^2-x^2)+\frac{b^2}{2}t}}{(1+\frac{b}{a}t)^{\nu+2}} \sum_{k=1}^{\infty} \frac{x^{-\nu} j_k \mathcal{J}_\nu(j_k \frac{x}{a})}{a^{2-\nu}\mathcal{J}_{\nu+1}(j_k)} e^{-\frac{j_k^2 t}{2a(a+bt)}} \tag{3.31}$$

where $j_k = j_{\nu,k}$.

Proof. A reformulation of the second identity (3.21) for Bessel processes provides the identity

$$K^{(a+b\cdot)} = \frac{\hat{H}_a^{(b/a)}}{1+\frac{b}{a}\hat{H}_a^{(b/a)}} \quad a.s.$$

where $\beta = b/a$. That allows us to write

$$\begin{aligned}
& \mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}K^{(a+b\cdot)}}\mathbb{I}_{\{K^{(a+b\cdot)}<\zeta^{(-\frac{b}{a})}\}}\right] \\
= \; & \mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}\frac{\hat{H}_a^{(b/a)}}{1+\frac{b}{a}\hat{H}_a^{(b/a)}}}\mathbb{I}_{\{\hat{H}_a^{(b/a)}<\zeta^{(-\frac{b}{a})}\}}\right] \\
= \; & \mathbb{E}_x\left[e^{-\frac{\lambda^2}{2}\frac{K^{(a)}}{1+\frac{b}{a}K^{(a)}}}(1+\frac{b}{a}K^{(a)})^{-\delta/2}e^{\frac{b}{2a}\left(\frac{a^2}{1+\frac{b}{a}K^{(a)}}-x^2\right)}\mathbb{I}_{\{K^{(a)}<\zeta^{(-\frac{b}{a})}\}}\right] \\
= \; & e^{\frac{b}{2a}(a^2-x^2)}\mathbb{E}_x\left[e^{-\frac{\bar{x}K^{(a)}}{1+\frac{b}{a}K^{(a)}}}(1+\frac{b}{a}K^{(a)})^{-\delta/2}\mathbb{I}_{\{K^{(a)}<\zeta^{(-\frac{b}{a})}\}}\right] \\
= \; & e^{\frac{b}{2a}(a^2-x^2)}\int_0^\infty p^\nu_{b/2a}(\bar{x},u)\mathbb{E}_x\left[e^{-uK^{(a)}}\mathbb{I}_{\{K^{(a)}<\zeta^{(-\frac{b}{a})}\}}\right]du
\end{aligned}$$

where we used identity (3.26). We conclude by using (3.30) to get the first assertion. Relation (3.31) is a consequence of the combination of Theorem 3.5.2 and (3.30). □

Remark 3.5.4 *The process* $R^{(\nu),b} = (R_t^{(\nu)} + bt, t \geq 0)$ *is easily seen to be inhomogeneous since it solves the equation*

$$R_t^{(\nu),b} = B_t + \frac{\delta-1}{2}\int_0^t \frac{ds}{R_s^{(\nu),b} - bs} + bt.$$

The reader should not confuse it with what is called a Bessel with a "naive" drift b, introduced in [123] and defined as the solution to

$$Q_t^{(\nu)} = B_t + \frac{\delta - 1}{2} \int_0^t \frac{ds}{Q_s^{(\nu)}} + bt.$$

Remark 3.5.5 *By setting $\delta = 1$ we are lead to the reflected Brownian motion. Amongst the consequences, we see that our analysis extends to the first passage time of the double barrier $(x \pm f(s), s \geq 0)$, i.e. $\inf\{s \geq 0; B_s = x \pm f(s)\}$, and the same kind of results prevails.*

Remark 3.5.6 *We mention that like for the Brownian motion case, the Mellin transform of the first passage time of Bessel processes to the square root boundary has been expressed by [29] in terms of Hypergeometric function. When the process starts below the curve, the density can be expressed as a series expansion in terms of the zeros of this function.*

3.6 Survey of Known Methods

Several methods appeared in the literature to solve the mentioned first passage time problem in some specific cases. We collect below the most significant ones and refer to Hobson et al. [55] for a similar survey.

3.6.1 Girsanov's Approach

The first method we describe below was formalized in the general setting by Salminen [107] but was previously used by Novikov [85] for asymptotic results and by Groeneboom [49] for the curve $f(t) = ct^2 + b$, c and b positive real numbers and $t \geq 0$. Assuming that $f \in C^2\left(\mathbb{R}_0^+, \mathbb{R}\right)$ such that $f(0) \neq 0$ then the law $\mathbb{P}_x^f$ of the process B^f, where $B_t^f = B_t - f(t) = B_t + f(0) - \int_0^t f'(s)\, ds$, for a fixed $t \geq 0$, is absolutely continuous with respect to the law of B denoted by $\mathbb{P}_x$. The Radon-Nikodym derivative being the martingale

$$M_t^{(f)} = \frac{d\mathbb{P}^f_{x+f(0)}}{d\mathbb{P}_{x+f(0)}}|_{\mathcal{F}_t} = \exp\big(-\int_0^t f'(s)dB_s - \frac{1}{2}\int_0^t f'(s)^2 ds\big). \quad (3.32)$$

A careful application of Doob's optional stopping Theorem combined with a device borrowed from [47] and [7], based on Williams' time reversal result, yields the following identity which is a slightly modification of [107, Theorem 2.1]

$$\frac{\mathbb{P}_x(T^{(f)} \in dt)}{\mathbb{P}_{x+f(0)}(T_0 \in dt)} = e^{f'(0)(x+f(0)) - \frac{1}{2}\int_0^t f'^2(s)ds} \mathbb{E}_{x+f(0)}\left[e^{\int_0^t r_s f''(s)\, ds}\right], \tag{3.33}$$

valid for $t \geq 0$, where we recall that r is a 3-dimensional Bessel bridge over the interval $[0,t]$ between $x + f(0)$ and 0.

We proceed by exploiting the absolute continuity relationship between $\mathbb{P}_x^{(\phi)}$, the law of the $GMOU^{(\phi)}$ process, and the Wiener measure in order to connect their first passage time distributions. Let r be a δ-dimensional Bessel bridge over $[0,t]$ between $x > 0$ and $z > 0$. We denote by $\mathbb{Q}_{x\to z}^{(\delta)}$ its law. To a given $\mu \in Mr^+$, we associate the quantity

$$h_x^{\delta,\mu}(y,t) = \mathbb{E}_{x\to 0}\left[e^{-\frac{1}{2}\int_0^t (r_s+y)^2 \mu(ds)}\right] \tag{3.34}$$

where $t > 0$ and $y \in \mathbb{R}$ and write simply $h^\mu = h^{3,\mu}$. In order to simplify notation, we assume for the rest of this Section that $\phi \in A_\infty^{-2,b}$. Now, we are ready to state the following.

Proposition 3.6.1 *For $x > y > 0$, we have*

$$\mathbb{P}_x^{(\phi)}(H_y^{(\phi)} \in dt) = \left(\frac{1}{\phi(t)}\right)^{1/2} e^{\frac{1}{2}\left(\frac{\phi'(t)}{\phi(t)}y^2 - \phi'(0)x^2\right)} h_{x-y}^{\phi''/\phi}(y,t)\, \mathbb{P}_x(T_y \in dt). \tag{3.35}$$

Proof. In remark 3.2.8 we obtained

$$d\mathbb{P}_{x|\mathcal{F}_t}^{(\phi)} = \frac{M_t^{(\phi)}(B)}{M_0^{(\phi)}(x)}\, d\mathbb{P}_{x|\mathcal{F}_t}, \quad t < \zeta^{(\phi)}.$$

Next, set

$$m(t) = \sqrt{\frac{\phi(0)}{\phi(t)}} e^{\frac{1}{2}\left\{\frac{\phi'(t)}{\phi(t)}y^2 - \frac{\phi'(0)}{\phi(0)}x^2\right\}}.$$

Then, Doob's optional stopping Theorem allows to get

$$\begin{aligned}\mathbb{P}_x^{(\phi)}(H_y^{(\phi)} \in dt) &= m(t)\,\mathbb{E}_x\left[e^{-\frac{1}{2}\int_0^t B_s^2 \frac{\phi''(s)}{\phi(s)}\,ds}, T_y \in dt\right] \\ &= m(t)\,\mathbb{E}_x\left[e^{-\frac{1}{2}\int_0^t B_s^2 \frac{\phi''(s)}{\phi(s)}ds} | T_y = t\right]\mathbb{P}_x(T_y \in dt).\end{aligned}$$

We conclude by following a line of reasoning similar to the proof of Theorem 1.5.1. □

We do not know how to compute explicitly the quantity (3.34) except for the particular value $y = 0$, see Pitman and Yor [97] or Remark 3.3.7 for a simple proof. The unpleasant presence of y in the expression forbids us to use the additive property possessed by the square Bessel processes. It breaks down the hope to extend the technique developed in [97]. In fact, we have a better understanding of this by switching to a generalized squared radial Ornstein-Uhlenbeck process with the help of a probability change of measure.

Proposition 3.6.2 *Set $F^{(\phi)} = \phi'/\phi$ and let x, $z > 0$ and $y \in \mathbb{R}$. We have*

$$h_{x,z}^{\delta,\mu}(y,t) = e^{-\frac{1}{2}\left\{F^{(\phi)}(t)(z+y)^2 - F^{(\phi)}(0)(x+y)^2 - \delta\log\frac{\phi(t)}{\phi(0)}\right\}}\frac{\mathbb{Q}_x^{(\delta,\phi)}(R_t \in dz)}{\mathbb{Q}_x^{(\delta,0)}(R_t \in dz)}$$

where R, under the probability $\mathbb{Q}_x^{(\delta,\phi)}$, satisfies the integral equation

$$R_t = x + B_t + y\log\phi(t) + \frac{\delta-1}{2}\int_0^t \frac{ds}{R_s} + \int_0^t F^{(\phi)}(s)R_s\,ds. \quad (3.36)$$

Proof. We easily check by Itô's formula that the process $(N^{(\phi)}, t \geq 0)$ defined, for a fixed $t \geq 0$, by

$$N_t^{(\phi)}(y) = e^{\frac{1}{2}\left[F^{(\mu)}(t)(R_t+y)^2 - F^{(\mu)}(0)(x+y)^2 - \delta\log\phi(t)\right] - \frac{1}{2}\int_0^t (R_s+y)^2\mu(ds)} \quad (3.37)$$

is a $\mathbb{P}$-martingale. By Girsanov's Theorem, under the probability measure $\mathbb{Q}^{(\delta,\phi)} = N^{(\phi)}\mathbb{P}$, the process R satisfies (3.36). Our assertion follows. □

3.6.2 Standard Method of Images

On $\{(x,t) \in \mathbb{R} \times \mathbb{R}^+;\ x \leq f(t)\}$ set $h(x,t)dx = \mathbb{P}\left(T^{(f)} > t, B_t \in dx\right)$ and observe that the space-time function h is the unique solution to the heat equation $\frac{\partial}{\partial t}h = \frac{1}{2}\frac{\partial^2}{\partial x^2}$ with boundary conditions

$$h\left(f(t),t\right) = 0, \quad h(.,0) = \delta_0(.) \quad \text{on} \quad]-\infty, f(0)] \tag{3.38}$$

where δ_0 stands for the Dirac function at 0. The standard method of images assumes the knowledge of h which admits the following representation, for some $a > 0$

$$h(x,t) = \frac{1}{\sqrt{t}}\eta\left(\frac{x}{\sqrt{t}}\right) - \frac{1}{a}\int_0^\infty \frac{1}{\sqrt{t}}\eta\left(\frac{x-s}{\sqrt{t}}\right)F(ds) \tag{3.39}$$

where $\eta(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ and $F(ds)$ is some positive, σ-finite measure with $\int_0^\infty \eta(\sqrt{\epsilon s})F(ds) < \infty$ for all $\epsilon > 0$. In Lerche [76, p.21], it is shown that if f is the unique root of the equation $h(x,t) = 0$ for t fixed and x unknown then we have

$$\mathbb{P}\left(T^{(f)} \leq t\right) = 1 - \eta\left(\frac{f(t)}{\sqrt{t}}\right) + \frac{1}{a}\int_0^\infty \eta\left(\frac{f(t)-s}{\sqrt{t}}\right)F(ds), \quad t > 0. \tag{3.40}$$

It is a challenging task to find further probabilistic links relating $F(ds)$ to f such as the following one. If $f \geq 0$ and $f(0) > 0$ we have

$$\int_0^\infty F(ds)e^{-\lambda s} = \mathbb{E}\left[e^{-\lambda f(T^{(f)}) - \frac{\lambda^2}{2}T^{(f)}}\right]$$

which easily seen to hold true, see [2]. Also, because f satisfies $h(f(.),.) = 0$, we see that

$$\int_0^\infty F(ds)e^{-\frac{s^2}{2t} + s\frac{f(t)}{t}} = a, \quad t > 0. \tag{3.41}$$

The drawback of this method is that the collected class of curves which can be treated using this tool must satisfy some criterions such as the concavity, see [76]. Finally, the above method extends to the case where the support of $F(ds)$ is any subset of $\mathbb{R}$. However, the corresponding boundary problem may be a two-sided one. As an instructive check we show that our method agrees with the method of images.

Proposition 3.6.3 *For a fixed $\beta > 0$ let $h^{(\beta)}$ be defined by (3.39) where $F(ds)$ is replaced by $F(ds)e^{-\beta s^2/2}$. Then, for a fixed $t > 0$, $f^{(\beta)}$ is the*

unique solution to $h^{(\beta)}(x,t) = 0$. *In other words, equation (3.40) is in agreement with Theorem 3.4.1 when* $F(ds)$ *and* f *are respectively replaced by* $F(ds)e^{-\beta s^2/2}$ *and* $f^{(\beta)}$.

Proof. The first assertion can easily be proved using (3.41) where t is replaced by $t/(1+\beta t)$, for $t > 0$. The second one is checked via some easy computations. □

3.6.3 Durbin's Approach

In [37], Durbin showed that, in the absolute continuous case, the problem reduces to the computation of a conditional expectation. That is, assuming that f is continuously differentiable and $f(0) \neq 0$ then, for any $t > 0$, holds the relationship

$$\mathbb{P}\left(T^{(f)} \in dt\right) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{f^2(t)}{2t}} h(t)dt \tag{3.42}$$

where

$$h(t) = \lim_{s \nearrow t} \frac{1}{t-s} \mathbb{E}\left[B_s - f(s); T^{(f)} > s \middle| B_t = f(t)\right].$$

There seems to be no-known way to compute the function h. Alternatively, another probabilistic representation for h, found in [2], says that

$$h(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{P}\left(T^{(f)} > t \middle| B_t = f(t) - \varepsilon\right).$$

This approach is compared to the standard method of images in [38].

Kendall [64] shows an intuitive interpretation involving the family of local times of B at f denoted by $l_\cdot^{B=f}$. Note that (3.42) may be written as $\mathbb{P}\left(T^{(f)} \in dt\right) = h(t)\mathbb{E}[dl_t^{B=f}]$, $t \geq 0$. Hence, integrating over $[0,a]$ yields

$$\mathbb{E}\left[\int_0^a h(s) dl_s^{B=f}\right] = \mathbb{P}\left(T^{(f)} < a\right).$$

Now, if $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ solves the equation

$$\mathbb{E}\left[\int_t^a g(s,a) dl_s^{B=f} \middle| B_t = f(t)\right] = 1, \quad 0 < t < a,$$

then we easily check that

$$\mathbb{E}\left[\int_0^a g(s,a)dl_s^{B=f}\right] = \mathbb{P}\left(T^{(f)} < a\right).$$

holds true as well. However, it is not clear how to express g in terms of h.

3.6.4 An Integral Representation

We end up this section with some integral equations satisfied by the density in the absolute continuous case. First, observe that if f is positive and does not vanish then B, when started at $B_0 = x > f(0)$, must hit f before 0. The strong Markov property gives then birth to

$$\frac{x}{\sqrt{2\pi t^3}}e^{-\frac{x^2}{2t}} = \int_0^t \mathbb{P}_x(T^{(f)} \in dr)\frac{f(r)}{\sqrt{2\pi(t-r)}}e^{-\frac{f(r)^2}{2(t-r)}}, \quad t > 0. \quad (3.43)$$

The above conditions on f can be relaxed leading to the conclusion that (3.43) holds for a larger class of curves, see [76]. Amongst other integral equations we quote the following one. Assuming that f is differentiable, we have, for $t \geq 0$,

$$\frac{\mathbb{P}_x(T^{(f)} \in dt)}{dt} = \frac{f(t)}{\sqrt{2\pi t^3}}e^{-\frac{f(t)^2}{2t}} - \int_0^t \frac{n_t(u)^{1/2}}{\sqrt{\pi}(t-u)}e^{-n_t(u)}\,\mathbb{P}_x\left(T^{(f)} \in du\right)$$

where $n_t(u) = \frac{(f(t)-f(u))^2}{2(t-u)}$. For these and other well-known integral equations we refer to [37], [41] and [76], for some new ones we refer to [93] and [27].

Chapter 4

On the First Passage Times of Generalized Ornstein-Uhlenbeck Processes

Поэтому мне хочется подчеркнуть законность и достоинтство позиции математика, понимающего место и роль науки в развитии естественных наук, техники, да и всей человеческой культуры, но спокойно продожающего развивать "чистую математику"в соответствии с внутренней логикой еч развития.

А.Н. Колмогоров

(I would like to emphasize the legitimacy and dignity of the position of a mathematician who understands the place and role of his science in the development of the natural sciences, technology, and the entire human culture, but calmly continues developing "pure mathematics" according to the inner logic of its development.

A.N. Kolmogorov)

4.1 Introduction

Let $Z := (Z_t,\ t \geq 0)$ be a spectrally negative Lévy process starting from 0 given on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ where the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions. For any $\lambda > 0$, we define a generalized Ornstein-Uhlenbeck (for short GOU) process $X := (X_t,\ t \geq 0)$, starting from $x \in \mathbb{R}$, with backward driven Lévy process (for short BDLP) Z as the unique solution to the following stochastic differential equation

$$dX_t = -\lambda X_t\, dt + dZ_t, \quad X_0 = x. \tag{4.1}$$

These are a generalization of the classical Ornstein-Uhlenbeck process constructed by simply replacing the driving Brownian motion with a Lévy process. In this Chapter we are concerned with the positive random variables H_a and the functional I_t defined by

$$H_a = \inf\{s \geq 0;\ X_s > a\} \qquad \text{and} \qquad I_t = \int_0^t X_s\, ds \tag{4.2}$$

respectively. The Laplace transform of H_a is known from Hadjiev [51]. There is an important literature regarding the distribution of additive functionals, stopped at certain random times, of diffusion processes, see for instance the book of [17] for a collection of explicit results. However, the law of such functionals for Markov processes with jumps are not known except in some special cases (e.g. the exponential functional of some Lévy processes, see [20] and the Hilbert transform of Lévy processes see [43] and [10]). The explicit form of the joint distribution (H_a, I_{H_a}), when X is the classical Ornstein-Uhlenbeck, is given by Lachal [70]. Here, the author exploits the fact that the bivariate process $(X_t, I_t, t \geq 0)$ is a Markov process. We shall extend his result by providing the Laplace transform of this two-dimensional distribution in the general case, i.e. when X is a GOU process as defined above. We recall that first passage time problems for Markov processes are closely related to the finding of an appropriate martingale associated to the process. We shall provide a methodology which allows us to build up the martingale used to compute the joint Laplace transform we are looking for. In a second step we shall combine martingales and Markovian techniques to derive the Laplace-Fourier transform.

GOU processes have found many applications in several fields. They are widely used in finance today to model the stochastic volatility of a

stock price process (see e.g. [9]) and for describing the dynamics of the instantaneous interest rate. The latter application, as a generalization of the Vasicek model, deserves particular attention, as these processes belong to the class of one factor affine term structure model. These are well known to be tractable, in the sense that it is easy to fit the entire yield curve by basically solving Riccati equations, see Duffie et al. [35] for a survey on affine processes. From the expression of the joint Laplace transform of (H_a, I_{H_a}), we provide an analytical formula for the price of a European call option on maximum on yields in the framework of GOU processes.

This Chapter is organized as follows. In Section 2, we recall some facts about spectrally negative Lévy and GOU processes and their first passage times above a constant level. In Section 3, we give an explicit form for the joint Laplace transform (H_a, I_{H_a}) in terms of new special functions. Sections 4 and 5 are devoted to some special cases. First, we study the stable OU processes, that is when Z is a stable process. Then, we consider the two sided case for which the positive jumps part is a compound Poisson process whose jumps are exponentially distributed. In the last Section, we apply the previous results to the pricing of a path-dependent option on yields with a more detailed study of the stable Vasicek case.

4.2 Preliminaries

4.2.1 Lévy Processes

Unless stated, throughout the rest of this Chapter $Z := (Z_t,\ t \geq 0)$ denotes a real-valued spectrally negative Lévy process starting from 0. It is a process with stationary and independent increments, whose Lévy measure ν charges only the negative real line ($\nu((0,\infty)) = 0$). Due to the absence of positive jumps, it is possible to extend analytically the characteristic function of Z to the negative imaginary line. Thus, one characterizes this process by its so-called Laplace exponent $\psi : [0,\infty) \to (-\infty,\infty)$ which is specified by the identity

$$\mathbb{E}\left[e^{uZ_t}\right] = e^{t\psi(u)}, \qquad t, u \geq 0,$$

and has the form

$$\psi(u) = bu + \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{0} (e^{ur} - 1 - ur\chi(r))\nu(dr)$$

where $\chi(r) := \mathbb{I}_{\{r>-1\}}$, $b \in \mathbb{R}$, $\sigma \geq 0$ and $\nu(.)$ is the Lévy measure on $(-\infty, 0]$ which satisfies the integrability condition $\int_{-\infty}^{0} (1 \wedge x^2)\, \nu(dx) < \infty$. It is known that ψ is a convex function with $\lim_{u\to\infty} \psi(u) = +\infty$. We assume that the process X does not drift to $-\infty$, which is the case when $\psi'(0^+)$ is non-negative, see Bertoin [11, Chapter VII] for a thorough description of these processes.
We introduce the first passage time process $T := (T_a, a \geq 0)$ defined, for a fixed $a \geq 0$, by $T_a = \inf\{s \geq 0;\ Z_s > a\}$. Denoting by ϕ the inverse function of ψ, the Laplace exponent of T is given by, see [11, Theorem VII.1],

$$\mathbb{E}\left[e^{-uT_a}\mathbb{I}_{\{T_a<\infty\}}\right] = e^{-a\phi(u)}, \quad u \geq 0. \tag{4.3}$$

4.2.2 GOU Processes

In this Section, we review some well-known facts concerning GOU processes and for the sake of completeness provide their proofs. By a technique of variation of constants, the solution of (4.1) can be written in terms of Z as follows

$$X_t = e^{-\lambda t}\left(x + \int_0^t e^{\lambda s}\, dZ_s\right), \quad t \geq 0. \tag{4.4}$$

From this representation, it is an easy exercise to derive the Laplace exponent of X, see Hadjiev [51].

Proposition 4.2.1 *For $u \geq 0$, we have*

$$\mathbb{E}_x\left[e^{uX_t}\right] = \exp\left(e^{-\lambda t}xu + \int_0^t \psi(e^{-\lambda r}u)\, dr\right)$$

where $\mathbb{E}_x$ is the expectation operator with respect to $\mathbb{P}_x$, the law of the process starting from x.

Proof. We consider an arbitrary subdivision $0 = t_0 < \ldots < t_n = t$ and introduce $\epsilon = \max_{i\leq n} |t_i - t_{i-1}|$. Let g be a bounded deterministic

function. Using the independency and the stationarity of the increments of the Lévy process Z, we get, for $u \geq 0$

$$\begin{aligned}
\mathbb{E}\left[\exp\left(u\int_0^t g(s)\,dZ_s\right)\right] &= \lim_{\epsilon\to 0}\prod_{i=1}^{n}\mathbb{E}\left[\exp\left(ug(t_i)(Z_{t_i}-Z_{t_{i-1}})\right)\right] \\
&= \lim_{\epsilon\to 0}\prod_{i=1}^{n}\mathbb{E}\left[\exp\left(ug(t_i)Z_{t_i-t_{i-1}}\right)\right] \\
&= \lim_{\epsilon\to 0}\prod_{i=1}^{n}\mathbb{E}\left[\exp\left(\psi\left(ug(t_i)\right)(t_i-t_{i-1})\right)\right] \\
&= \exp\left(-\int_0^t \psi(ug(r))\,dr\right).
\end{aligned}$$

Finally, choosing $g(t) = e^{-\lambda t}$, the statement follows. □

From the representation (4.4), we get that $X_t \longrightarrow \int_0^\infty e^{-\lambda s}\,dZ_s$ a.s. as t tends to ∞. Consequently, the Laplace transform of the limiting distribution of X, denoted by $\widehat{\rho}^X(u)$, $u \geq 0$, is given by

$$\widehat{\rho}^X(u) = \exp\left(\int_0^\infty \psi(e^{-\lambda r}u)\,dr\right),$$

whenever the Lévy measure satisfies the condition

$$\int_{r<-1} \log|r|\,\nu(dr) < \infty, \tag{4.5}$$

see Sato [108, Chapter III]. We assume that this condition holds throughout this Chapter. A nice feature, for both practice and theory, is the fact that ρ is a selfdecomposable distribution. Conversely, any selfdecomposable distribution can be viewed as the limiting distribution of a GOU process. For interesting papers on this relationship and applications, we refer to Jeanblanc et al. [60], Jurek [61] and Sato [108] and the references therein.

The process X is a Feller process. Its infinitesimal generator $\mathcal{A}$ is an integro-differential operator acting on $\mathcal{C}_c^2(\mathbb{R})$, the space of twice continuously differentiable functions with compact support. It is defined

by

$$\begin{aligned}\mathcal{A}f(x) &= \frac{1}{2}\sigma^2 f''(x) + (b-\lambda x)f'(x) + \\ &\quad \int_{-\infty}^{0} \left(f(x+r) - f(x) - f'(x)r\chi(r)\right)\nu(dr).\end{aligned}$$

To complete the description, we mention that X is a special semimartingale with triplet of predictable characteristics given by

$$\left(bt - \lambda\int_0^t X_s\,ds, \frac{1}{2}\sigma^2 t, \nu(dr)dt\right). \tag{4.6}$$

Next, we deal with the Laplace transform of the first passage time of a fixed level $y \geq x$ of the GOU process which appeared in Hadjiev [51] and Novikov [86]. For sake of completeness, we give a detail proof of this result and we follow the approach of Novikov [86]. We construct an exponential family of martingales and we estimate the Laplace transform by applying Doob's optional stopping Theorem. Before stating the main result of this Section, we set up some notation and give some Lemmas. We define the function φ by

$$\varphi(u) = \frac{1}{\lambda}\int_0^u \frac{\psi(r)}{r}\,dr, \quad u \geq 0,$$

and decompose it as follows

$$\varphi(u) = \frac{1}{\lambda}\left(mu + \frac{\sigma^2}{2}u^2 + I_1(u) + I_2(u)\right)$$

where

$$I_1(u) = \int_0^\infty \int_0^u r^{-1}(e^{rw} - rwI_{\{w>-1\}} - 1)I_{\{w>-1\}}\,dr\,\nu(dw),$$

and

$$\begin{aligned}I_2(u) &= \int_0^\infty \int_0^u r^{-1}(e^{rw} - 1)I_{\{w\geq -1\}}\,dr\,\nu(dw) \\ &= A - \int_0^\infty \left[\log(u) + \int_{-wu}^\infty r^{-1}e^{-r}\,dr\right] I_{\{w\geq -1\}}\,\nu(dw),\end{aligned}$$

with $A = \int_0^\infty [e_\gamma + \log(-w)]\,I_{\{w\geq -1\}}\,\nu(dw)$ and e_γ is the Euler constant, see [72].

Remark 4.2.2 *Note that the Laplace transform of X can be expressed in terms of the function φ. Indeed, for $u \geq 0$, we have*

$$\mathbb{E}_x\left[e^{uX_t}\right] = \exp\left(e^{-\lambda t}xu + \varphi(u) - \varphi(ue^{-\lambda t})\right).$$

Moreover, we have the identity $\varphi(u) = \log(\hat{\rho}^X(u))$.

Lemma 4.2.3 *If*

$$\sigma > 0 \text{ or } \int_0^\infty |w| I_{\{-1<w<0\}}\, \nu(dw) = \infty, \tag{4.7}$$

then

$$\lim_{u\to\infty} \frac{\varphi(u)}{u} = \infty. \tag{4.8}$$

If

$$\sigma = 0 \text{ or } \int_0^\infty |w| I_{\{-1<w<0\}}\, \nu(dw) < \infty, \tag{4.9}$$

then

$$\lim_{u\to\infty} \frac{\varphi(u)}{u} = \frac{1}{\lambda}\left(\int_0^\infty |w| I_{\{-1<w<0\}}\, \nu(dw)\right). \tag{4.10}$$

Proof. It is clear, from the condition (4.5), that the integral I_1 is finite. For I_2 we use the following asymptotic result, as u tends to ∞

$$I_2(u) = -\log(u)\nu((-\infty,-1]) + \frac{1}{\lambda}\left(\int_0^\infty |w| I_{\{-1<w<0\}}\, \nu(dw)\right) + O(1).$$

That is $I_2(u) \asymp O(log(u))$. By the inequality $e^w - wI_{\{|w|<1\}} - 1 \geq \frac{w^2}{2} I_{\{w<0\}}$, we find that

$$I_1(u) \geq \frac{u^2}{4}\int_0^\infty w^2 I_{\{w>0\}}\, \nu(dw) + O(1).$$

Hence,

$$\lambda\varphi(u) \geq mu + \frac{u^2}{4}\left(\sigma^2 + \int_0^\infty w^2 I_{\{w>0\}}\, \nu(dw)\right) + O(log(u)).$$

Thus if $\sigma > 0$ we obtain the limit (4.8). Now, we assume that condition (4.9) holds. Then,

$$\lambda\varphi(u) = mu + \int_0^\infty \left[\int_0^u \frac{e^{rw} - rw - 1}{r}\, dr\right] I_{\{-1<w<0\}}\, \nu(dw) + I_2(u).$$

Note that for $x < 0$ and $v > 0$ the following inequalities hold

$$0 \leq \frac{e^{rw} - rw - 1}{r} \leq -w.$$

Taking into account the assumption $\int_0^\infty |w| I_{\{-1<w<0\}} \nu(dw) < \infty$, using the dominated convergence Theorem and the l'Hospital rule, we find

$$\lim_{u\to\infty} \frac{1}{u} \int_0^\infty \left[\int_0^u \frac{e^{rw} - rw - 1}{r} dr\right] I_{\{-1<w<0\}}\, \nu(dw) = \int_0^\infty |w| I_{\{-1<w<0\}}\, \nu(dw).$$

(4.10) follows.

If $\int_0^\infty |w| I_{\{-1<w<0\}}\, \nu(dw) = \infty$, a similar argument leads to the following estimate, with any $\epsilon > 0$

$$\lim_{u\to\infty} \frac{\varphi(u)}{u} \geq \frac{1}{\lambda}\left(m + \int_0^\infty |w| I_{\{-1<w<-\epsilon\}}\, \nu(dw)\right).$$

Letting $\epsilon \to 0$, we obtain (4.8). □

Fix $a > x$. For the remainder of the Chapter, we shall impose the following condition.

Assumption 1

$$\text{Either } \sigma > 0 \quad \text{or} \quad \int_{-1}^0 r\nu(dr) = \infty \quad \text{or} \quad b - \int_{-1}^0 r\nu(dr) > \lambda a.$$

We proceed by introducing the following function, for $x \in \mathbb{R}$,

$$\mathcal{H}_\nu(x) \quad = \quad \int_0^\infty \exp\left(xr - \varphi(r)\right) r^{\nu-1} dr.$$

Theorem 4.2.4 *For any $\gamma > 0$, the process $\left(e^{-\gamma t}\mathcal{H}_{\frac{\gamma}{\lambda}}(X_t),\ t \geq 0\right)$ is a martingale.*

Proof. The martingale property follows from an application of Fubini's Theorem, justified by Lemma 4.2.3, together with remark 4.2.2. □

Finally, we derive the Laplace transform of H_a.

Proposition 4.2.5 *For $a > x$, and $\gamma > 0$, we have*

$$\mathbb{E}_x\left[e^{-\gamma H_a}\right] = \frac{\mathcal{H}_{\frac{\gamma}{\lambda}}(x)}{\mathcal{H}_{\frac{\gamma}{\lambda}}(a)}.$$

Proof. It is a straightforward application of Doob's optional stopping Theorem to the bounded stopping time $H_a \wedge t$. The passage to the limit $t \to \infty$ is justified by Lemma 4.2.3 and by dominated convergence. □

We end up this Section with the following limit result.

Proposition 4.2.6 *Let $x, a \in \mathbb{R}$ and $\gamma > 0$, then we have*

$$\lim_{\lambda \to 0} \frac{\mathcal{H}_{\frac{\gamma}{\lambda}}(x)}{\mathcal{H}_{\frac{\gamma}{\lambda}}(a)} = e^{-(x-a)\phi(\gamma)}.$$

Proof. We can rewrite $\mathcal{H}$ by considering the following change of variable $r = \phi(s)$ and denoting $z = \lambda^{-1}$

$$\begin{aligned}
\mathcal{H}_{z\gamma}(x) &= \int_0^\infty \exp\left(x\phi(r) - z\int_1^{\phi(r)} \psi(u)\,\frac{du}{u}\right)\phi(r)^{z\gamma-1}\phi'(r)dr \\
&= \int_0^\infty f_x(r)\exp\left(-zp(r;\gamma)\right)dr
\end{aligned}$$

where $f_x(r) = e^{x\phi(r)}\phi(r)^{-1}\phi'(r)$ and $p(t;\gamma) = \int_1^r \frac{\phi'(u)}{\phi(u)}u\,du - \gamma\log(\phi(r))$. We use the Laplace's method to derive an asymptotic approximation for large value of the parameter z, see [89, Theorem 2.1]. We get the following approximation

$$\mathcal{H}_{z\gamma}(x) \sim f_x(\gamma)e^{-zp(\gamma)}\left(\frac{2\pi}{xp''(\gamma)}\right)^{\frac{1}{2}} \quad \text{as } z \to \infty,$$

where $p'(t) = \frac{\phi'(t)}{\phi(t)}(t-\gamma)$ and $p''(\gamma) = \frac{\phi'(\gamma)}{\phi(\gamma)} \neq 0$. The result follows.

□

4.3 Study of the Law of (H_a, I_{H_a})

Our aim in this Section is to characterize the joint law of the couple $(H_a, \int_0^{H_a} X_s\, ds)$ through transform techniques. We shall start with computing the following joint Laplace transform, for any $x \leq a$,

$$\Lambda_a^{(\gamma,\theta)}(x) = \mathbb{E}_x\left[e^{-\gamma H_a + \theta I_{H_a}}\right], \quad \gamma, \theta > 0.$$

To this end, we introduce the GOU process, denoted by $X^{\frac{\theta}{\lambda}}$, with the triplet of predictable characteristics $\left(b't - \lambda\int_0^t X_s\, ds, \frac{\sigma^2}{2}t, e^{\frac{\theta}{\lambda}r}\nu(dr)dt\right)$ where $b' := b + \frac{\theta}{\lambda}\sigma^2 + \int_{-\infty}^{-1}(e^{\frac{\theta}{\lambda}r} - 1)r\nu(dr)$. Before stating our main result we note the following intermediate result.

Lemma 4.3.1 *For $\gamma, \theta > 0$ such that $\eta := \gamma - \psi(\frac{\theta}{\lambda}) > 0$, and $a > x$, we have*

$$\Lambda_a^{(\gamma,\theta)}(x) = e^{-\frac{\theta}{\lambda}(a-x)}\, \mathbb{E}_x\left[e^{-\eta\, H_a^{(\frac{\theta}{\lambda})}}\right]$$

where $H_a^{(\frac{\theta}{\lambda})} = \inf\left\{s \geq 0;\ X_s^{\frac{\theta}{\lambda}} > a\right\}$.

Remark 4.3.2 *We note that this Lemma can easily be extended to compute the joint law of the couple $(H_a, \int_0^{H_a} \Lambda(X_s)\, ds)$ where X is the solution to the SDE $dX_t = \Lambda(X_t)dt + dZ_t$, $X_0 = x < a$, and where $\Lambda(x)$ is any locally integrable function on $\mathbb{R}$ and $H_a = \inf\{s \geq 0;\ X_s > a\}$ such that a is regular for itself.*

Proof. Fix $a > x$. Exploiting the fact that X has non-positive jumps, we get

$$\int_0^{H_a} X_s\, ds = \frac{1}{\lambda}\left(Z_{H_a} + x - a\right)$$

which yields

$$\Lambda_a^{(\gamma,\theta)}(x) = e^{-\frac{\theta}{\lambda}(a-x)}\, \mathbb{E}_x\left[e^{-\gamma H_a + \frac{\theta}{\lambda} Z_{H_a}}\right].$$

We recall that $\mathcal{F}_t = \sigma(Z_s, s \leq t)$ denotes the natural filtration of Z up to time t. We now consider the Girsanov's transform $\mathbb{P}^{(\xi)}$ of the probability measure $\mathbb{P}$ which is defined by $d\mathbb{P}^{(\xi)}_{|\mathcal{F}_t} = \exp\left(\xi Z_t - t\psi(\xi)\right)\, d\mathbb{P}_{|\mathcal{F}_t}$, $t, \xi \geq 0$. Under $\mathbb{P}^{(\xi)}$, Z, denoted by $Z^{(\xi)}$, is again a Lévy process with the following Laplace exponent, for $u \geq 0$

$$\begin{aligned}
\psi^{(\xi)}(u) &:= \log\left(\mathbb{E}\left[e^{uZ_1^{(\xi)}}\right]\right) \\
&= \log\left(\mathbb{E}\left[e^{(u+\xi)Z_1}\right]\right) - \psi(\xi) \\
&= \left(b + \sigma^2\xi + \int_{-\infty}^{-1} (e^{\xi r} - 1) r\nu(dr)\right) u + \frac{1}{2}\sigma^2 u^2 \\
&\quad + \int_{-\infty}^{0} (e^{ur} - 1 - ur\chi(r)) e^{\xi r} \nu(dr).
\end{aligned}$$

By choosing $\xi = \frac{\theta}{\lambda}$ and using the representation (4.6), it is straightforward to deduce the triplet of predictable characteristics of the associated GOU process $X^{\frac{\theta}{\lambda}}$. We point out that $X^{\frac{\theta}{\lambda}}$ has again non-positive jumps, since the two probability measures are absolutely continuous. Finally, our relationship follows from the computations

$$\begin{aligned}
\Lambda_a^{(\gamma,\theta)}(x) &= e^{-\frac{\theta}{\lambda}(a-x)}\, \mathbb{E}_x\left[e^{-\gamma H_a + \frac{\theta}{\lambda} Z_{H_a}}\right] \\
&= e^{-\frac{\theta}{\lambda}(a-x)}\, \mathbb{E}_x\left[e^{-(\gamma - \psi(\frac{\theta}{\lambda}))H_a + \frac{\theta}{\lambda} Z_{H_a} - \psi(\frac{\theta}{\lambda}) H_a}\right] \\
&= e^{-\frac{\theta}{\lambda}(a-x)}\, \mathbb{E}_x\left[e^{-(\gamma - \psi(\frac{\theta}{\lambda}))H_a^{(\frac{\theta}{\lambda})}}\right].
\end{aligned}$$

□

Introduce the function φ_β defined by

$$\varphi_\beta(u) = \frac{1}{\lambda}\int_0^u \frac{\psi(r+\beta)}{r}\, dr, \quad u \geq 0.$$

We are now ready to state the main result of this Section.

Theorem 4.3.3 *For $\gamma, \theta > 0$ and $a > x$, we have*

$$\Lambda_a^{(\gamma,\theta)}(x) \quad = \quad e^{-\frac{\theta}{\lambda}(a-x)} \frac{\mathcal{H}_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(x)}{\mathcal{H}_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(a)},$$

where

$$\mathcal{H}_{\nu,\beta}(x) = \int_0^\infty \exp\left(xr - \varphi_\beta(r)\right) r^{\nu-1} dr.$$

Proof. Combining the results of the previous Lemma and Proposition 4.2.5, with the obvious notation, we obtain

$$\Lambda_a^{(\gamma,\theta)}(x) \quad = \quad e^{-\frac{\theta}{\lambda}(a-x)} \frac{\mathcal{H}_{\frac{\eta}{\lambda}}^{(\frac{\theta}{\lambda})}(x)}{\mathcal{H}_{\frac{\eta}{\lambda}}^{(\frac{\theta}{\lambda})}(a)}.$$

Next, since

$$\begin{aligned}
\mathcal{H}_{\frac{\eta}{\lambda}}^{(\frac{\theta}{\lambda})}(x) &= \int_0^\infty \exp\left(xr - \frac{1}{\lambda}\int_1^r \psi^{(\frac{\theta}{\lambda})}(v)\,\frac{dv}{v}\right) r^{\frac{\eta}{\lambda}-1} dr \\
&= \int_0^\infty \exp\left(xr - \frac{1}{\lambda}\int_1^r \psi(v+\frac{\theta}{\lambda})\,\frac{dv}{v}\right) r^{\frac{\gamma}{\lambda}-1} dr \\
&= \exp\left(\frac{1}{\lambda}\int_0^1 \psi(v+\frac{\theta}{\lambda})\,\frac{dv}{v}\right) \mathcal{H}_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(x),
\end{aligned}$$

we obtain the identity

$$\frac{\mathcal{H}_{\frac{\eta}{\lambda}}^{(\frac{\theta}{\lambda})}(x)}{\mathcal{H}_{\frac{\eta}{\lambda}}^{(\frac{\theta}{\lambda})}(a)} = \frac{\mathcal{H}_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(x)}{\mathcal{H}_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(a)}.$$

By using the convexity of ψ and the fact that $\lim_{u\to\infty}\psi(u) = +\infty$, we have for a fixed $\theta > 0$ and large u, $\psi(u+\theta) \geq \psi(u)$. Moreover, under the assumption 1, Novikov [86] shows that $\lim_{u\to\infty} u^{-1}\int_0^u \psi(r) r^{-1} dr = +\infty$. Therefore, by following a line of reasoning similar to [86, Theorem 2] the proof is completed. □

In Section 4.4, the special case with stable BDLP's is studied in detail. In what follows, we provide the Laplace-Fourier transform of the joint distribution. We first show the following Lemma.

Lemma 4.3.4 *The bivariate process $(X_t, I_t, t \geq 0)$ is a Markov process. Its infinitesimal generator is defined on $\mathcal{C}_c^{2,1}(\mathbb{R} \times \mathbb{R})$ by*

$$\begin{aligned}\mathcal{A}^* f(x,y) &= \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}(x,y) + (b - \lambda x)\frac{\partial f}{\partial x}(x,y) + x\frac{\partial f}{\partial y}(x,y) + \\ &\quad \int_{-\infty}^{0} \left(f(x+r,y) - f(x,y) - \frac{\partial f}{\partial x}(x,y) r\chi(r) \right) \nu(dr).\end{aligned}$$

Proof. We start by recalling that, although the additive functional I_t is not Markovian, the bivariate process $(I_t, X_t, t \geq 0)$ is a strong Markov process, see [16]. The second part of the Lemma is a consequence of Itô's formula. Indeed, for any function $f \in \mathcal{C}_c^{2,1}(\mathbb{R} \times \mathbb{R})$, we have

$$\begin{aligned}f(X_t, I_t) &= f(x,0) + \int_0^t \frac{\partial f}{\partial x}(X_{s-}, I_s)\, dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(X_s, I_s)\, d\langle X^c \rangle_s \\ &\quad + \sum_{0<s\leq t} f(X_s, I_s) - f(X_{s-}, I_s) - \frac{\partial f}{\partial x}(X_{s-}, I_s)\Delta X_s + \\ &\quad \int_0^t \frac{\partial f}{\partial y}(X_s, I_s)\, dI_s \\ &= f(x,0) - \lambda \int_0^t \frac{\partial f}{\partial x}(X_s, I_s) X_s\, ds + \int_0^t \frac{\partial f}{\partial x}(X_{s-}, I_s)\, dZ_s + \\ &\quad \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(X_s, I_s)\, d\langle Z^c \rangle_s + \int_0^t \frac{\partial f}{\partial y}(X_s, I_s) X_s\, ds + \\ &\quad \sum_{0<s\leq t} f(X_{s-} + \Delta Z_s, I_s) - f(X_{s-}, I_s) - \frac{\partial f}{\partial x}(X_{s-}, I_s)\Delta Z_s\end{aligned}$$

where X^c denotes the continuous martingale part of X. Finally, taking into consideration that $d\langle Z^c \rangle_s = \sigma^2 ds$, we obtain $\mathcal{A}^*$. □

Corollary 4.3.5 *For $\gamma, \theta, \lambda > 0$ and $a > x$, we have*

$$\Lambda_a^{(\gamma, i\theta)}(x) = \frac{\overline{\mathcal{H}}_{\frac{\gamma}{\lambda}, \frac{i\theta}{\lambda}}(x)}{\overline{\mathcal{H}}_{\frac{\gamma}{\lambda}, \frac{i\theta}{\lambda}}(a)}$$

where $\overline{\mathcal{H}}_{\nu,\beta}(x) = e^{\beta x}\mathcal{H}_{\nu,\beta}(x)$.

Proof. In order to simplify the notation in the proof we assume that $\sigma = 0$. We consider the process $M := (M_t,\ t \geq 0)$ defined, for a fixed $t \geq 0$, by

$$M_t = \exp\left(-\gamma t + i\theta \int_0^t X_s\, ds\right) \overline{\mathcal{H}}_{\frac{\gamma}{\lambda}, \frac{i\theta}{\lambda}}(X_t).$$

We shall prove that M is a complex martingale. From the integral representation (4.11), it follows that the function $\mathcal{H}_{\nu,\beta}(x)$ is analytic in the domain $\Re(\nu) > 0$, $\Re(\beta) > 0$, $x \in \mathbb{R}$. Set $u(t,x,y) := e^{-\gamma t + i(\theta y + \frac{\theta}{\lambda} x)}$, $g(x) := \mathcal{H}_{\frac{\gamma}{\lambda}, \frac{i\theta}{\lambda}}(x)$ and $f(t,x,y) := u(t,x,y)g(x)$. Thanks to the remark following Proposition 4.2.5, we see that g is a solution of the following integro-differential equation

$$\mathcal{A}^{(i\frac{\theta}{\lambda})} g(x) = (\gamma - \psi(i\frac{\theta}{\lambda}))g(x) \tag{4.11}$$

with

$$\begin{aligned} \mathcal{A}^{(\xi)} f(x) &= (\bar{b} - \lambda x) f'(x) + \\ &\quad \int_{-\infty}^0 \left(f(x+r) - f(x) - f'(x) r \chi(r)\right) e^{\xi r} \nu(dr) \end{aligned}$$

where we recall that $\bar{b} := b + \int_{-\infty}^0 (e^{\xi r} - 1) r \chi(r) \nu(dr)$. We observe that

$$\begin{aligned} \frac{\partial f}{\partial y}(t,x,y) &= i\theta u(t,x,y) g(x) \\ \frac{\partial f}{\partial x}(t,x,y) &= u(t,x,y)\left(i\frac{\theta}{\lambda} g(x) + g'(x)\right). \end{aligned}$$

By applying the change of variables formula for processes with finite

variation, we get

$$\begin{aligned}
df(t,X_t,I_t) &= \left(\frac{\partial f}{\partial t}(t,X_t,I_t) - \lambda X_t \frac{\partial f}{\partial x}(t,X_t,I_t) + \frac{\partial f}{\partial y}(t,X_t,I_t)\right) dt \\
&\quad + \int_{-\infty}^{0} f(x+r,y) - f(x,y) - \frac{\partial f}{\partial x}(x,y) r\nu(dr)dt \\
&\quad + \frac{\partial f}{\partial x}(t,X_{t-},I_t)dZ_t \\
&= u(t,x,y)\left(\left(b + \int_{-\infty}^{0}(e^{\frac{\theta}{\lambda}r}-1)r\chi(r)\nu(dr) - \lambda X_t\right) g'(x)\right. \\
&\quad \int_{-\infty}^{0} (g(x+r) - g(x) - g'(x)r\chi(r))\, e^{\frac{\theta}{\lambda}r}\nu(dr) + \\
&\quad \left.\left(-\gamma + ib\frac{\theta}{\lambda} + \int_{-\infty}^{0} e^{\frac{\theta}{\lambda}r} - 1 - i\frac{\theta}{\lambda} r\chi(r)\nu(dr)\right) g(x)\right) dt \\
&\quad + N_t,
\end{aligned}$$

where $(N_t,\ t \geq 0)$ is a $\mathcal{F}$-martingale. Consequently, by using the fact that g is a solution of the equation (4.11), we have shown that $(M_t,\ t \geq 0)$ is also a purely discontinuous martingale with respect to the natural filtration of X.
Next, we derive the following estimates, for any $t \geq 0$

$$\begin{aligned}
\mathbb{E}\left[|M_{H_a\wedge t}|\right] &\leq \mathbb{E}\left[|\mathcal{H}_{\frac{\gamma}{\lambda},\frac{i\theta}{\lambda}}(X_{H_a\wedge t})|\right] \\
&\leq \mathbb{E}\left[|\mathcal{H}_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(X_{H_a\wedge t})|\right] \\
&\leq \mathbb{E}\left[|\mathcal{H}_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(a)|\right] < \infty.
\end{aligned}$$

We complete the proof of the corollary by applying the Doob's optional sampling Theorem at the bounded stopping time $H_a \wedge t$ and the dominated convergence Theorem. □

In the sequel, we assume that the exponential moments of the BDLP Z are finite, that is

Assumption 2 $\int_{r<-1} e^{vr}\nu(dr) < \infty$ for every $v \in \mathbb{R}$.

Theorem 4.3.6 *For $\gamma, \theta, \lambda > 0$ and $a > x$ we have:*

$$\Lambda_a^{(\gamma,-\theta)}(x) \quad = \quad e^{\frac{\theta}{\lambda}(a-x)} \frac{\mathcal{H}_{\frac{\gamma}{\lambda},-\frac{\theta}{\lambda}}(x)}{\mathcal{H}_{\frac{\gamma}{\lambda},-\frac{\theta}{\lambda}}(a)}.$$

Proof. It is well known that when the Lévy measure of Z satisfies the assumption 2, its Laplace exponent is an entire function, see [115]. Then we can follow the same route as for the proof of the Theorem 4.3.3, but using the martingale $(\exp(-\xi Z_t - t\psi(-\xi)),\ t \geq 0)$, for any $\xi > 0$, in the Girsanov transform. □

Remark 4.3.7 *Note that if for a fixed $\delta > 0$, we assume only that $\int_{r<-1} e^{-\frac{\delta}{\lambda}r}\nu(dr) < \infty$, then $\Lambda^{(\gamma,-\theta)}$ is well defined for any $\theta < \delta$, since the Laplace exponent is analytic in a convex domain.*

4.4 The Stable Case

We investigate the stable OU processes, that is the GOU processes with stable BDLP's, in more detail. We recall that a stable process $Z := (Z_t,\ t \geq 0)$ with index $\alpha \in (0, 2]$ is a Lévy process which enjoys the selfsimilarity property $(Z_{kt}, t \geq 0) \overset{(d)}{=} (k^{1/\alpha} Z_t, t \geq 0)$, for any $k > 0$.

If the stable process has non-positive jumps, excluding the negative of stable subordinator, its Laplace exponent is given, for $1 < \alpha \leq 2$, by

$$\psi(u) = cu^\alpha, \quad u \geq 0,$$

where $c = \tilde{c} \mid \cos(\frac{1}{2}\pi\alpha) \mid^{-1}$ and $\tilde{c} > 0$, see [108, Example 46.7]). Finally, it is worth noting that if Z is a stable process, with index α, we have the following representation for X, for any $t \geq 0$,

$$X_t = e^{-\lambda t}\left(x + \tilde{Z}_{\tau(t)}\right) \tag{4.12}$$

where $\tilde{Z}$ is an α-stable Lévy process defined on the same probability space as Z and $\tau(t) = \frac{e^{\alpha\lambda t}-1}{\alpha\lambda}$.

We now compute the Laplace transform of the first passage time of a constant level by the stable OU process ($\alpha \in (1, 2]$). As we have said,

another proof exists of this result, see [51]. However, we shall describe a methodology which can be extended to more general selfsimilar Markov processes with one sided jumps and for which singleton is regular for itself. For instance, we refer to [71] for a characterization of selfsimilar processes in $\mathbb{R}^+$, the so-called semi-stable processes. Our proof is based on the selfsimilarity property of Z. We shall proceed in two steps. First, we give the Mellin transform of the first passage time of the BDLP to a specific curve, see [109] and [123] for selfsimilar diffusions with continuous paths and [85] for spectrally negative Lévy processes. Using a deterministic time change we then derive the Laplace transform of H_a. It is clear that the first passage time of a constant level of these processes inherits the selfsimilarity property. Consequently, a unique monotone and continuous function φ exists such that, for $\gamma > 0$

$$\mathbb{E}_x\left[e^{-\gamma T_a}\right] = \frac{\varphi\left(\gamma^{1/\alpha}x\right)}{\varphi\left(\gamma^{1/\alpha}a\right)}$$

where $x \lessgtr a$ depending on the side of the jumps of X. We recall that in the stable case $\varphi(x) = e^{-c^{-1/\alpha}x}$. In order to emphasize the role played by the scaling property in the proof of the following result we shall keep the notation φ. We introduce the following positive random variable

$$T_y^{(\alpha,d)} = \inf\left\{s \geq 0;\ Z_s \geq a(s+d)^{1/\alpha}\right\}, \qquad (a > x),$$

which is the first passage time of the process Z above the curve $a(t+d)^\alpha$.

Theorem 4.4.1 *For $1 < \alpha \leq 2$ and $m > 0$, the Mellin transform of the random variable $T_a^{(\alpha,d)}$ is given by*

$$\mathbb{E}_x\left[(T_a^{(\alpha,d)}+d)^{-m}\mathbb{I}_{\{T_a^{(\alpha,d)}<\infty\}}\right] = d^{-m}\frac{\mathcal{H}_m(d^{-1/\alpha}x)}{\mathcal{H}_m(a)} \tag{4.13}$$

where

$$\begin{aligned}\mathcal{H}_m(-x) &= \int_0^\infty \varphi\left(-xr^{1/\alpha}\right)e^{-r}r^{m-1}\,dr \\ &= \alpha\sum_{k=0}^\infty \frac{(-1)^k\Gamma(\frac{k}{\alpha}+m)}{c^{k/\alpha}k!}x^k.\end{aligned}$$

Proof. From (4.3) and the selfsimilarity of Z, it is clear that the process $(e^{-\gamma t}\varphi(\gamma^{1/\alpha}Z_t),\ t \geq 0)$ is a $\mathcal{F}$-martingale. By an application of Doob's

optional sampling Theorem, we have (using the bounded stopping time $T_y^{(\alpha,d)} \wedge t$ and then applying the dominated convergence Theorem)

$$\mathbb{E}_x\left[e^{-\gamma T_a^{(\alpha,d)}}\varphi\left(\gamma^{1/\alpha}Z_{T_a^{(\alpha,d)}}\right)\right] = \varphi\left(\gamma^{1/\alpha}x\right) \tag{4.14}$$

where by integrating both sides of (4.14) by the measure $e^{-d\gamma}\gamma^{m-1}d\gamma$, and using Fubini's Theorem we get

$$\begin{aligned}&\mathbb{E}_x\left[\int_0^\infty e^{-\gamma(T_a^{(\alpha,d)}+d)}\varphi\left(\gamma^{1/\alpha}Z_{T_a^{(\alpha,d)}}\right)\gamma^{m-1}d\gamma\right]\\ &= \int_0^\infty \varphi\left(\gamma^{1/\alpha}x\right)e^{-d\gamma}\gamma^{m-1}d\gamma.\end{aligned}$$

Using the fact that Z has non-positive jumps, it follows that $Z_{T_a^{(\alpha,d)}} = y(T_a^{(\alpha,d)}+d)^{1/\alpha}$. Thus,

$$\begin{aligned}&\mathbb{E}_x\left[\int_0^\infty e^{-\gamma(T_a^{(\alpha,d)}+d)}\varphi\left(\gamma^{1/\alpha}y(T_a^{(\alpha,d)}+d)^{1/\alpha}\right)\gamma^{m-1}d\gamma\right]\\ &= d^{-m}\mathcal{H}_m(d^{-1/\alpha}x).\end{aligned}$$

The change of variable $r = \gamma(T_a^{(\alpha,d)}+d)$ yields

$$\mathbb{E}_x\left[\int_0^\infty e^{-r}\varphi\left(r^{1/\alpha}a\right)r^{m-1}(T_a^{(\alpha,d)}+d)^{-m}dr\right] = d^{-m}\mathcal{H}_m(d^{-1/\alpha}x).$$

Thus, we have

$$\mathbb{E}_x\left[(T_a^{(\alpha,d)}+d)^{-m}\right] = d^{-m}\frac{\mathcal{H}_m(d^{-1/\alpha}x)}{\mathcal{H}_m(a)}.$$

Next, note that

$$\mathcal{H}_m(-x) \quad = \quad \sum_{k=0}^\infty \frac{(-1)^k(c^{-1/\alpha}x)^k}{k!}\int_0^\infty e^{-r}r^{m+\frac{k}{\alpha}-1}dr.$$

The proof is then completed by using the representation of the Gamma function $\Gamma(z) = \int_0^\infty e^{-r}r^{z-1}dr$, $\Re(z) > 0$. □

Remark 4.4.2 *Let $Z = (Z_t, t \geq 0)$ denotes a real-valued spectrally negative Lévy process starting from 0. Introduce the time-changed process*

$Y_t = \exp(Z_{A_t})$, *where* $A_t = \inf\{s \geq 0; \vartheta(s) := \int_0^s \exp(\alpha Z_u)\, du > t\}$. *Lamperti showed that* $(Y_t,\ t \geq 0)$ *is a* $\mathbb{R}^+$*-valued càdlàg* α *selfsimilar Markov process. Let us denote by* H_a^Y *(resp.* T_y*) the first passage time of* Y *(resp.* Z*) at the level* $a > 0$*. We have the following identity*

$$\begin{aligned} H_y^Y &\overset{(d)}{=} \vartheta(T_{\log(y)}) \\ &= \int_0^{T_{log(y)}} \exp(\alpha Z_u)\, du. \end{aligned}$$

We point out that there is an error in the statement of Theorem 2 in [122]. Indeed, Itô's formula for processes with finite variation together with the fact that X *has no positive jumps yield*

$$\begin{aligned} \vartheta(T_x) &= \vartheta(T_0) + \int_0^x e^{\alpha u}\, dT_u + \sum_{0 \leq u \leq x} (\vartheta(T_u) - \vartheta(T_{u^-}) - e^{\alpha u} \Delta D_u) \\ &= \int_0^x e^{\alpha u}\, dT_u + \sum_{0 \leq u \leq x} (\int_{T_{u-}}^{T_u} \exp(\alpha Z_u)\, du - e^{\alpha u} \Delta D_u). \end{aligned}$$

The last term on the right hand-side was forgotten by the author in his proof.

For more information on the property of the function H, we refer to [85]. As a consequence we state the following result.

Theorem 4.4.3 *The Laplace transform of the random variable* H_a *is given by*

$$\mathbb{E}_x\left[\exp\left(-\gamma H_a\right)\right] = \frac{\mathcal{H}_{\frac{\gamma}{\alpha\lambda}}((\alpha\lambda)^{1/\alpha} x)}{\mathcal{H}_{\frac{\gamma}{\alpha\lambda}}((\alpha\lambda)^{1/\alpha} a)}, \quad a > x.$$

Proof. Fix $a > x$. We have the following relationship between first

passage times

$$\begin{aligned}
H_a &= \inf\{s \geq 0;\ X_s > a\} \\
&= \inf\left\{s \geq 0;\ e^{-\lambda t}\left(x + \int_0^t e^{\lambda s}\, dZ_s\right) > a\right\} \\
&= \inf\left\{s \geq 0;\ e^{-\lambda t}\left(x + \tilde{Z}_{\tau(s)}\right) > a\right\} \\
&= A\left(\inf\left\{s \geq 0;\ x + \tilde{Z}_s > a\,(\alpha\lambda s + 1)^{1/\alpha}\right\}\right) \\
&= A\left(T^{(\alpha,(\alpha\lambda)^{-1})}_{(\alpha\lambda)^{1/\alpha}a}\right)
\end{aligned}$$

where we have performed the deterministic time change $A(t) = \tau^{-1}(t)$, i.e. $A(t) = \frac{1}{\alpha\lambda}\ln(\alpha\lambda t + 1)$. Therefore,

$$\begin{aligned}
\mathbb{E}_x\left[e^{-\gamma H_a}\right] &= \mathbb{E}_x\left[(\alpha\lambda T^{(\alpha,(\alpha\lambda)^{-1})}_{(\alpha\lambda)^{1/\alpha}a} + 1)^{-\frac{\gamma}{\lambda\alpha}}\right] \\
&= (\alpha\lambda)^{-\frac{\gamma}{\lambda\alpha}}\mathbb{E}_x\left[(T^{(\alpha,(\alpha\lambda)^{-1})}_{(\alpha\lambda)^{1/\alpha}a} + (\alpha\lambda)^{-1})^{-\frac{\gamma}{\lambda\alpha}}\right] \\
&= \frac{\mathcal{H}_{\frac{\gamma}{\alpha\lambda}}((\alpha\lambda)^{1/\alpha}x)}{\mathcal{H}_{\frac{\gamma}{\alpha\lambda}}((\alpha\lambda)^{1/\alpha}a)}.
\end{aligned}$$

□

Finally, we mention the expression of $\Lambda_a^{(\gamma,\theta)}$ in this case.

Theorem 4.4.4 *For $\gamma, \theta > 0$ and $a > x$, we have*

$$\Lambda_y^{(\gamma,\theta)}(x) = e^{-\frac{\theta}{\lambda}(a-x)}\frac{\mathcal{H}_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(x)}{\mathcal{H}_{\frac{\gamma}{\lambda},\frac{\theta}{\lambda}}(a)}$$

where $\mathcal{H}_{\nu,\beta}(x) = \int_0^\infty \exp\left(xr - \frac{c}{\lambda}\int_0^r (v+\beta)^\alpha \frac{dv}{v}\right) r^{\nu-1}dr.$

Remark 4.4.5 *When Z is a Brownian with drift b (i.e. $\alpha = 2, c = \frac{1}{2}$), we obtain*

$$\Lambda_a^{(\gamma,\theta)}(x) = e^{\lambda/2(x^2-a^2)-\lambda b(x-a)}\frac{\mathcal{D}_{\overline{\nu}}\left(-\sqrt{2\lambda}(x-\overline{\theta})\right)}{\mathcal{D}_{\overline{\nu}}\left(-\sqrt{2\lambda}(a-\overline{\theta})\right)} \tag{4.15}$$

where $\overline{\nu} := \frac{\theta^2}{2\lambda^3} + \frac{b\theta}{\lambda^2} - \frac{\gamma}{\lambda}$, $\overline{\theta} = \frac{b}{\lambda} + \frac{\theta}{\lambda^2}$, *and*

$$\mathcal{D}_\nu(x) = \frac{e^{-x^2/2}}{\Gamma(-\nu)} \int_0^\infty \exp\left(-xr - \frac{1}{2}r^2\right) r^{-\nu-1}\, dr$$

denotes the parabolic cylinder function, see Chapter 1. We also note that, by taking $b = 0$ *in (4.15), we recover the result of Lachal [70].*

4.5 The Compound Poisson Case with Exponential Jumps

In this part, we extend the results of the previous sections by including positive jumps in the dynamics of X. More precisely, we add an independent component which is a compound Poisson process whose jump sizes have an exponential distribution. Let $Z^+ := (Z_t^+,\ t \geq 0)$ be a compound Poisson process, that is for $t \geq 0$,

$$Z_t^+ = \sum_{k=1}^{N_t(q)} \xi_k$$

where $(N_t,\ t \geq 0)$ is a Poisson process with parameter $p > 0$ and $(\xi_k, k \geq 1)$ is a sequence of i.i.d. random variables. Further, we assume that ξ_1 is exponentially distributed with positive parameter p. The law of Z^+ is characterized by its Laplace transform

$$\log \mathbb{E}\left[e^{uZ_1^+}\right] = \psi^+(u), \quad u \in C_p,$$

where $\psi^+(u) = \frac{uq}{p-u}$ and $C_p = \{u \in \mathbb{C} : \Re(u) < p\}$. Next, we introduce the spectrally negative Lévy process Z_t^-, with Lévy measure denoted by ν^-. It is a measure with support on $\mathbb{R}^-$ which satisfies the integrability condition $\int_{-\infty}^0 (1 \wedge r^2)\nu^-(dr) < \infty$. Set $\theta^- = \inf\{u \leq 0 : \mathbb{E}[e^{uZ_1^-}] < \infty\}$, with the convention that $\inf\{\emptyset\} = \infty$, we have

$$\log \mathbb{E}\left[e^{uZ_1^-}\right] = \psi^-(u), \quad u \in C_{\theta^-},$$

where $\psi^-(u) = mu + \frac{\sigma}{2}u^2 + \int_{-\infty}^0 (e^{ur} - 1 - urI_{\{r>-1\}})\nu^-(dr)$, $m \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$ and $C_{\theta^-} = \{u \in \mathbb{C} : \Re(u) > \theta^-\}$. Finally we consider the Lévy process $Z := (Z_t,\ t \geq 0)$ defined, for $t \geq 0$, by

$$Z_t = Z_t^+ + Z_t^-.$$

Since the two components are independent, we have the identity

$$\log \mathbb{E}\left[e^{uZ_1}\right] = \psi(u), \quad u \in C = C_p \cap C_{\theta^-},$$

where ψ has the form

$$\psi(u) = \psi^+(u) + \psi^-(u).$$

For any $\lambda > 0$, we define the GOU process $X := (X_t, t \geq 0)$ associated to Z as the unique solution of the following stochastic differential equation

$$dX_t = -\lambda X_t \, dt + dZ_t, \quad X_0 = x \in \mathbb{R}.$$

We introduce the first passage time κ_a defined by

$$\kappa_a = \inf\{s \geq 0;\ X_s > a\}, \quad x < a.$$

We denote by Δ_a the overshoot of X over the level a, i.e. $\Delta_a = X_{\kappa_a} - a$. In what follows, we shall compute the law of the couple $(\kappa_a, \int_0^{\kappa_a} X_s \, ds)$ by evaluating its joint Laplace transform which we denote as follows

$$\Lambda_a^{(\gamma,\theta)}(x) := \mathbb{E}_x\left[e^{-\gamma\kappa_a + \theta \int_0^{\kappa_a} X_s \, ds}\right].$$

As for the one sided case, the Laplace transform of the first passage time κ_a can also be computed with the help of martingales techniques, see Novikov et al [88]. Indeed, define

$$\varphi(u) = \frac{1}{\lambda}\int_1^u \frac{\psi^-(r)}{r} \, dr, \quad u > \theta^-,$$

and introduce, for $x \in \mathbb{R}$, the function

$$\mathcal{H}_\gamma(q; x) = \int_0^1 e^{pxu - \varphi(pu)} u^{\frac{\gamma}{\lambda} - 1}(1-u)^{\frac{q}{\lambda} - 1} \, du.$$

We recall the result of [88] and for sake of completeness we sketch the proof.

Proposition 4.5.1 *For $x < a$ and for any $\gamma > 0$,*

$$\mathbb{E}_x\left[e^{-\gamma\kappa_a}\right] = \frac{\mathcal{H}_\gamma(q+\lambda; x)}{\mathcal{H}_\gamma(q; a)},$$

and the law of the overshoot is given by

$$\mathbb{E}_x\left[e^{u\Delta_a}\right] = \frac{1}{1 - u/p}, \quad u < p.$$

Proof. Since the proof is similar to the one of Proposition 4.2.5, we just describe the main steps. In [88], it is shown that the process $(e^{-\gamma t}\mathcal{H}_\gamma(q;X)_t,\ t\geq 0)$ is a $\mathcal{F}$-martingale. Then, thanks to the Wald identity we have

$$\mathbb{E}_x\left[e^{-\gamma\kappa_a}\mathcal{H}_\gamma(q+\lambda;X_{\kappa_a})\right]=\mathcal{H}_\gamma(q+\lambda;x).$$

Now using the facts that the random variables κ_a and Δ_a are independent and

$$\mathbb{E}_x\left[e^{u\Delta_a}\right]=\frac{1}{1-u/p},\quad u<p,$$

we deduce the Laplace transform of κ_a

$$\mathbb{E}_x\left[e^{u\Delta_a}\right]=\frac{\mathcal{H}_\gamma(q+\lambda;x)}{\mathcal{H}_\gamma(q;a)},\quad x<a.$$

□

Next, we deal with the computation of Λ. Let us introduce, for $\theta\in C$, the function

$$\varphi_\theta(u)=\frac{1}{\lambda}\int_1^u\frac{\psi^-(r+\frac{\theta}{\lambda})}{r}\,dr,\quad u<p.$$

Set $\hat{p}=p-\frac{\theta}{\lambda}$, and define the function $\mathcal{H}_{\gamma,\theta}$, for $x\in\mathbb{R}$, as follows

$$\mathcal{H}_{\gamma,\theta}(q;x)=\int_0^1 e^{\hat{p}ux-\varphi_\theta(\hat{p}u)}u^{\frac{\gamma}{\lambda}-1}(1-u)^{\frac{q}{\lambda}-1}\,du.$$

We are ready to extend the results of Section 4.3.

Proposition 4.5.2 *For $\gamma,\lambda>0$, and $\frac{\theta}{\lambda}\in C$, we have*

$$\Lambda_a^{(\gamma,\theta)}(x)=e^{\frac{\theta}{\lambda}(x-a)}\frac{\hat{p}}{p}\frac{\mathcal{H}_{\gamma,\theta}(\hat{q}+\lambda;x+\frac{\hat{m}}{\hat{p}})}{\mathcal{H}_{\gamma,\theta}(\hat{q};a+\frac{\hat{m}}{\hat{p}})},\quad x<a,$$

where $\hat{q}=\frac{qp}{\hat{p}}$ and $\hat{m}=\frac{1}{\theta/\lambda-1}e^{\theta/\lambda-q}(\frac{1}{\theta/\lambda-q}+1)-e^{-q}(\frac{1}{q}-1)$.

Proof. Fix $a>x$, we have

$$\int_0^{\kappa_a}X_s\,ds=\frac{1}{\lambda}\left(Z_{\kappa_a}+x-X_{\kappa_a}\right)$$

which yields

$$\Lambda_a^{(\gamma,\theta)}(x) = e^{\frac{\theta}{\lambda}x}\,\mathbb{E}_x\left[e^{-(\gamma-\psi(\frac{\theta}{\lambda}))\kappa_a+\frac{\theta}{\lambda}Z_{\kappa_a}-\psi(\frac{\theta}{\lambda})\kappa_a-\frac{\theta}{\lambda}X_{\kappa_a}}\right].$$

For $\psi(\xi) < \infty$, that is for $\xi \in C \cap \mathbb{R}$, the process $(\exp(\xi Z_t - t\psi(\xi)), t \geq 0)$ is a $\mathcal{F}$-martingale. We now consider the Girsanov's transform $\mathbb{P}^{(\xi)}$ of the probability measure $\mathbb{P}$ which is defined by

$$d\mathbb{P}^{(\xi)}_{|\mathcal{F}_t} = \exp(\xi Z_t - t\psi(\xi))\; d\mathbb{P}_{|\mathcal{F}_t},\; t \geq 0.$$

Under $\mathbb{P}^{(\xi)}$, Z is again a Lévy process with the following Laplace exponent, for $u \geq 0$,

$$\psi^{(\xi)}(u) \quad := \quad \psi(u+\xi) - \psi(\xi).$$

We have for $\frac{\theta}{\lambda} \in C \cap \mathbb{R}$,

$$\Lambda_a^{(\gamma,\theta)}(x) \quad = \quad e^{\frac{\theta}{\lambda}(x-a)}\,\mathbb{E}_x^{(\frac{\theta}{\lambda})}\left[e^{-(\gamma-\psi(\frac{\theta}{\lambda}))\kappa_a-\frac{\theta}{\lambda}\Delta_a}\right].$$

After some easy algebra, one finds that, under $\mathbb{P}^{(\frac{\theta}{\lambda})}$, Z^+ is again compound Poisson process with exponential jumps of parameter $\hat{p} = p + \frac{\theta}{\lambda}$ and drift $\hat{m} = \frac{1}{\theta/\lambda-1}e^{\theta/\lambda-q}(\frac{1}{\theta/\lambda-q}+1) - e^{-q}(\frac{1}{q}-1)$. The Poisson process has parameter $\hat{q} = \frac{pq}{\hat{p}}$. Thus, we have

$$\mathbb{E}_x^{(\frac{\theta}{\lambda})}\left[e^{-(\gamma-\psi(\frac{\theta}{\lambda}))\kappa_a}\right] \quad = \quad \frac{\mathcal{H}_{\gamma,\theta}(\hat{q}+\lambda; x+\frac{\hat{m}}{\hat{p}})}{\mathcal{H}_{\gamma,\theta}(\hat{q}; a+\frac{\hat{m}}{\hat{p}})}$$

and

$$\mathbb{E}_x^{(\frac{\theta}{\lambda})}\left[e^{u\Delta_a}\right] = \frac{1}{1-u/\hat{p}}, \quad u < \hat{p}.$$

Using the fact that the overshoot and the first passage time are independent random variables, we obtain

$$\Lambda_a^{(\gamma,\theta)}(x) \quad = \quad e^{\frac{\theta}{\lambda}(x-a)}\,\mathbb{E}_x^{(\frac{\theta}{\lambda})}\left[e^{-(\gamma-\psi(\frac{\theta}{\lambda}))\kappa_a}\right]\mathbb{E}_x^{(\frac{\theta}{\lambda})}\left[e^{-\frac{\theta}{\lambda}\Delta_a}\right].$$

The statement follows from Proposition 4.5.1. □

We close this Section by investigating the case when the BDLP Z is simply a compound Poisson process, i.e. $Z = Z^-$. We recall that the Laplace transform of the first passage time κ_a is given by, see [88],

$$\mathbb{E}_x\left[e^{-\gamma\kappa_a}\right] = \frac{\Phi\left(\frac{\gamma}{\lambda}, \frac{p+\gamma}{\lambda}+1; q(x-\frac{m}{\lambda})\right)}{\Phi\left(\frac{\gamma}{\lambda}, \frac{p+\gamma}{\lambda}; q(a-\frac{m}{\lambda})\right)}, \quad x < a, \tag{4.16}$$

where Φ denotes the Kummer function.

Proposition 4.5.3 *For $\gamma, \theta > 0$ such that $\eta := \gamma - \psi(\frac{\theta}{\lambda}) > 0$, and $a > x$, we have*

$$\Lambda_a^{(\gamma,\theta)}(x) = \frac{e^{\frac{\theta}{\lambda}x}\hat{p}}{\hat{p}+\gamma_\theta} \frac{\Phi\left(\frac{\gamma_\theta}{\lambda}, \frac{\hat{p}+\gamma_\theta}{\lambda}+1; (q+\frac{\theta}{\lambda})(x-\frac{\hat{m}}{\lambda})\right)}{\Phi\left(\frac{\gamma_\theta}{\lambda}, \frac{\hat{p}+\gamma_\theta}{\lambda}; q(a-\frac{\hat{m}}{\lambda})\right)} \frac{q+\frac{\theta}{\lambda}}{q}$$

where $\gamma_\theta = \gamma - \frac{\theta}{\lambda}(m + \frac{p}{q-\theta/\lambda})$, $\hat{p} = p - \frac{\lambda}{\theta}$ *and* $\hat{m} = m + \frac{1}{\theta/\lambda - 1}e^{\theta/\lambda - q}(\frac{1}{\theta/\lambda - q} + 1) - e^{-q}(\frac{1}{q} - 1)$.

Proof. Fix $a > x$, by combining the results of Proposition 4.5.1 with the expression (4.16), we get

$$\begin{aligned}
\Lambda_a^{(\gamma,\theta)}(x) &= e^{\frac{\theta}{\lambda}x}\,\mathbb{E}_x^{(\frac{\theta}{\lambda})}\left[e^{-(\gamma-\psi(\frac{\theta}{\lambda}))\kappa_a}\right]\mathbb{E}_x^{(\frac{\theta}{\lambda})}\left[e^{-\frac{\theta}{\lambda}\Delta_a}\right] \\
&= e^{\frac{\theta}{\lambda}x}\frac{\hat{p}}{\hat{p}+\gamma_\theta}\frac{\Phi\left(\frac{\gamma_\theta}{\lambda}, \frac{\hat{p}+\gamma_\theta}{\lambda}+1; (q+\frac{\theta}{\lambda})(x-\frac{\hat{m}}{\lambda})\right)}{\Phi\left(\frac{\gamma_\theta}{\lambda}, \frac{\hat{p}+\gamma_\theta}{\lambda}; q(a-\frac{\hat{m}}{\lambda})\right)}\frac{q+\frac{\theta}{\lambda}}{q}.
\end{aligned}$$

□

4.6 Application to Finance

We apply the results of the previous Sections to the pricing of a European call option on maximum on yields in the generalized Vasicek framework. We extend the results of [73] by allowing jumps in the interest rate dynamics. We refer to their paper for the motivation and the description of the financial problems.

In our framework, that is when the interest rate dynamics is given as the solution of (4.1), it is an easy task to derive the current price of the discount bond

$$\begin{aligned} P_x(0,T) &:= \mathbb{E}_x\left[\exp\left(-\int_0^T X_s\,ds\right)\right] \\ &= \exp\left(A(T)x + D(T)\right) \end{aligned}$$

where $A(t) = \frac{1}{\lambda}\left(1 - e^{-\lambda t}\right)$ and $D(t) = -\int_0^t \psi(A(r))\,dr$, where ψ stands for the Laplace exponent of Z. The price of the option is given by

$$C^X\left(0, T^*, K; x, T\right) := \mathbb{E}_x\left[e^{-\int_0^T X_s\,ds}\left(\sup_{u\in[0,T^*]} X_u - K\right)^+\right]$$

where $K \in \mathbb{R}^+$ denotes (resp. $T^* \in \mathbb{R}^+$) the strike (resp. the time to maturity). Next, we shall give a closed form expression for the Laplace transform with respect to time to maturity of this functional. For $\gamma > 0$, we introduce the notation

$$L_\gamma(K; x, T) := \int_0^\infty e^{-\gamma T^*} C^X(0, T^*, K; x, T)\,dT^*.$$

Proposition 4.6.1 *We assume that* $\int_{r<-1} e^{-\frac{1}{\lambda}r}\nu(dr) < \infty$. *Then, for* $x \le K$, *we have*

$$L_\gamma(K; x, T) = \mathcal{H}_{\frac{\gamma}{\lambda}, -\frac{1}{\lambda}}(x) \int_K^\infty e^{y/\lambda} \frac{P_\gamma(a)}{\mathcal{H}_{\frac{\gamma}{\lambda}, -\frac{1}{\lambda}}(a)}\,da \tag{4.17}$$

where $P_\gamma(a) := \int_0^\infty e^{-\gamma T} P_a(0,T)\,dT$.

Proof. Observing that $\{\sup_{u\in[0,T^*]} X_u < a\} = \{H_a < T^*\}$, and using

the strong Markov property of the process X we obtain

$$\begin{aligned} L_\gamma(K;x,T) &= \mathbb{E}_x\left[\int_K^\infty da \int_{H_a}^\infty dT^* \exp\left(-\gamma T^* - \int_0^{T^*} X_s\, ds\right)\right] \\ &= \mathbb{E}_x\left[\int_K^\infty da \int_{H_a}^\infty dT^* \exp\big(-\gamma(T^*-H_a) - \gamma H_a \right. \\ &\quad \left. - \int_0^{H_a} X_s\, ds - \int_{H_a}^{T^*} X_s\, ds\Big)\right] \\ &= \int_K^\infty da\, \mathbb{E}_x\left[\exp\left(-\gamma H_a - \int_0^{H_a} X_s\, ds\right)\right] P_\gamma(a). \end{aligned}$$

To get the desired expression for the Laplace transform of the option price it remains to compute $\mathbb{E}_x\left[\exp\left(-\gamma H_a - \int_0^{H_a} X_s\, ds\right)\right]$. From Theorem 4.3.6 and Remark 4.3.7, choosing $\theta = 1$, we obtain

$$\mathbb{E}_x\left[\exp\left(-\gamma H_a - \int_0^{H_a} X_s\, ds\right)\right] = e^{\frac{1}{\lambda}(a-x)} \frac{\mathcal{H}_{\frac{\gamma}{\lambda},-\frac{1}{\lambda}}(x)}{\mathcal{H}_{\frac{\gamma}{\lambda},-\frac{1}{\lambda}}(a)}.$$

The identity (4.17) follows. □

Finally, we conclude this Section by investigating the possibility that the interest rates in the mean reverting stable Vasicek model, become negative. In this case, we have $\psi(u) = bu + c\delta^\alpha u^\alpha$, $\delta > 0$. The Laplace transform of the limiting distribution of the process X is given by

$$\begin{aligned} E\left[\exp\left(u\rho^X\right)\right] &= \exp\left(c\delta^\alpha u^\alpha \int_0^\infty e^{-\lambda\alpha r}\, dr + bu \int_0^\infty e^{-\lambda r}\, dr\right) \\ &= \exp\left(\frac{c\delta^\alpha}{\lambda\alpha} u^\alpha + \frac{b}{\lambda} u\right). \end{aligned}$$

We recognize the Laplace transform of a α-stable random variable with $\frac{\delta^\alpha}{\lambda\alpha}$, $\beta = -1$ and $\frac{b}{\lambda}$. In Table 1, we show the probability of a negative long-term interest rate p^n and the mean value $\bar{r}$ for different values of the index but with the other parameter being constant ($b = 0.01$, $\lambda = 0.1$ and $\delta = 0.00025$). These results are the outcomes of Monte Carlo simulation. We recall that for $\alpha = 2$ the mean value is simply given by the coefficient of the drift term $\frac{b}{\lambda}$, whereas for $1 < \alpha < 2$ the stable

random variables without drift are not centered. We observe that the probability of negative interest rate decreases with the index α, but remains very small for moderate values of α. Moreover, the mean value of X stays almost unchanged for the same value of the index and equals the ratio $\frac{b}{\lambda} = 0.1$, which is a realistic level, for instance, for an annual interest rate. It is worth noting that it is possible to get both very small values for p^n and reasonable values for long-term interest rates y for any α by playing with the family of the parameters (λ, b, δ).

Table 4.1:

α	$p^n(\approx)$	y
2	0	1
1.8	1.1×10^{-7}	0.0996
1.5	6.4×10^{-5}	0.099
1.2	0.015	0.086

Chapter 5

On the Resolvent Density of Regular α-Stable Ornstein-Uhlenbeck Processes

Earth is not a gift from our parents, it is a loan from our children.
Amerindian Proverb

5.1 Introduction

The fluctuation theory for Lévy processes has proved enormously fruitful in both theory and application. It originates in an analytical argument such as the Wiener-Hopf factorization. Unfortunately, except for the stable and the completely asymmetric case, explicit expressions for these factors can not be found. In very recent work, much effort

has been devoted to reprove these factors, in the spectrally negative case, by means of probabilistic devices such as excursion theory, see Bertoin [11] or martingale techniques, see Kyprianou and Palmowski [69], Nguyen-Ngoc and Yor [82]. Moreover, it is well known that the law of the exit times associated to the completely asymmetric Lévy processes is characterized in terms of these factors. In this Chapter, we aim to solve some exit problems associated to a spectrally negative α-stable Ornstein-Uhlenbeck process X via the resolvent density (or Green function). It seems rather difficult to use directly the techniques developed for Lévy processes since the properties of the stationarity and independency of the increments are required at some stage, properties which are not fulfilled by X. However, the connection with its underlying Lévy process allows to use some devices which bring us to explicit results. Our approach consists on computing the resolvent density by a combination of martingales techniques and potential theory. More precisely, we compute the law of the hitting time of a fixed level by the α-stable Ornstein-Uhlenbeck process. Then, we derive its resolvent density in terms of the q-scale function associated to X which is given explicitly in terms of a generalization of the Mittag-Leffler function. It turns out that the knowledge of the hitting time distribution is sufficient to characterize the Laplace transform of the exit from above of an interval for X. The rest of the Chapter is organized as follows. The next Section is devoted to some recalls about the properties of the resolvent density of X. In Section 3, we derive the resolvent density at 0 of general α-stable Ornstein-Uhlenbeck processes. Focusing on the spectrally negative case, in Section 4, we give an explicit expression of the resolvent density. In Section 4, we introduce several processes to X, obtained as Doob's h-transform and compute the Laplace transform of their first passage times. More precisely, we characterize the process X conditioned to stay positive and the bridges associated to X. Finally, in the last Section we make the connection with the well known results for the Lévy case.

5.2 Preliminaries

Let $Z := (Z_t, t \geq 0)$ be a stable process with index $\alpha \in (1, 2]$ defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We recall that Z is a càdlàg process with stationary and independent increments which fulfils the

scaling property $(Z_{ct}, t \geq 0) \stackrel{(d)}{=} (c^{1/\alpha} Z_t, t \geq 0)$, for any $c > 0$, where $\stackrel{(d)}{=}$ denotes equality in distribution. The characteristic function of Z has the following form

$$\Psi(u) = c^{-1} \mid u \mid^{\alpha} (1 - i\beta \mathrm{sgn}(u) \tan(\pi\alpha/2)), \quad u \in (-\infty, +\infty),$$

where $c > 0$ and $\beta \in [-1, 1]$ is the skewness parameter. With the choice $\beta = -1$, the process is spectrally negative, i.e. Z does not jump upwards. In this case, it is possible to extend Ψ on the negative imaginary line to derive the Laplace exponent of Z

$$\psi(u) = c^{-1} u^{\alpha}, \quad u \geq 0. \tag{5.1}$$

The distribution of Z_1 is absolutely continuous with a continuous density denoted by p^c, i.e. $\mathbb{P}(Z_1 \in dx) = p^c(x)dx$. Note that, by the scaling property, we have $\mathbb{P}(Z_t \in dx) = t^{-1/\alpha} p^c(t^{-1/\alpha} x)dx$. Doob [34] introduced the α-stable Ornstein-Uhlenbeck process $(X_t,\ t \geq 0)$,with parameter $\lambda > 0$ which is defined by

$$X_t = e^{-\lambda t}\ Z_{\tau(t)}, \quad t \geq 0, \tag{5.2}$$

where $\tau(t) = \frac{e^{\alpha\lambda t} - 1}{\alpha\lambda}$. Note that for $t > 0$, X is governed by the stochastic differential equation

$$dX_t = -\lambda X_t\, dt + dZ_t, \tag{5.3}$$

with $X_0 = 0$. X is ergodic with unique invariant measure $p^{\lambda c}(x)dx$, see Sato [108]. We point out, that in this case, we have $p^{\lambda c}(x) = \rho^X(x)$, where we recall that $\rho^X(x)$ is the density of the limiting distribution of X, see Chapter 4. Without loss of generality, in the following, we assume that $c = 1$ in (5.1), unless stated. Its semigroup is specified by the kernel $\mathbb{P}_x(X_t \in dy) = p_t(x, y) p^{\lambda}(y)dy$, $t > 0$, with

$$p_t(x, y) \quad = \quad \frac{\tau(t)^{-1/\alpha}}{p^{\lambda}(y)} e^{\lambda t} p\left(\tau(t)^{-1/\alpha}(e^{\lambda t} y - x)\right),\ x, y \geq 0. \tag{5.4}$$

It is known that each point of the real line is regular (for itself), that is for any $x \in \mathbb{R}$, $\mathbb{P}_x(H_x = 0) = 1$, where $H_x = \inf\{s > 0; X_s = x\}$ denotes the first hitting time of x by X, see Shiga [111]. As a consequence, for each singleton $\{y\} \in \mathbb{R}$, X admits a local time, denoted by L_t^y. The continuous additive functional L^y is determined by its q-potential, u^q, which is finite for any $q > 0$ and given by

$$u^q(x, y) = \mathbb{E}_x \left[\int_0^{\infty} e^{-qt}\, dL_t^y \right].$$

From the definition of L^y, we derive the following

$$\begin{aligned} u^q(x,y) &= \mathbb{E}_x\left[\int_0^{H_y} e^{-qt}\, dL_t^y\right] + \mathbb{E}_x\left[\int_{H_y}^{\infty} e^{-qt}\, dL_t^y\right] \\ &= \mathbb{E}_x\left[\int_0^{\infty} e^{-q(u+H_y)}\, dL_{u+H_y}^y\right] \\ &= \mathbb{E}_x\left[e^{-qH_y}\right]\mathbb{E}_y\left[\int_0^{\infty} e^{-qu}\, dL_u^y\right] \end{aligned}$$

where the last line follows from the strong Markov property. Thus, we obtain the following identity

$$\mathbb{E}_x\left[e^{-qH_y}\right] = \frac{u^q(x,y)}{u^q(y,y)}, \quad x,y \in \mathbb{R}. \tag{5.5}$$

For any $q > 0$, let R^q be the q-resolvent of X which is defined, for every measurable function $f \geq 0$, by

$$R^q f(x) = \int_0^{\infty} e^{-qt}\mathbb{E}_x\left[f(X_t)\right]\, dt, \quad x \in \mathbb{R}.$$

Note that we have, for any $x \in \mathbb{R}$,

$$\begin{aligned} R^q f(x) &= \mathbb{E}_x\left[\int_{y\in\mathbb{R}} dy f(y) \int_0^{\infty} e^{-qt}\, dL_t^y\right] \\ &= \int_{y\in\mathbb{R}} dy f(y) u^q(x,y). \end{aligned} \tag{5.6}$$

Finally, we summarize in the following some properties of the resolvent of X.

Lemma 5.2.1 *1. R^q has the strong Feller property. There exists a jointly borel measurable function, denoted by r^q such that for $x, y \in \mathbb{R}$, $R^q(x, dy) = r^q(x,y)p^\lambda(y)dy$.*

2. For $q > 0$, the mapping $(x,y) \mapsto r^q(x,y)$ is continuous and bounded by $\max(q^{-1}, 0)$ on $\mathbb{R}\times\mathbb{R}$.

3. The q-potential of L^y is related to the resolvent density of X as follows

$$u^q(x,y)p^\lambda(y) = r^q(x,y), \quad x,y \in \mathbb{R}.$$

Proof. Since each point of the real line is regular (for itself) and X is recurrent the fine topology coincides with the initial topology of $\mathbb{R}$, see Bally and Stoica [8]. The first assertion follows. The second assertion follows from Proposition 3.1 in [8]. The last assertion follows from equation (5.6). □

Remark 5.2.2 *Note that, for $x, y \in \mathbb{R}$, we also have*

$$\mathbb{E}_x\left[e^{-qH_y}\right] = \frac{r^q(x,y)}{r^q(y,y)}. \tag{5.7}$$

5.3 The General α-Stable OU Process

Throughout this Section we consider the general α-stable Ornstein Uhlenbeck process with $\alpha \in (1,2]$. We will compute the resolvent density at the origin. In what follows r^q stands for $r^q(0,0)$.

Theorem 5.3.1 *Let $q > 0$, we have*

$$r^q = \left(p(0)p^\lambda(0)(\alpha\lambda)^{1/\alpha-1}\mathcal{B}\left(\frac{q}{\alpha\lambda}, 1-\frac{1}{\alpha}\right)\right) \tag{5.8}$$

where $\mathcal{B}$ stands for the Beta function and

$$p(0) = \frac{\cos\left(\frac{1}{\alpha}\arctan(\beta\tan\frac{\pi\alpha}{2})\right)}{\alpha\sin\frac{\pi}{\alpha}\Gamma(1-\frac{1}{\alpha})}\left(1+\beta^2\tan^2\frac{\pi\alpha}{2}\right)^{-1/2\alpha}.$$

In particular, for the spectrally negative case ($\beta = -1$), the expression reduces to

$$r^q = \frac{p^\lambda(0)\lambda^{1/\alpha-1}\alpha^{1/\alpha}\Gamma(\frac{q}{\alpha\lambda})}{\Gamma(\frac{q}{\alpha\lambda}+1-\frac{1}{\alpha})} \tag{5.9}$$

where Γ denotes the Gamma function.

Proof. From the expression of the density of the semigroup (5.4), we get

$$\begin{aligned} r^q &= \int_0^\infty e^{-qt} p_t(0,0) p^\lambda(0) dt \\ &= p(0)p^\lambda(0) \int_0^\infty e^{-qt} \kappa_\lambda(t)\, dt \\ &= p(0)(\alpha\lambda)^{1/\alpha-1} \mathcal{B}(\frac{q}{\alpha\lambda}, 1-\frac{1}{\alpha}) \end{aligned}$$

where $\kappa_\lambda(t) = \tau(t)^{-1/\alpha} e^{\lambda t/2}$ and the expression follows from [48, p. 376]. We have for $\beta = -1$, see [119, Formula 4.9.1],

$$p(0) = \frac{\sin(\frac{\pi}{\alpha})}{\pi c^{1/\alpha}} \Gamma\left(1+\frac{1}{\alpha}\right).$$

Finally in this case we get

$$\begin{aligned} r^q &= \frac{\lambda^{1/\alpha-1}\alpha^{1/\alpha}\sin(\frac{\pi}{\alpha})}{\pi c^{1/\alpha}} \Gamma\left(1+\frac{1}{\alpha}\right) \mathcal{B}\left(\frac{q}{\alpha\lambda}, 1-\frac{1}{\alpha}\right) \\ &= \frac{\lambda^{1/\alpha-1}\alpha^{1/\alpha}}{c^{1/\alpha}} \frac{\Gamma(\frac{q}{\alpha\lambda})}{\Gamma(\frac{q}{\alpha\lambda}+1-\frac{1}{\alpha})} \end{aligned}$$

where we have used the following identities

$$\Gamma(\nu+1) = \nu\Gamma(\nu),\ \Gamma(1-\nu) = -\nu\Gamma(-\nu) \text{ and } \Gamma(\nu)\Gamma(-\nu) = -\frac{\pi}{\nu\sin(\pi\nu)}.$$

□

Next, we introduce $\sigma = (\sigma_l, l \geq 0)$ the right continuous inverse of the continuous and increasing functional $(L_t^0, t \geq 0)$. It is plain that σ, as the inverse local time of a standard process, is a subordinator, see e.g. Blumenthal and Getoor [16]. It is also well known that its Laplace exponent is expressed in terms of the q-potential of the local time. More precisely, we have

$$-\log\left(\mathbb{E}\left[e^{-q\sigma_l}\right]\right) = \frac{l}{u^q} \tag{5.10}$$

where $u^q = u^q(0,0)$. Let us also introduce the recurrent δ-dimensional radial Ornstein-Uhlenbeck process with drift parameter $\mu > 0$, which is defined, for $0 < \delta < 1$, as the non-negative solution of

$$dR_t = \left(\frac{\delta-1}{2R_t} - \mu R_t\right) dt + dB_t$$

where B is a Brownian motion. Denote by $\sigma^{(\delta,\mu)}$ the inverse local time at 0 of R. Next, we compute the density of the length of excursions away from 0 for X, which we denote by h.

Corollary 5.3.2

$$h(s) = \left(\frac{c}{\alpha}\right)^{1/\alpha} \frac{\lambda^{-1-1/\alpha}}{\Gamma(\frac{1}{\alpha})} (e^{\alpha\lambda s} - 1)^{1/\alpha - 1}$$

and we have

$$(\sigma_l, l \geq 0) \stackrel{(d)}{=} (\sigma_l^{(1/\alpha,\alpha\lambda)}, l \geq 0) \tag{5.11}$$

where the positive constant of the stable process, in (5.1), is $c = \frac{\lambda^{2\alpha+1}\alpha^{\alpha+1}}{\Gamma^{\alpha}(1-\frac{1}{\alpha})}$.

Proof. Since the transition probabilities of X are diffuse, σ is a driftless subordinator see [44]. Thus, its Laplace exponent has the following form

$$\frac{1}{u^q} = \int_0^\infty (e^{-qs} - 1) h(s) ds. \tag{5.12}$$

Denoting $A_\alpha = \left(\frac{c}{\alpha\lambda}\right)^{1/\alpha} \lambda^{-1}$, we have

$$\frac{A_\alpha \Gamma(\frac{q}{\alpha\lambda} + 1 - \frac{1}{\alpha})}{\Gamma(\frac{q}{\alpha\lambda})} = q \int_0^\infty e^{-qs} h(s)\, ds.$$

Next, using the following integral representation of the ratio of gamma function

$$\frac{\Gamma(\frac{q}{\alpha\lambda} + 1 - \frac{1}{\alpha})}{\Gamma(\frac{q}{\alpha\lambda} + 1)} = \frac{\alpha\lambda}{\Gamma(\frac{1}{\alpha})} \int_0^\infty e^{-qs} (e^{\alpha\lambda s} - 1)^{1/\alpha - 1}\, ds,$$

we deduce the expression for h. Finally, from the formula (59) in Pitman and Yor [98], we notice that h is also the density of the Lévy measure of the inverse local time of R with dimension $\delta = \frac{1}{\alpha}$ and parameter $\mu = \alpha\lambda$. □

Remark 5.3.3 *1. Let* $v^q = \lim_{\lambda\to 0} r^q$ *the resolvent density at* 0 *of the BDLP. Since* $\lim_{\lambda\to 0} \kappa_\lambda(t) = t^{-1/\alpha}$, *we get*

$$\begin{aligned} v^q &= p(0) \int_0^\infty e^{-qt} t^{-1/\alpha}\, dt \\ &= p(0)\, \Gamma(1 - \frac{1}{\alpha}) q^{1/\alpha - 1}. \end{aligned}$$

In particular, we recover the well known result in the case of spectrally negative α-stable Lévy process, see e.g. [14],

$$v^q = \frac{q^{1/\alpha-1}}{\alpha c^{1/\alpha}}.$$

2. *In the case of the classical Ornstein-Uhlenbeck process, we have, see Hawkes and Truman [52],*

$$v^q = \frac{\sqrt{\lambda\pi}}{\Gamma(\frac{q}{\lambda})\mathcal{D}_{-\frac{q}{\lambda}}(0)\mathcal{D}_{-\frac{q}{\lambda}}(0)}$$

where we recall that $\mathcal{D}$ denotes the parabolic cylinder function, and

$$\mathcal{D}_{-\nu}(0) = \frac{\sqrt{\pi}}{2^{\nu/2}\Gamma(\nu/2+1/2)}.$$

Then, by using the identity $2^{2\nu-1}\Gamma(\nu)\Gamma(\nu+1/2) = \sqrt{\pi}\,\Gamma(2\nu)$, we get

$$v^q = \frac{\sqrt{\lambda}\,\Gamma(\frac{q}{2\lambda}+1/2)}{2\Gamma(\frac{q}{2\lambda})}$$

which corresponds to the formula (5.9) with $\alpha = 2$ and $c = 1/2$.

5.4 The Spectrally Negative Case

In this Section, we focus on *spectrally negative* α-stable Ornstein-Uhlenbeck processes, $\alpha \in (1,2]$, starting from $x \in \mathbb{R}$. Let us give some notation. Let H_y^- (H_y) and T_y^-, denotes the downward (upward) hitting time of y by X and Z respectively. We also introduce $T_y^{(f),-}$ the downward hitting time of Z to the boundary $f(t) = y(1+\alpha\lambda t)^{1/\alpha}$, $t \geq 0$. Finally, η_y (resp. $\eta_y^{(f)}$) denotes the downward first passage time of Z over y (resp. f). Next, we denote the Mittag-Leffler function of parameter $\alpha > 0$ by

$$\mathcal{E}_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(1+\alpha n)}, \quad x \in \mathbb{C},$$

and its derivative by $\mathcal{E}'_\alpha$. We recall that the θ-scale function of Z, denoted by $\mathcal{W}^\theta$, see Takács [118] or Bertoin [12] is given by

$$\mathcal{W}^\theta(x) = \alpha x^{\alpha-1}\mathcal{E}'_\alpha((\theta^{1/\alpha}x)^\alpha), \quad \Re(x) \geq 0 \text{ for } 1 < \alpha < 2$$

and we write simply $\mathcal{V}(\theta^{1/\alpha}x) = \alpha\theta^{1-1/\alpha}\mathcal{W}^\theta(x)$. Finally, we introduce the functions

$$\begin{aligned}\mathcal{N}^q(x) &= \int_0^\infty \mathcal{V}\left((u\alpha\lambda)^{1/\alpha}x\right)e^{-u}u^{\frac{q}{\alpha\lambda}-1}du, \quad x \in \mathbb{R}^+, \Re(q) > \frac{1}{\alpha} - 1, \\ &= \alpha^{-1}\sum_{n=0}^\infty (\alpha\lambda)^{\frac{q}{\alpha\lambda}+n+1-\frac{1}{\alpha}}\frac{\Gamma(n+\frac{q}{\alpha\lambda}-\frac{1}{\alpha})}{\Gamma(\alpha(n+1))}x^{\alpha(n+1)-1}. \qquad (5.13)\end{aligned}$$

and

$$\bar{\mathcal{N}}^q(x) = \int_0^x \mathcal{N}^q(y)\,dy. \qquad (5.14)$$

Next, we recall the following function, introduced in Chapter 4,

$$\begin{aligned}\mathcal{H}_q(x) &= \int_0^\infty e^{ux-\frac{u^\alpha}{\alpha\lambda}}u^{\frac{q}{\lambda}-1}du, \quad x \in \mathbb{R} \text{ and } \Re(q) > 0, \qquad (5.15) \\ &= \alpha^{-1}\sum_{n=0}^\infty (\alpha\lambda)^{\frac{\frac{q}{\lambda}+n}{\alpha}}\frac{\Gamma(\frac{n+\frac{q}{\lambda}}{\alpha})}{n!}x^n. \qquad (5.16)\end{aligned}$$

We recall that the function H has been introduced and studied by Novikov [85]. H is well defined and continuous on $\mathbb{R}$. With respect to the parameter q, $\mathcal{H}_\cdot(x)$ has an analytical continuation to the right half-plane. In particular, we have

$$\mathcal{H}_q(0) = (\alpha\lambda)^{\frac{q}{\alpha\lambda}}\frac{1}{\alpha}\Gamma(\frac{q}{\alpha\lambda})$$

and for $\alpha = 2$, we have

$$\mathcal{D}_{\frac{q}{\lambda}}(x) = \frac{1}{\Gamma(\frac{q}{\lambda})}\mathcal{H}_q(x).$$

Next, we define the positive constant w_q by

$$\begin{aligned}w_q &= \frac{\mathcal{H}_q(0)\mathcal{H}_q(0)}{r^q} \\ &= \frac{\Gamma(\frac{q}{\alpha\lambda}+1-\frac{1}{\alpha})\Gamma(\frac{q}{\alpha\lambda})}{(\alpha\lambda)^{\frac{q}{\alpha\lambda}}\frac{1}{\alpha}}.\end{aligned}$$

Note that in the Brownian case, w_q is identified as the wronskrian of the parabolic cylinder functions $\mathcal{D}_{\frac{q}{\lambda}}(-.)$ and $\mathcal{D}_{\frac{q}{\lambda}}(+.)$. We are now ready to state the main result of this Section.

Theorem 5.4.1 *For any $x, y \in \mathbb{R}$, and $q > 0$, we have*

$$\begin{aligned} r^q(x,y) &= w_q^{-1}\mathcal{H}_q(x)\mathcal{H}_q(-y), \quad x \le y, \\ r^q(x,0) &= w_q^{-1}\mathcal{H}_q(0)\left(\mathcal{H}_q(x) - \mathcal{N}^q(x)\right), \quad x \ge 0, \\ \int_0^x r^q(x,y)\,dy &= 1 + \mathcal{N}^q(x)\left(\frac{1}{\Gamma(\frac{q}{\alpha\lambda})} - qw_q^{-1}H_{q-\alpha\lambda}(-x)\right) - \frac{\bar{\mathcal{N}}^q(x)}{\Gamma(\frac{q}{\alpha\lambda})}. \end{aligned}$$

Remark 5.4.2 *It would be interesting to compute explicitly the resolvent density for any $x \ge y$. It is linked to the problem of expressing the q-scale function of the Lévy process, $\mathcal{W}^q(x-y)$ as a combination of a function of x and y.*

Remark 5.4.3 *1. Note that if we consider the OU process driven by a stable process with a linear drift $\mu \in \mathbb{R}$, the results of this Chapter can be readily extended. Indeed, denoting by $\kappa^{\mu}_{x\to y}$ the first passage time of this process starting at x to the level y, we have the following relationship between first passage times*

$$\begin{aligned} \kappa^{\mu\lambda}_{x\to y} &= \inf\{s \ge 0;\ e^{-\lambda s}\left(x - \mu(e^{\lambda s} - 1) + Z_{\tau(s)}\right) < y\} \\ &= \inf\{s \ge 0;\ e^{-\lambda s}\left(x - \mu + Z_{\tau(s)}\right) < y - \mu\} \\ &= \kappa_{x-\mu\to y-\mu}. \end{aligned}$$

2. It is also possible to get the results for the case when $\lambda < 0$ by using the fact that the invariant measure p^λ is λ-excessive. Indeed, we have the following absolute continuity relationship, with the obvious notation,

$$\frac{d\mathbb{P}_x^{(-\lambda)}}{d\mathbb{P}_x^{(\lambda)}}|\mathcal{F}_t = e^{-\lambda t}\frac{p^\lambda(X_t)}{p^\lambda(x)}, \quad t > 0.$$

3. In the case $\alpha = 2$, that is when X is the classical Ornstein-Uhlenbeck process, we get

$$r^q(x,y) = w_q^{-1}e^{\lambda(x^2+y^2)}\mathcal{D}_{-\frac{q}{\lambda}}((x\wedge y)\sqrt{2\lambda})\mathcal{D}_{-\frac{q}{\lambda}}(-(x\vee y)\sqrt{2\lambda}).$$

The proof of the Theorem is based on the identity (5.26). It is clear that the Laplace transform of the hitting time of X is required. That will be the focus of the following subsections. The proof of the Theorem is then split into two main steps. The first one consists in computing the distribution of the downward hitting time to 0 and relies on martingales techniques. At a second stage we compute the law of the downward hitting for the process starting from 0 by using the potential theory. Finally, we conclude the proof by computing the first passage time below the level 0.

5.4.1 The law of H_0^-: a Martingale Approach

Proposition 5.4.4 *For any $x \in \mathbb{R}$, and $q > -\nu_\alpha(y)(< -\alpha^{-1})$, we have*

$$\mathbb{E}_x[e^{-qH_0^-}] \quad = \quad \frac{\mathcal{H}_q(x) - \mathcal{N}^q(x)\mathbb{I}_{\{x\geq 0\}}}{\mathcal{H}_q(0)}. \tag{5.17}$$

First we recall that, for $x \leq 0$, the law of H_0 has been evaluated by Hadjiev [51], see Chapter 4, as follows.

Lemma 5.4.5 *Let $\nu_\alpha(0)$ be the smallest positive zero of $\mathcal{H}_\cdot(0)$, then for any $x \leq 0$, and $q > -\nu_\alpha(0)$, we have*

$$\mathbb{E}_x[e^{-qH_0}] \quad = \quad \frac{\mathcal{H}_q(x)}{\mathcal{H}_q(0)}. \tag{5.18}$$

Next, we aim to compute the law of the downward hitting time H_0^- for $x \geq 0$. To this end, we introduce the process Y defined for each $t \geq 0$ by $Y_t = (\alpha\lambda t + 1)^{-1/\alpha} Z_t$. We have the following.

Lemma 5.4.6 *For all $q > 0$ and $x \geq 0$, the process*

$$N_t = (\alpha\lambda t + 1)^{-\frac{q}{\alpha\lambda}} \left(\mathcal{H}_q(Y_t) - \mathcal{N}^q(Y_t))\mathbb{I}_{\{t\leq\eta_0\}}\right) \tag{5.19}$$

is a $\mathbb{P}_x$-martingale.

Proof. Let us introduce the process

$$M_t^\theta = e^{-\theta t}\left(e^{\theta^{1/\alpha} Z_t} - \mathcal{V}(\theta^{1/\alpha} Z_t)\mathbb{I}_{\{t\leq\eta_0\}}\right).$$

Then, we recall from Doney [33] that the law of the downward hitting time T_0^- of Z is given by

$$\int_0^\infty e^{-\beta x}\mathbb{E}_x[e^{-\theta T_0^-}]\,dx = \frac{1}{\beta-\theta^{1/\alpha}} - \frac{\alpha\theta^{1-1/\alpha}}{\beta^\alpha-\theta}.$$

The right hand side is defined by continuity for $\beta,\theta>0$. Noting that for $\beta>\theta^{1/\alpha}$, $\frac{1}{\beta^\alpha-\theta}=\sum_{n=1}^\infty \beta^{-\alpha n}\theta^{n-1}$, so inverting the Laplace transform yields, for $x\geq 0$

$$\mathbb{E}_x[e^{-\theta T_0^-}] = e^{\theta^{1/\alpha}x} - \mathcal{V}(\theta^{1/\alpha}x).$$

The strong Markov property entails that M^θ is a $\mathbb{P}_x$-martingale on $[0,T_0^-]$. Further, note that for $\alpha=2$, the Brownian case, $\mathcal{V}(\theta^{1/\alpha}(Z_{T_0^-}))=0$. For the other cases, since Z does not creep downwards, we have $T_0^-\geq\eta_0$ a.s. which implies that M^θ is a $\mathbb{P}_x$-martingale on $[0,\eta_0]$. The martingale property follows then by observing that the remaining part of M^θ, after η_0, is a $\mathbb{P}_x$-martingale. Note also that the first passage time over 0 of Y is a.s. η_0. Set $\bar\theta=\alpha\lambda\theta$, then by integrating $M^{\bar\theta}$ by the measure $e^{-\theta}\theta^{q-1}d\theta$ we get

$$\begin{aligned}
&\int_0^\infty M_t^{\bar\theta}e^{-\theta}\theta^{q-1}d\theta\\
=&\int_0^\infty e^{-\theta(\alpha\lambda t+1)}\left(e^{\bar\theta^{1/\alpha}Z_t}-\mathcal{V}(\bar\theta^{1/\alpha}Z_t)\mathbb{I}_{\{t\leq\eta_0\}}\right)\theta^{q-1}d\theta\\
=&\ N_t
\end{aligned}$$

where we have set $u=\theta(\alpha\lambda t+1)$. Since $M^{\bar\theta}$ is a $\mathbb{P}_x$-martingale, the martingale property for N follows by Fubini's Theorem. □

The proof of the Proposition is completed by an application of Doob's optional stopping Theorem to the stopping time T_0^- and by observing from (5.2) that $T_0^-=\tau(H_0^-)$ a.s..

Remark 5.4.7 *Let $0\leq x\leq a$. By an application of Doob's optional stopping Theorem to the martingale $M^{\bar\theta}$, using the linearity of the expectation sign and the fact that $\mathbb{E}_x[e^{-\theta T_a}]=e^{\theta^{1/\alpha}(x-a)}$, we recover from the previous Lemma the following well known result*

$$\mathbb{E}_x[e^{-\theta T_a}\mathbb{I}_{\{T_a<\eta_0\}}] = \frac{\mathcal{W}(\theta^{1/\alpha}x)}{\mathcal{W}(\theta^{1/\alpha}a)}.$$

Similarly, since for $0 \leq a \leq x$, $\mathbb{E}_x[e^{-\theta T_a^-}] = e^{\theta^{1/\alpha}(x-a)} - \mathcal{V}(\theta^{1/\alpha}(x-a))$, *we get*

$$\mathbb{E}_x[e^{-\theta T_a^-}\mathbb{I}_{\{T_a^- < \eta_0\}}] = \frac{\mathcal{W}(\theta^{1/\alpha}x) - \mathcal{W}(\theta^{1/\alpha}(x-a))}{\mathcal{W}(\theta^{1/\alpha}a)}.$$

5.4.2 The law of H_y^-: a Potential Approach

We introduce the dual process of X, denoted by $\hat{X}$, relative to the invariant measure $p^\lambda(x)dx$. Since the dual of the Lévy process Z is $-Z$, we note from (5.3) that $\hat{X}$ has the same law that the spectrally positive stable Ornstein-Uhlenbeck process. It is solution to the stochastic differential equation

$$d\hat{X}_t = -\lambda \hat{X}_t\, dt - dZ_t.$$

Its semigroup with respect to the invariant measure is given by $\hat{p}_t(x,y) = p_t(y,x)$, $x, y \geq 0$. Recall that we have the following duality between the resolvent densities

$$\hat{r}^q(x,y) = r^q(y,x), \quad x, y \in \mathbb{R}. \tag{5.20}$$

The law of the first hitting time of y by $\hat{X}$, denoted by $\hat{T}_y$, is characterized by its Laplace transform as follows. For $q \geq 0$, $x \geq y$, we have

$$\mathbb{E}_x[e^{-q\hat{T}_y}] \quad = \quad \frac{\mathcal{H}_q(-x)}{\mathcal{H}_q(-y)}.$$

Remark 5.4.8 *Since the resolvent density is jointly continuous, all the points are co-regular (regular for the dual). Thus, one can define, for the dual, the local time and its associate q-potential at all points. Therefore the dual identity of (5.26) holds also for* $\hat{X}$.

Proposition 5.4.9 *For* $y \leq 0$, *we have*

$$\mathbb{E}_0[e^{-qH_y^-}] \quad = \quad \frac{1}{\mathcal{H}_q(y)}\left(\mathcal{H}_q(0) - \mathcal{H}_q(0)\frac{\mathcal{N}^q(-y)}{\mathcal{H}_q(-y)}\right)$$

and

$$\begin{aligned} r^q(0,y) &= w_q^{-1}\mathcal{H}_q(0)\left(\mathcal{H}_q(-y) - \mathcal{N}^q(-y)\right) \\ r^q(y,y) &= w_q^{-1}\mathcal{H}_q(y)\mathcal{H}_q(-y). \end{aligned}$$

Proof. Fix $y \leq 0$. First, note that from the duality relationship, we have $r^q(0,y) = \hat{r}^q(y,0)$. Next, the identity $\hat{r}^q(y,0) = \mathbb{E}_y[e^{-q\hat{T}_0}]r^q$ and formula (5.9) yield

$$r^q(0,y) = w_q^{-1}\mathcal{H}_q(0)\left(\mathcal{H}_q(-y) - \mathcal{N}^q(-y)\right).$$

Then, observing that $\hat{r}^q(y,y)\mathbb{E}_0[e^{-q\hat{T}_y}] = r^q(y,0)$, we obtain

$$r^q(y,y) = w_q^{-1}\mathcal{H}_q(y)\mathcal{H}_q(-y).$$

Finally, for $y \leq 0$, we have

$$\begin{aligned}
\mathbb{E}_0[e^{-qH_y^-}] &= \frac{r^q(0,y)}{r^q(y,y)} \\
&= \frac{\mathcal{H}_q(0)}{\mathcal{H}_q(y)\mathcal{H}_q(-y)}\left(\mathcal{H}_q(-y) - \mathcal{N}^q(-y)\right) \\
&= \frac{1}{\mathcal{H}_q(y)}\left(\mathcal{H}_q(0) - \mathcal{H}_q(0)\frac{\mathcal{N}^q(-y)}{\mathcal{H}_q(-y)}\right).
\end{aligned}$$

The proof is completed. □

5.4.3 The Stable OU Process Killed at κ_0

We introduce the first passage time over the level 0 by X

$$\kappa_0 = \inf\{s \geq 0;\ X_s < 0\}$$

and the first exit time from the interval $(0,a]$

$$H_{0,a} = \inf\{s \geq 0;\ X_s \notin (0,a]\}.$$

First, we give an expression of the Laplace transform of κ_0.

Theorem 5.4.10 *For $x > 0$, the Laplace transform of κ_0 is given by*

$$\mathbb{E}_x[e^{-q\kappa_0}] = \frac{\overline{\mathcal{N}}^q(x) - \mathcal{N}^q(x)}{\Gamma(\frac{q}{\alpha\lambda})}$$

where we recall that $\overline{\mathcal{N}}^q(x) = \int_0^\infty \mathcal{E}_\alpha(u\alpha\lambda x^\alpha)e^{-u}u^{\frac{q}{\alpha\lambda}-1}du$.

Proof. It is clear that $\kappa_0 = A(\eta_0)$ a.s.. Thus,

$$\mathbb{E}_x\left[e^{-q\kappa_0}\right] = \mathbb{E}_x\left[(\alpha\lambda\eta_0+1)^{-\frac{q}{\alpha\lambda}}\right].$$

But, setting $\bar{\theta} = \theta\alpha\lambda$, we know that

$$\mathbb{E}_x\left[e^{-\bar{\theta}\eta_0}\right] = \mathcal{E}_\alpha(\bar{\theta}x^\alpha) - \alpha\bar{\theta}^{1-1/\alpha}x^{\alpha-1}\mathcal{E}'_\alpha(\bar{\theta}x^\alpha).$$

Finally, by integrating both sides of the latter equation by the measure $e^{-\theta}\theta^{\frac{q}{\alpha\lambda}-1}d\theta$, and use Fubini's Theorem, we obtain the result. □

Next, denote by r_0^q (resp. $\hat{r}_0^q$), the resolvent density of the process X (resp. $\hat{X}$) killed at time κ_0. We end up by giving two nice consequences of the previous results and the proof of Theorem 5.4.1 will be completed by the Remark following this Corollary.

Corollary 5.4.11 *Let $0 \le x \le a$ and $q \ge 0$. Then,*

$$\mathbb{E}_x[e^{-qH_a}\mathbb{I}_{\{H_a<\kappa_0\}}] = \frac{\mathcal{N}^q(x)}{\mathcal{N}^q(a)}.$$

In particular,

$$\mathbb{P}_x[H_a < \kappa_0] = \frac{\mathcal{N}(x)}{\mathcal{N}(a)}.$$

Moreover, we have

$$\mathbb{E}_x[e^{-q\kappa_0}\mathbb{I}_{\{\kappa_0<H_a\}}] = \frac{1}{\Gamma(\frac{q}{\alpha\lambda})}\left(\overline{\mathcal{N}}^q(x) - \frac{\mathcal{N}^q(x)}{\mathcal{N}^q(a)}\overline{\mathcal{N}}^q(a)\right).$$

Consequently,

$$\mathbb{E}_x[e^{-qH_{0,a}}] = \frac{1}{\Gamma(\frac{q}{\alpha\lambda})}\left(\overline{\mathcal{N}}^q(x) - \frac{\mathcal{N}^q(x)}{\mathcal{N}^q(a)}\overline{\mathcal{N}}^q(a)\right).$$

Proof. First, note the following identity, for $x \le a$

$$\mathbb{E}_x[e^{-qH_a}\mathbb{I}_{\{H_a<\kappa_0\}}] = \frac{r_0^q(x,a)}{r_0^q(a,a)}.$$

We proceed by giving an expression of the resolvent density of X killed upon entering the negative half-line. By the strong Markov property and the absence of negative jumps for $\hat{X}$, we get, for $x, y \ge 0$,

$$\hat{r}_0^q(y,x) \quad = \quad \hat{r}^q(y,x) - \mathbb{E}_y[e^{-q\hat{T}_0}]\hat{r}^q(0,x).$$

The switch identity for Markov processes, see [16, Chap. VI], tells us that $\hat{r}_0^q(x,y) = r_0^q(y,x)$. Hence,

$$r_0^q(x,y) \quad = \quad r^q(x,y) - \frac{\mathcal{H}_q(-y)}{\mathcal{H}_q(0)} r^q(x,0).$$

The first assertion follows. The second assertion is obtained by passage to the limit. Moreover, the Strong Markov property yields

$$\mathbb{E}_x[e^{-q\kappa_0}] = \mathbb{E}_x[e^{-q\kappa_0}\mathbb{I}_{\{\kappa_0<H_a\}}] + \mathbb{E}_x[e^{-qH_a}\mathbb{I}_{\{H_a<\kappa_0\}}]\mathbb{E}_a[e^{-q\kappa_0}].$$

Using the result of the previous Theorem, we deduce the third assertion. The last one follows readily from the identity

$$\mathbb{E}_x[e^{-qH_{0,a}}] = \mathbb{E}_x[e^{-q\kappa_0}\mathbb{I}_{\{\kappa_0<H_a\}}] + \mathbb{E}_x[e^{-qH_a}\mathbb{I}_{\{H_a<\kappa_0\}}].$$

□

Remark 5.4.12 *1. Note that for $x \leq y$*

$$r_0^q(x,y) \quad = \quad w_q^{-1}\mathcal{N}^q(x)\mathcal{H}_q(-y).$$

2. From the Strong Markov property, we also have, for $x \geq 0$

$$\mathbb{E}_x\left[e^{-q\kappa_0}\right] \quad = \quad 1 - q\int_0^\infty r_0^q(x,y)dy.$$

It is from this identity that we compute the last expression in Theorem 5.4.1.

3. Finally we characterize the law of the first passage time below a lower level. To this end, let us observe, from (5.3), that the process of jumps of X, denoted by ΔX, is identical to the one of Z. It is a Poisson point process valued in $(-\infty,0)$ with characteristic measure $\nu(dx) = x^{-\alpha-1}dx$. Let $y \leq x \leq z$. Thus,

$$\mathbb{E}_x\left[\mathbb{I}_{\{X_{\kappa_0^-}\in dy\}}\mathbb{I}_{\{\Delta X_{\kappa_0}\in dz\}}e^{-q\kappa_0}\right] = r_0^q(x,y)\nu(dz).$$

This result is a straightforward consequence of the compensation

formula applied to ΔX. Indeed, we have

$$\begin{aligned}
& \mathbb{E}_x\left[\mathbb{I}_{\{X_{\tau_0-}\in dy\}}\mathbb{I}_{\{\Delta X_{\tau_0-}\in dz\}}e^{-q\tau_0}\right] \\
= \; & \mathbb{E}_x\left[\sum_{t\geq 0}\mathbb{I}_{\{X_{t-}\in dy\}}\mathbb{I}_{\{\Delta X_t\in dz\}}e^{-qt}\mathbb{I}_{\{X_s\in(0,\infty)\forall s<t\}}\right] \\
= \; & \mathbb{E}_x\left[\int_0^\infty dt\,\mathbb{I}_{\{X_t\in dy\}}e^{-qt}\mathbb{I}_{\{X_s\in(0,\infty)\forall s<t\}}\right]\nu(dz) \\
= \; & \mathbb{E}_x\left[\int_0^\infty dt\,e^{-qt}p_t^0(x,y)\right]\nu(dz)
\end{aligned}$$

where $p_t^0(x,y)$ stands for the transition densities of the X killed when entering the negative real line.

5.5 Some Related First Passage Times

In this Section, we study the law of the first passage time of some (Markov) processes, the laws of which are constructed from the one of X. In order to simplify the notation we shall work in the canonical setting. That is, we denote by $D([0,\infty))$ (resp. $D([0,t])$ for $t>0$) the space of càdlàg paths $\omega:[0,\infty)\longrightarrow\mathbb{R}$ (resp. $\omega:[0,t]\longrightarrow\mathbb{R}$). $D([0,\infty))$ will be equipped with the Skohorod topology, with its Borel σ-algebra $\mathcal{F}$, and the natural filtration $(\mathcal{F}_t)_{t\geq 0}$. We keep the notation X for the coordinate process. Let $\mathbb{P}_x$ (resp. $\mathbb{E}_x$) be the law (resp. the expectation operator) of the stable OU process starting at $x\in\mathbb{R}$.

5.5.1 The Stable OU Process Conditioned to Stay Positive

Note from the previous subsection that the function $\mathcal{N}$ is a positive invariant function for the stable OU process killed at κ_0. Thus, we introduce the new probability measure $\mathbb{P}^\uparrow$ on $D([0,\infty))$ defined as a Doob's h-transform of this latter process.

$$\mathbb{P}_x^\uparrow(A)=\frac{1}{\mathcal{N}(x)}\mathbb{E}_x\left[\mathcal{N}(X_t),A,t<\kappa_0\right],\quad A\in\mathcal{F}_t.$$

It turns out that $\mathbb{P}_x^{\uparrow}$ can be identified as the conditional law $\mathbb{P}_x(\,|\,\kappa_0 = \infty)$. Indeed denoting by θ the shift operator, we have, from the strong Markov property, for any Borel set $A \in \mathcal{F}_t$

$$\begin{aligned}
\mathbb{P}_x(A \mid \kappa_0 = \infty) &= \mathbb{P}_x(A, \kappa_0 = \infty)/\mathbb{P}_x(\kappa_0 = \infty) \\
&= \mathbb{P}_x(A, t < \kappa_0, \kappa_0 \circ \theta_t = \infty)/\mathbb{P}_x(\kappa_0 = \infty) \\
&= \mathbb{E}_x\left[\mathbb{P}_{X_t}(\kappa_0 = \infty)/\mathbb{P}_x(\kappa_0 = \infty, A, t < \kappa_0)\right] \\
&= \frac{1}{\mathcal{N}(x)}\mathbb{E}_x\left[\mathcal{N}(X_t), A, t < \kappa_0\right].
\end{aligned}$$

Corollary 5.5.1 *Let $0 \le x \le a$. Then,*

$$\mathbb{E}_x^{\uparrow}[e^{-qH_a}] = \frac{\mathcal{N}(a)\mathcal{N}^q(x)}{\mathcal{N}(x)\mathcal{N}^q(a)}.$$

Proof. It is a direct consequence of the definition of $\mathbb{P}_x^{\uparrow}$, the Doob's optional stopping Theorem and the previous corollary. □

5.5.2 The Law of the Maximum of Bridges

Recall that the stable OU process is a strong Markov process with right continuous paths. It has transition densities $p_t(x, y)$ with respect to the σ-finite measure p^{λ}. Moreover, there exists a second right process $\hat{X}$ in duality with X relative to the measure p^{λ}. Under these conditions Fitzsimmons et al. [42] construct the bridges of X by using Doob's method of h-transform. Let us denote by $\mathbb{P}_{x,y}^{l}$ the law of X started at x and conditioned to be at y at time t. We have the following absolute continuity relationship, for $l < t$,

$$d\mathbb{P}_{x,y}^{t}\,|_{\mathcal{F}_l} = \frac{p_{t-l}(X_l, y)}{p_t(x, y)}\, d\mathbb{P}_x\,|_{\mathcal{F}_l}\,. \tag{5.21}$$

Let us still denote by H_a the first passage time of the canonical process X at the level $a > x$. We assume that X has only negative jumps. The law of the maximum of the bridge of X, denoted by M, is given in the following

Theorem 5.5.2 *For $q > 0$, $x, y, a \in \mathbb{R}$ with $x \le a$, we have*

$$\int_0^{\infty} e^{-qt}\mathbb{P}_{x,y}^{t}(M_t \ge a)p_t(x, y)\, dt = r^q(a, y)\frac{\mathcal{H}_q(x)}{\mathcal{H}_q(a)}. \tag{5.22}$$

Proof. Thanks to the absolute continuity relationship (5.21) and Doob's optional stopping Theorem, we have

$$\mathbb{P}^t_{x,y}(H_a \in dl)p_t(x,y) = p_{l-t}(a,y)\mathbb{P}_x(H_a \in dl).$$

Then, by integrating, we get

$$\mathbb{P}^t_{x,y}(H_a \geq t)p_t(x,y) = \int_0^t p_{l-t}(a,y)\mathbb{P}_x(\tau_a \in dl). \tag{5.23}$$

Next, we use the fact that $\mathbb{P}^t_{x,y}(H_a \leq t) = \mathbb{P}^t_{x,y}(M_t \geq a)$. Finally by taking the Laplace transform with respect to t, and by noticing the convolution on the right hand side of (5.23), we complete the proof. □

5.6 The Lévy Case: $\lambda \to 0$

We end up by showing that when considering the limit $\lambda \to 0$, we recover the results for spectrally negative Lévy processes. Let us denote $\phi(q) = q^{1/\alpha}$.

Proposition 5.6.1 *We have the following limit results*

$$\lim_{\lambda\to 0} \frac{\mathcal{H}_{\frac{q}{\lambda}}(x)}{\mathcal{H}_{\frac{q}{\lambda}}(0)} = e^{-x\phi(q)}, \tag{5.24}$$

$$\lim_{\lambda\to 0} \frac{\mathcal{N}^{\frac{q}{\lambda}}(x)}{\mathcal{H}_{\frac{q}{\lambda}}(0)} = \mathcal{V}(\phi(q)x). \tag{5.25}$$

As a consequence, we have the following results for the underlying α-stable Lévy process

$$\begin{aligned} v^q(x,y) &= v^q(y-x,0) \\ &= \phi'(q)e^{\phi(q)(x-y)} - \mathcal{W}^q(x-y)\mathbb{I}_{\{x\geq y\}}. \end{aligned}$$

Also, for $x, y > 0$, we get

$$v^q_0(x,y) = e^{-\phi(q)y}\mathcal{W}^q(x) - \mathcal{W}^q(x-y)\mathbb{I}_{\{x\geq y\}}.$$

For $q > 0$, $x, y \in \mathbb{R}$, we have

$$\mathbb{E}_x\left[e^{-qT_y}\right] = e^{-\phi(q)(x-y)} - \frac{1}{\phi'(q)}\mathcal{W}^q(x-y)\mathbb{I}_{\{x\geq y\}}.$$

Let $0 \leq x \leq a$.

$$\mathbb{E}_x[e^{-qT_a}\mathbb{I}_{\{T_a<\eta_0\}}] = \frac{\mathcal{W}^q(x)}{\mathcal{W}^q(a)}.$$

For any $x, q > 0$, *we have*

$$\begin{aligned}\mathbb{E}_x\left[e^{-q\eta_0}\right] &= 1 - q\int_0^\infty v_0^q(x,y)dy \\ &= 1 + q\int_0^x \mathcal{W}^q(y)dy - \frac{q}{\phi(q)}\mathcal{W}^q(x).\end{aligned}$$

Finally, for $a \in \mathbb{R}$ *with* $x \leq a$,

$$\begin{aligned}&\int_0^\infty e^{-qt}\mathbb{P}^t_{x,y}(M_t \geq a)p_t(x,y)\,dt \\ &= \left(\phi'(q)e^{\phi(q)(a-y)} - \mathcal{W}^q(a-y)\mathbb{I}_{\{y\leq a\}}\right)e^{\phi(q)(x-a)} \\ &= \phi'(q)e^{\phi(q)(x-y)} - e^{\phi(q)(x-a)}\mathcal{W}^q(a-y)\mathbb{I}_{\{y\leq a\}}.\end{aligned}$$

Proof. We can rewrite H by considering the following change of variable $r = \phi(s)$ and denoting $z = \lambda^{-1}$

$$\begin{aligned}\mathcal{H}_{z\gamma}(x) &= \int_0^\infty \exp\left(x\phi(r) - z\int_1^{\phi(r)} \psi(u)\,\frac{du}{u}\right)\phi(r)^{z\gamma-1}\phi'(r)dr \\ &= \int_0^\infty f_x(r)\exp\left(-zp(r;\gamma)\right)dr\end{aligned}$$

where $f_x(r) = e^{x\phi(r)}\phi(r)^{-1}\phi'(r)$ and $p(t;\gamma) = \int_1^r \frac{\phi'(u)}{\phi(u)}u\,du - \gamma\log(\phi(r))$. We use the Laplace's method to derive an asymptotic approximation for large values of the parameter z, see [89, Theorem 2.1]. We get the following approximation

$$\mathcal{H}_{z\gamma}(x) \sim f_x(\gamma)e^{-zp(\gamma)}\left(\frac{2\pi}{xp''(\gamma)}\right)^{\frac{1}{2}} \qquad (z \to \infty)$$

where $p'(t) = \frac{\phi'(t)}{\phi(t)}(t-\gamma)$ and $p''(\gamma) = \frac{\phi'(\gamma)}{\phi(\gamma)} \neq 0$. The second limit is obtained in a similar way. The rest follows after some easy computations. □

Remark 5.6.2 *The last assertion of the proposition holds for any spectrally negative Lévy process.*

Glossary

$\wedge$	$x \wedge y = \inf(x,y)$	34
$\vee$	$x \vee y = \sup(x,y)$	34
$\mathcal{B}$	Beta function	72
Γ	Gamma function	23
$\mathcal{C}_b^2(\mathbb{R})$	Space of twice continuously differentiable and bounded functions	11
Mr^+	Space of positive Radon measures defined on $\mathbb{R}_0^+$	50
	Bessel processes	9
K_y	First passage time of a Bessel process to a fixed level y	42
L_y	Last passage time of R to the level y	17
$m(dy)$	Speed measure of the 3-dimensional Bessel process	29
$N^{(1)}(r)$	L^1-norm of the 3-dimensional Bessel bridge	27
$N^{(2)}(r)$	L^2-norm of the 3-dimensional Bessel bridge	27
Q^ν	Square Bessel process of index ν	67
$q_t(x,y)$	Density of the semigroup of the 3-dimensional Bessel process	29
R	3-dimensional Bessel process	17
r	3-dimensional Bessel bridge process	17
$\Upsilon^{\lambda,\alpha}_{x\to y}(t)$	Joint Laplace transform of the L^1 and L^2-norms of the 3-dimensional Bessel bridge	30
	Brownian motion	9
Ai	Airy function	38
B, W	Standard Brownian motion	11
$\zeta^{(\beta)}$	$=\zeta^{(1+\beta t)}$	62
$f^{(\alpha,\beta)}$	$=S^{(\alpha,\beta)}f$	49

$\Pi^{(\alpha,\beta)}$	Transformation of Sturm-Liouville solutions	49
$p_{\cdot\to a}(t)$	Density of T_a	12
$q_{\cdot\to\alpha}(t)$	Density of $T_\alpha^{(2)}$	38
Σ	Doob's transform	50
σ	Inverse local time of $\lvert B\rvert$	44
$S^{(\alpha,\beta)}$	Family of transformation of curves	49
$S^{(\beta)}$	$=S^{(1,\beta)}$, family of elementary transformations	49
τ	Deterministic time change of B, see Doob's transform	11
$T_\alpha^{(1)}$	First passage time of the Brownian motion over the linear curve	36
$T_\alpha^{(2)}$	First passage time of the Brownian motion over the quadratic curve	36
$T^{(\surd)}$	First passage time of B over the square root boundary	14
$T^{(f)}$	First passage time of B over the curve f	47
$T^{(f,\mu)}$	First passage time of B with drift μ over the curve f	36
T_a	First passage time of the Brownian motion of the fixed level a	12
υ_k	Decreasing sequence of negative zeros of the Airy function	38
φ	Decreasing solution of a Sturm-Liouville equation	30
ψ	Increasing solution of a Sturm-Liouville equation	13
w	Wronskrian of the solutions of a Sturm-Liouville equation	30
	Classical Ornstein-Uhlenbeck process	9
$\mathcal{D}_\nu$	Parabolic cylinder function of index ν	12
$\mathcal{G}$	Infinitesimal generator of U	11
$H_a^{(\mu)}$	First passage time of $U^{(\mu)}$ to a fixed level a	39
H_a	First passage time of the OU process of the fixed level a	12
$\mathcal{H}_\nu$	Hermite function of index ν	12
λ	Parameter of the OU process	11
$\mathbb{P}^{(\lambda)}$	Law of the OU process with parameter λ	11
$p_{\cdot\to a}^{(\lambda,\mu)}(t)$	Density of $H_a^{(\mu)}$	39
$p_{\cdot\to a}^{(\lambda)}(t)$	Density of H_a	12
ϱ	Deterministic time change, inverse of τ	49
τ	Deterministic time change	49
U	Classical Ornstein-Uhlenbeck process	11
$U^{(\mu)}$	Ornstein-Uhlenbeck process with drift μ	39
${}^{(\mu)}U$	Mean reverting OU process with parameter μ	14
Φ	Confluent Hypergeometric function	23

	Generalized Ornstein-Uhlenbeck processes	78
$\mathcal{E}_\alpha$	Mittag-Leffler function of parameter α	112
H_y^-	Downward hitting time of X to the level y	112
H_a	First passage time above of X to the level a	78
$\eta^{(f)}$	First passage time below of Z over the curve f	112
η_y	First passage time below of Z to the level y	112
I	Primitive of X	78
κ_b	First passage time below of X to the level b	118
L^y	Local time of X at y	107
$\mathcal{N}^q$	q-scale function of X	113
ν	Lévy measure of Z	79 38
R^q	q-resolvent of X	108
r^q	q-resolvent density of X	108
σ	Inverse local time of X	110
T_y^-	Downward hitting time of Z to the level y	112
$T^{(f),-}$	Downward hitting time of Z to the curve f	112
T_y	First passage time above of Z to the level y	80
$T_y^{(\alpha,d)}$	First passage time of Z over the curve $y(t+d)^{1/\alpha}$	93
u^q	q-potential of the local time of X	107
v^q	q-resolvent density of Z	111
ϕ	Pseudo-inverse of the Laplace exponent ψ	80
φ	Primitive of the Laplace exponent of Z	82
$\mathcal{W}^\theta$	θ-scale function of Z	113
X	Generalized Ornstein-Uhlenbeck process	78
$\hat{X}$	Dual process of X	116
ψ	Laplace exponent of Z	79
Z	Spectrally negative Lévy process	78

Bibliography

[1] Abate, J. and Whitt, W. The Fourier-series method for inverting transforms of probability distributions. *Queueing Syst.*, 10(1):5–88, 1992.

[2] Alili, L. Fonctionnelles exponentielles et certaines valeurs principales des temps locaux browniens. Thèse doctorale de l'Université Paris VI, 1995.

[3] Alili, L. and Patie, P. On the first crossing times of selfsimilar diffusions and a family of continuous curves. *Working paper, ETH Zürich*, 2004.

[4] Alili, L. and Patie, P. On the joint law of the L^1 and L^2 norms of a 3-dimensional Bessel bridge. *Submitted*, 2004.

[5] Alili, L. and Patie, P. Study of some functional transformations with an application to moving boundary crossing problems. *Working paper, ETH Zürich*, 2004.

[6] Alili, L and Patie, P. On the first crossing times of a Brownian motion and a family of continuous curves. *Comptes Rendus de l'Académie des Sciences - Mathématiques*, 340(3):225–228, 2005.

[7] Alili, L., Patie, P. and Pedersen, J.L. Representations of first hitting time density of an Ornstein-Uhlenbeck process. *Stochastic Models*, 21(4), 2005.

[8] Bally, V. and Stoica, L. A class of Markov processes which admit a local time. *The Annals of Probability*, 15(1):241–262, 1987.

[9] Barndorff-Nielsen, O.E. and Shephard, N. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *Journal of the Royal Statistical Society*, 63(2):167–241, 2001.

[10] Bertoin, J. On the Hilbert transform of the local times of a Lévy process. *Bull. Sci. Math.*, 119(2):147–156, 1995.

[11] Bertoin, J. *Lévy Processes.* Cambridge University Press, Cambridge, 1996.

[12] Bertoin, J. On the first exit time of a completely asymmetric stable process from a finite interval. *Bull. London Math. Soc.*, 28:514–520, 1996.

[13] Biane, Ph. and Yor, M. Valeurs pricipales associées aux temps locaux browniens. *Bull. Sci Maths.*, 2ème série(111):23–101, 1987.

[14] Bingham, N.H. Fluctuation theory in continuous time. *Adv. in Appl. Probab. 7*, pages 705–766, 1975.

[15] Blake, I.F. and Lindsey, W.C. Level-crossing problems for random processes. *IEEE Trans. Information Theory*, IT-19:295–315, 1973.

[16] Blumenthal, R.M. and Getoor, R.K. *Markov Processes and Potential Theory.* Academic Press, New York, 1968.

[17] Borodin, A.N. and Salminen, P. *Handbook of Brownian Motion - Facts and Formulae.* Probability and its Applications. Birkhäuser Verlag, Basel, 2nd edition, 2002.

[18] Borovkov, K. and Novikov, A. Explicit bounds for approximation rates for boundary crossing probabilities for the Wiener process. *Preprint*, 2004.

[19] Breiman, L. First exit times from a square root boundary. *In Fifth Berkeley Symposium*, 2(2):9–16, 1967.

[20] Carmona, Ph., Petit, F. and Yor, M. Exponential functionals of Lévy processes. *Lévy processes Theory and Applications*, pages 41–55, 2001.

[21] Daniels, H.E. Saddle point approximation in statistics. *Ann. Math. Statist.*, 25:631–650, 1954.

[22] Daniels, H.E. The minimum of a stationary Markov process superimposed on a U-shaped trend. *Journal of Applied Probability*, 6:399–408, 1969.

[23] Daniels, H.E. The maximum size of closed epidemic. *Add. Appl. Prob.*, 6:607–621, 1974.

[24] Daniels, H.E. Sequential tests constructed from images. *Ann. Statist.*, 10:394–400, 1982.

[25] Daniels, H.E. The first crossing time density for Brownian motion with a pertubated linear boundary. *Bernoulli*, (6):571–580, 2000.

[26] Darling, D.A. and Siegert, A.J.F. The first passage problem for a continuous Markov process. *Ann. Math. Stat.*, (24):624–639, 1953.

[27] De-La-Peña, V. and Hernandez-Del-Valle, G. First-passage time densities of Brownian motion over a nondecreasing right-continuous barrier. *Submitted*, 2005.

[28] Delbaen, F. and Yor, M. Passport options. *Math. Finance*, 12(4):299–328, 2002.

[29] Delong, D.M. Crossing probabilities for a square root boundary for a Bessel process. *Comm. Statist. A–Theory Methods*, 10(21):2197–2213, 1981.

[30] Di Nardo, E., Nobile, A.G., Pirozzi, E. and Ricciardi, L.M. A computational approach to first-passage-time problems for Gauss-Markov processes. *Adv. in Appl. Probab.*, 33(2):453–482, 2001.

[31] Doering, D.A., Hagan, P.S. and Levermore, C.D. Mean exit times for particles driven by weakly colored noise. *SIAM J. Appl. Math.*, 49(5):1480–1513, 1989.

[32] Donati-Martin, C., Matsumoto, H. and Yor, M. Some absolute continuity relationships for certain anticipative transformations of geometric Brownian motions. *Publ. Res. Inst. Math. Sci.*, 37(3):295–326, 2001.

[33] Doney, R. Hitting probabilities for spectrally positive Lévy processes. *J. London Math. Soc.*, 2(44):566–576, 1991.

[34] Doob, J.L. The Brownian movement and stochastic equations. *Annals of Mathematics*, 43(2):351–369, 1942.

[35] Duffie, D., Filipović, D. and Schachermayer, W. Affine processes and applications in finance. *Ann. Appl. Probab.*, 13(3):984–1053, 2003.

[36] Durbin, J. Boundary crossing probabilities for the Brownian motion and Poisson processes and techniques for computing the power of Kolmogorov-Smirnov test. *Journal of Appl. Prob.*, 8:431–453, 1971.

[37] Durbin, J. The first-passage density of a continuous Gaussian process to a general boundary. *Journal of Appl. Prob.*, 22:99–122, 1985.

[38] Durbin, J. A reconciliation of two different expressions for the first-passage density of Brownian motion to a curved boundary. *J. Appl. Probab.*, 25(4):829–832, 1988.

[39] Durbin, J. The first passage time density of the Brownian motion process to a curved boundary. *J. Appl. Prob.*, (29):291–304, 1992. with an App. by D. Williams.

[40] Ferebee, B. The tangent approximation to one-sided Brownian exit densities. *Z. Wahr. Verw. Gebiete*, 61(3):309–326, 1982.

[41] Ferebee, B. An asymptotic expansion for one-sided Brownian densities. *Z. Wahr. Verw. Gebiete*, 63(1):1–15, 1983.

[42] Fitszimmons, P.J., Pitman, J. and Yor, M. Markovian bridges: Constuction, Palm intepretation, and splicing. *Seminar on Stochatic Processes*, pages 101–134, 1992.

[43] Fitzsimmons, P.J. and Getoor, R.K. On the distribution of the Hilbert transform of the local time of a symmetric Lévy process. *Ann. Probab.*, 20:1484–1497, 1992.

[44] Getoor, R.K. and Sharpe, M.J. Last exit times and additive functionals. *Annals of Probability*, 1:550–569, 1973.

[45] Ghomrasni, R. On distributions associated with the generalized Lévy stochastic area formula. *Studia Scientiarum Mathematicarum Hungarica. A Quarterly of the Hungarian Academy of Sciences*, 41(1):83–100, 2004.

[46] Giraudo, M.T., Sacerdote, L. and Zucca, C. A Monte Carlo method for the simulation of first passage times of diffusion processes. *Methodology and Computing in Applied probability*, 3:215–231, 2001.

[47] Göing-Jaeschke, A. and Yor, M. A clarification note about hitting times densities for Ornstein-Uhlenbeck processes. *Finance and Stochastics*, 7(3):413–415, 2003.

[48] Gradshteyn, I.S. and Ryshik, I.M. *Table of Integrals, Series and Products.* Academic Press, San Diego, 6th edition, 2000.

[49] Groeneboom, P. Brownian motion with a parabolic drift and Airy functions. *Prob. Theory and Related Fields*, 81(1):79–109, 1989.

[50] Guo, X. and Shepp, L. Some optimal stopping problems with non trivial boundaries for pricing exotic options. *Journal of Appl. Prob.*, (38):1–12, 2001.

[51] Hadjiev, D.I. The first passage problem for generalized Ornstein-Uhlenbeck processes with non-positive jumps. *Séminaire de Probabilités, XIX, 1983/84, Lecture Notes in Mathematics*, 1123:80–90, 1985.

[52] Hawkes, J. and Truman, A. Statistics of local time and excursions for the Ornstein-Uhlenbeck process. In M.T. Barlow and N.H. Bingham, editors, *Proceedings of Durham Symposium on Stochastic Analysis*, Series 167, pages 91–102. Cambridge University Press, 1991.

[53] Henrici, P. *Applied and Computational Complex Analysis: Special Functions-Integral Transforms-Asymptotics-Continued Fractions*, volume 2. Wiley Interscience, New York, 1977.

[54] Hille, E. *Lectures on ordinary differential equations.* Addison-Wesley Publishing Company, Reading, Massachussetts, 1969.

[55] Hobson, D.G., Williams, D. and Wood, A.T.A. Taylor expansions of curve crossing boundaries. *Bernoulli*, 5(5):779–795, 1999.

[56] Hu, Y., Shi, Z. and Yor, M. *Some Applications of Lévy Area Formula to Pseudo-Brownian and pseudo-Bessel bridges*, pages 181–209. Exponentials Functionals and Principal Values Related to Brownian Motion. 1997.

[57] Itô, K. and McKean, H.P. *Diffusion Processes and their Sample Paths*. Springer-Verlag Berlin Heidelberg, 1996.

[58] Jeanblanc, M. and Rutkowski, M. *Modelling of Default Risk: An Overview*, pages 171–269. Higher education press, beijing edition, 2000.

[59] Jeanblanc, M., Pitman, J. and Yor, M. Feynman-Kac's formula and decomposition of Brownian paths. *Mat. Apl. Comput.. Sociedade Brasiliera de Matemática Applicada e Computacional*, 16(1):27–52, 1997.

[60] Jeanblanc, M., Pitman, J. and Yor, M. Self similar processes with independent increments associated with Lévy and Bessel processes. *Stochastic Processes and Applications*, (100):223–232, 2002.

[61] Jurek, Z.J. Selfdecomposability: An exception or a rule? *Annales Univer. M. Curie-Sklodowska, Lublin-Polonia*, LI, Sectio A:93–107, 1997. A Special volume dedicated to Professor Dominik Szynal.

[62] Karatzas, I. and Shreve, S.E. *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics, 113. Springer-Verlag, New York, 2nd edition edition, 1991.

[63] Keilson, J. and Ross, H.F. First passage time of gaussian markov (Ornstein-Uhlenbeck) statistical processes. *Selected Tables in mathematical studies, Amer. Math. Soc., Providence R.I.*, III:233–327, 1975.

[64] Kendall, W.S. Boundary crossing for Brownian motion. *Personnal communication*, 2004.

[65] Kent, J.T. Eigenvalue expansions for diffusion hitting times. *Z. Wahrsch. Verw. Gebiete*, 52:309–319, 1980.

[66] Kent, J.T. The spectral decomposition of a diffusion hitting time. *Ann. Prob.*, 10(1):207–219, 1982.

[67] Kou, S.G. and Wang, H. First passage times of a jump diffusion process. *Adv. Appl. Prob.*, 35:504–531, 2003.

[68] Kunitomo, N. and Ikeda, N. Pricing options with curved boundaries. *Math. Finance*, 2:275–298, 1992.

[69] Kyprianou, A.E. and Palmowski, Z. A martingale review of some fluctuation theory for spectrally negative Lévy processes. *Séminaire de Probabilités*, XXXVIII:16–29, 2004.

[70] Lachal, A. Quelques martingales associées à l'intégrale du processus d'Ornstein-Uhlenbeck. Application à l'étude des premiers instants d'atteinte. *Stochastic and Stochactic Report*, 3-4:285–302, 1993.

[71] Lamperti, J.W. Semi-stable Markov processes. *Z. Wahrsch. Verw. Geb.*, 22:205–225, 1972.

[72] Lebedev, N.N. *Special functions and their applications.* Dover Publications, New York, 1972.

[73] Leblanc, B. and Scaillet, O. Path dependent options on yields in the affine term structure. *Finance and Stochastics*, 2:349–367, 1998.

[74] Leblanc, B., Renault, O. and Scaillet, O. A correction note on the first passage time of an Ornstein-Uhlenbeck process to a boundary. *Finance and Stochastics*, 4:109–111, 2000.

[75] Lefebvre, M. First-passage densities of a two-dimensional process. *SIAM Journal on Applied Mathematics*, 49(5):1514–1523, 1989.

[76] Lerche, H.R. Boundary crossing of Brownian motion: Its relation to the law of the iterated logarithm and to sequential analysis. *Lecture Notes in Statistics*, 40, 1986.

[77] Lévy, P. *Processus Stochastiques et Mouvement Brownien.* Gauthier-Villars, Paris, 1948.

[78] Linetsky, V. Computing hitting time densities for OU and CIR processes: Applications to mean-reverting models. *Journal of Computational Finance*, 7:1–22, 2004.

[79] Longstaff, F.A. and Schwartz, E.S. Valuing credit derivatives. *The Journal of Fixed Income*, 5:6–12, 1995.

[80] Louchard, G. The Brownian excursion area. *Comput. Math. Appl.*, 10:413–417, 1984. Erratum:A12 (1986) 375.

[81] Mckean, H.P. *Stochastic Integrals.* 1969.

[82] Nguyen-Ngoc, L. and Yor, M. Some martingales associated to reflected Lévy processes. *To appear in Séminaire de Probabilités*, 2005.

[83] Novikov, A.A. The stopping times of a Wiener process. *Teor. Verojatnost. i Primenen.*, 16:458–465, 1971. Translation in Theory Probab. Appl. 16 (1971), 449-456.

[84] Novikov, A.A. Estimates for and asymptotic bahaviour of the probabilities of a Wiener process not crossing a moving boundary. *Mat. Sb. (N.S.)*, 110(152)(4):539–550, 1979.

[85] Novikov, A.A. A martingale approach in problems on first crossing time of non linear boundaries. *Proceedings of the Steklov Institute of Mathematics*, 4:141–163, 1981 (english edition 1983).

[86] Novikov, A.A. Martingales and first passage times for Ornstein-Uhlenbeck processes with a jump component. *Teor. Veroyatnost. i Primenen*, 48(2):340–358, 2003.

[87] Novikov, A.A., Frishling, V. and Kordzakhia, N. Approximations of boundary crossing probabilities for a Brownian motion. *J. Appl. Prob.*, 36(4):1019–1030, 1999.

[88] Novikov, A.A., Melchers, R.E., Shinjikashvili, E. and Kordzakhia, N. First passage time of filtered Poisson process with exponential shape function. *Preprint University of sydney*, 2003.

[89] Olver, F.W.J. *Introduction to Asymptotics and Special Functions.* Academic Press, 1974.

[90] Patie, P. On the resolvent density of regular α-stable Ornstein-Uhlenbeck processes. *Working paper, ETH Zürich*, 2004.

[91] Patie, P. On a martingale associated to generalized Ornstein-Uhlenbeck processes and an application to finance. *Stochastic Processes and their Applications*, 115(4):593–607, 2005.

[92] Perman, M. and Wellner, J.A. On the distribution of the Brownian areas. *The Annals of Applied Probability*, 6(4):1091–1111, 1996.

[93] Peskir, G. Limit at zero of the Brownian first-passage density. *Prob. Theory Related Fields*, Vol. 124(1):100–111, 2002.

[94] Peskir, G. On integral equations arising in the first-passage problem for Brownian motion. *J. Integral Equations Appl.*, Vol. 14(4):397–423, 2002.

[95] Pistorius, M. *Exit Problems of Lévy Processes with Applications in Finance.* PhD thesis, Utrecht University, 2003.

[96] Pitman, J. and Yor, M. Bessel processes and infinitely divisible laws. *Lect. Notes in Maths.*, 850:285–370, 1981.

[97] Pitman, J. and Yor, M. A decomposition of Bessel bridges. *Z. Wahrsch. Verw. Gebiete*, 59:425–457, 1982.

[98] Pitman, J. and Yor, M. On the lengths of excursions of some Markov processes. *Séminaire de Probabilités*, XXXI(1655):272–286, 1997.

[99] Pötzelberger, K. and Wang, L. Boundary crossing probability for Brownian motion. *J. Appl. Prob.*, 38:152–164, 2001.

[100] Revuz, D. and Yor, M. *Continuous Martingales and Brownian Motion*, volume 293. Springer-Verlag, Berlin-Heidelberg, 3rd edition, 1999.

[101] Ricciardi, L.M. and Sato, S. First-passage time density and moments of the Ornstein-Uhlenbeck process. *J. Appl. Prob.*, 25:43–57, 1988.

[102] Ricciardi, L.M., Sacerdote, L. and Sato, S. On an integral equation for first passage time probability densities. *J. Appl. Prob.*, 21:302–314, 1984.

[103] Robbins, H. and Siegmund, D. Statistical tests of power one and the integral representation of solutions of certain parabolic differential equations. *Bull. Inst. Math. Acad. Sinica (Taipei)*, 1:93–120, 1973.

[104] Roberts, G.O. Asymptotic approximations for Brownian motion boundary hitting times. *The Annals of Probability*, 19(4):1689–1731, 1991.

[105] Roberts, G.O. and Shortland, C.F. Pricing barrier options with time-dependent coefficients. *Math. Finance*, 7:83–93, 1997.

[106] Rogers, L.C.G. Evaluating first-passage probabilities for spectrally one-sided Lévy processes. *J. Appl. Probab.*, 37(4):1173–1180, 2000.

[107] Salminen, P. On the first hitting time and the last exit time for a Brownian motion to/from a moving boundary. *Adv. in Appl. Probab.*, 20(2):411–426, 1988.

[108] Sato, K. *Lévy Processes and Infinetely Divisible Distributions.* Cambridge University Press, Cambridge, 1999.

[109] Shepp, L.A. A first passage problem for the Wiener process. *Annals of Mathematics and Statistics*, 38:1912–1914, 1967.

[110] Shepp, L.A. On the integral of the absolute value of the pinned Wiener process. *The Annals of Probability*, 10(1):234–239, 1982.

[111] Shiga, T. A recurrence criterion for Markov processes of Ornstein-Uhlenbeck type. *Probab. Theory Related Fields*, 85(4):425–447, 1990.

[112] Siegert, A.J.F. On the first passage time probability problem. *Physical Rev.*, 81(2):617–623, 1951.

[113] Siegmund, D. Boundary crossing probabilities and statistical applications. *Ann. Statis.*, 14:361–404, 1986.

[114] Siegmund, D. and Yuh, Yih Shyh. Brownian approximations to first passage probabilities. *Z. Wahr. Verw. Gebiete*, 59:239–248, 1982.

[115] Skorohod, A.V. *Random Processes with Independent Increments*, volume 47 of *Mathematics and its Application.* Kluwer, Dordrecht, Netherlands, 1991.

[116] Smith, C.E. A Laguerre series approximation to the probability density of the first passage time of the Ornstein-Uhlenbeck. *Noise in Physical Systems and 1/f Fluctuations*, 1991.

[117] Strassen, V. Almost sure behavior of sums of independent random variables and martingales. *Proc. Fifth Berkeley Symp. Math. Statis. Prob. (Berkeley 1965/66)*, Vol. II, 1967.

[118] Takács, L. *Combinatorial Methods in the Theory of Stochastic Processes.* Wiley, New York.

[119] Uchaikin, V. and Zolotarev, M. *Chance and Stability - Stable Distributions and their Applications.* Modern Probability and Statistics. VSP, Utrecht, 1999.

[120] Uchiyama, K. Brownian first exit from and sojourn over one sided moving boundary and application. *Z. Wahrsch. Verw. Gebiete*, 54:75–116, 1980.

[121] Uhlenbeck, G.E. and Ornstein, L.S. On the theory of the Brownian motion. *Physical Review*, 36:823–841, 1930.

[122] Winkel, M. Limit clusters in the inviscid Burgers turbulence with certain random initial velocities. *J. Stat. Phys.*, 107:893–917, 2002.

[123] Yor, M. On square-root boundaries for Bessel processes and pole seeking Brownian motion. *Stochastic analysis and applications (Swansea, 1983). Lecture Notes in Mathematics*, 1095:100–107, 1984.

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Research Interests

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