ASIAN OPTIONS UNDER ONE-SIDED LÉVY MODELS

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Abstract. We generalize, in terms of power series, the celebrated Geman-Yor formula for the pricing of Asian options in the framework of spectrally negative Lévy-driven assets. We illustrate our result by providing some new examples.

1. Introduction

Asian options are path-dependent contingent claims whose settlement price is calculated with reference to the average price of the underlying security over a prescribed time period. In this paper, we are concerned with the pricing of fixed-strike Asian call options in a market driven by a spectrally negative Lévy process, that is a process with stationary and independent increments having no positive jumps. The motivation for studying such financial contracts in Lévy-driven asset models with not positive jumps are two-fold. On the one hand, a commonly accepted remedy to the imperfections of the geometric Brownian motion as a model for asset prices is the use of exponential Lévy type dynamics, see e.g. Schoutens [35]. Moreover, over the last years, it has been observed by several authors that the structure of the class of spectrally negative Lévy processes is relevant for modeling the dynamics of the prices of financial assets. For instance, Eberlein and Madan [16] provide a variety of economical reasons to support the consistency of processes with no positive jumps in the context of long maturity stock price distributions embedded in option prices. Schoutens and Madan [27] also argue that spectrally negative Lévy processes are sufficient for long dated options. In this regard, we mention that the markets for long-term options have witnessed an explosive growth over the last decade. Currently, liquid prices for maturities up to thirty years and beyond are shown for these type of products, see e.g. [9]. On the other hand, as we shall see in this paper, this class of models including the Black-Scholes dynamics is flexible and simple enough to provide a tractable expression for the Laplace transform with respect to time to maturity of the price of fixed-strike Asian options.

In this framework, it turns out that the issue of pricing Asian options is a great mathematical challenge. Indeed, it is already a difficult problem to determine the law of an additive functional of a diffusion process, such as the arithmetic average of the exponential of a Brownian motion, to be convinced that the case of Lévy processes might not be straightforward. This is probably

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a reason why most of the literature studies only the pricing of Asian options in Black-Scholes type models.

Using stochastic calculus, and specifically the Bessel processes, Geman and Yor [18], see also the excellent monograph of Yor [39], obtained an analytical formula for the Laplace transform in time of the Asian option price. Their approach reveals that the issue of evaluating Asian options amounts to finding the law of the so-called exponential functional of the Brownian motion with drift taken at some independent exponential time. Then, many authors have been interested in characterizing the law of the exponential functional in the more general framework of Lévy processes. Beside some isolated cases and until very recently, only information regarding some transformations, such as the entire moments, or the tail behavior of the distribution has been identified, see e.g. [6], [7], [28] [21]. We refer to the survey paper of Bertoin and Yor [4] for a very nice description of these kinds of results. However, Patie [33] and [32] offers a power series and a contour integral representation of the law of this exponential functional for the class of spectrally negative Lévy processes. In this paper, relying on this result, we provide a generalization of the Geman-Yor formula in the context of spectrally negative Lévy processes.

Coming back to the Black-Scholes framework, we mention that there is a substantial literature devoted to the issue of pricing Asian options. In particular, Rogers and Shi [34] have formulated a one-dimensional partial differential equation that can model both floating and fixed strike Asian options. Donati-Martin et al. [11] express the prices of Asian options in terms of the resolvent density of some diffusions. We also indicate that Carr and Schröder [8] and more recently Schröder [37] used complex analysis techniques for inverting numerically or analytically the Geman-Yor Laplace transform. Dufresne [13], see also Schröder [36] and Linetsky [26], resorts to Laguerre polynomials for deriving an analytical expression for Asian call options. We also refer to Fu et al. [17] for a description of numerical methods developed for approximating the price of these type of options in the Black-Scholes model. Beyond the diffusion case, we would like to mention that Večeř and Xu [38] provide an interesting formulation of Asian option prices in the general framework of special semimartingales as the solution of a boundary value problem associated to a partial integro-differential equation. Finally, the difficulty of getting analytical expressions for this problem have lead many authors to find some interesting upper and lower bounds for the prices of essentially discrete monitored Asian options. We refer to Albrecher et al. [1] where such bounds are derived for implementing a static super-hedge for fixed-strike Asian call options.

The remaining part of the paper is organized as follows. In the next Section, after describing the financial market model, we discuss some basic ideas on the pricing of Asian call options. We also recall a recent result regarding the representation in terms of power series of the law of the exponential functional of spectrally negative Lévy processes. In Section 3, we state and proof the generalization of the Geman-Yor formula. Finally, we end the paper by providing three examples illustrating our main result. We also mention that parts of the results stated in Theorem 3.1 below were announced in the note [33].

2. **Preliminaries**

2.1. **The market model.** Let $\xi = (\xi_t)_{t\geq 0}$ be a spectrally negative Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ where $(\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by ξ satisfying the usual conditions. For any $x \in \mathbb{R}$, $\mathbb{P}_x$ stands for the law of ξ when started at x, i.e.

$\mathbb{P}_x$ is the law of $\xi + x$ under $\mathbb{P} = \mathbb{P}_0$. Accordingly, we shall write $\mathbb{E}_x$ and $\mathbb{E}$ for the associated expectation operators. Next, we consider a financial market where two assets are traded. There is the riskless security whose price grows at the continuously compounding positive interest rate r. The dynamics of the risky asset $S = (S_t)_{t\geq 0}$ is governed by the exponential of ξ, that is, for any $t \geq 0$,

$$S_t = e^{\xi_t}. \tag{2.1}$$

We exclude the case when ξ is degenerate, that is when it is the negative of a subordinator, i.e. a process with increasing paths, or a pure drift process. In this setting, it is well known that the characteristic exponent Ψ, defined by

$$\Psi(z) = \log\left(\mathbb{E}[e^{iz\xi_1}]\right),\ z \in \mathbb{R},$$

admits an analytical continuation to the lower half-plane and we set $\psi(u) = \Psi(-iu), u \geq 0$. It means that, for any $u \geq 0$, ψ admits the following Lévy-Khintchine representation

$$\psi(u) = \delta u + \frac{\sigma}{2}u^2 + \int_{-\infty}^{0} \left(e^{uy} - 1 - uy\mathbb{I}_{\{|y|<1\}}\right) m(dy) \tag{2.2}$$

where $\sigma \geq 0$ is the Gaussian coefficient, $\delta \in \mathbb{R}$ is the drift and the Lévy measure m satisfies the integrability condition $\int_{-\infty}^{0} \left(1 \wedge y^2\right)\ m(dy) < \infty$. We refer to the books of Bertoin [3] and Kyprianou [24] for backgrounds on Lévy processes. It is easily seen that the discounted asset price $\tilde{S} = (\tilde{S}_t = e^{-rt}S_t)_{t\geq 0}$ is also a spectrally negative Lévy process with Laplace exponent $\tilde{\psi}(u) = \psi(u) - ru$. Note that, by changing the value of the drift, one may also consider the case when the risky asset pays a continuous compound dividend yield at some fixed rate per annum. Next, we recall that the fundamental theorem of asset pricing, see Delbaen and Schachermayer [10], requires that $\tilde{S}$ is a (local)-martingale under a probability measure which is equivalent to the historical one. For the sake of simplicity, by assuming that

$$\psi(1) = r$$

we set $\mathbb{P}$ to be a risk-neutral probability measure. We mention that Eberlein et al. [14] give a complete description of the set of equivalent local martingale measures in the setting of Lévy-driven assets.

Let us now recall some basic properties of the Laplace exponent ψ which will be useful in the sequel. First, ψ is continuous on $\mathbb{R}^+$ with $\psi(0) = 0$ and $\lim_{u\to\infty} \psi(u) = +\infty$. By monotone convergence, one gets $\mathbb{E}[\xi_1] = \psi'(0^+) = \delta + \int_{-\infty}^{-1} y\, m(dy) \in [-\infty, \infty)$. Let us write

$$R = \lim_{u\to\infty} \frac{\psi(u)}{u}. \tag{2.3}$$

Then, R may take different values depending upon the coefficients of ψ. Indeed, if $\sigma = 0$ and $\int_{-\infty}^{0} 1 \wedge |y|\, m(dy) < \infty$, that is the Lévy process ξ has paths of bounded variation then we get, see [3, Corollary VII.5], $R = \bar{\delta}$ with

$$\bar{\delta} = \delta - \int_{-1}^{0} y\, m(dy). \tag{2.4}$$

Since we have excluded the degenerate cases, it is plain that $\bar{\delta} > 0$. For the other cases, $R = +\infty$. To summarize, we have

$$R = \begin{cases} \bar{\delta} & \text{if } \sigma = 0 \text{ and } \int_{-\infty}^{0} 1 \wedge |y| \, m(dy) < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

Note that 0 is always a root of the equation $\psi(u) = 0$. However, in the case $\mathbb{E}[\xi_1] < 0$, this equation admits another positive root, which we denote by θ. Moreover, for any $\mathbb{E}[\xi_1] \in [-\infty, \infty)$, the function $u \mapsto \psi(u)$ is continuous and increasing on $[\max(\theta, 0), \infty)$. Thus, it has a well-defined inverse function $\phi : [0, \infty) \to [\max(\theta, 0), \infty)$ which is also continuous and increasing.

2.2. **Asian options.** Let us start by introducing the so-called exponential functional of the Lévy process ξ which is defined, for any $0 \leq t_0 \leq t$, by

$$\Sigma_{t_0,t} = \int_{t_0}^{t} e^{\xi_s} ds. \tag{2.5}$$

Next, we set, for any $t_0, t \geq 0$,

$$\overline{\Sigma}_{t_0,t} = \frac{\Sigma_{t_0,t}}{t - t_0}.$$

We simply write $\Sigma_t = \Sigma_{0,t}$ and $\overline{\Sigma}_t = \overline{\Sigma}_{0,t}$. The payoff of the arithmetic Asian call option written at time $t_0 > 0$, with maturity $T > 0$ and fixed-strike price K is given by

$$\left(\overline{\Sigma}_{t_0,T} - K\right)_+ .$$

By an arbitrage argument, the value at time t of the Asian call option is

$$C_t(t_0, T) = e^{-r(T-t)} \mathbb{E}_x \left[\left(\overline{\Sigma}_{t_0,T} - K\right)_+ \middle| \mathcal{F}_t \right].$$

In the Black-Scholes model, Geman and Yor [19] showed that this conditional expectation could be factorized into simple terms. In what follows, we state the extension of their result to the general framework of Lévy processes whose proof is straightforward.

Proposition 2.1. *Let us assume that $\psi(1) = r$. Then, for any $t_0 \leq t < T$, we have*

$$C_t(t_0, T) = \frac{e^{-r(T-t)}}{T - t_0} S_t \, \mathbb{E} \left[\left(\widehat{\Sigma}_{T-t} - K' \right)_+ \right]$$

where $\widehat{\Sigma}_{T-t}$ is a copy of Σ_{T-t} independent of $\mathcal{F}_t$ and

$$K' = \frac{K(T - t_0) - \Sigma_{t_0,t}}{S_t}.$$

A direct consequence of the previous proposition is that the price of an Asian option depends on the first moment of the random variable $(\Sigma_t - K')_+$. Unfortunately, it is challenging mathematical problem to derive a tractable expression for this quantity. Instead, Geman and Yor [19] suggested to compute such a moment but for the exponential functional considered at some random time. More precisely, by replacing the time-dependent strike K' by a constant $a > 0$, we consider the function

$$\mathbb{E}\left[(\Sigma_{\mathbf{e}_q} - a)_+\right] = q \int_0^\infty e^{-qt} \mathbb{E}\left[(\Sigma_t - a)_+\right] dt$$

where $\mathbf{e}_q$ is an exponentially distributed random variable of parameter $q > 0$ and is taken independent of ξ. The value of the option is then obtained by inverting the above Laplace transform in time and by choosing $a = K'$.

2.3. **Law of the exponential functional.** It is now clear that to generalize the Geman-Yor formula to spectrally negative Lévy processes one has to compute the first truncated moment of the random variable $\Sigma_{\mathbf{e}_q}$. In this part, we recall a recent result obtained by Patie [33], [32] regarding the distribution of this positive random variable. To this end, we proceed by introducing some notation taken from [31]. First, let ψ be of the form (2.2) with $\psi'(0^+) \geq 0$. Then, set $a_0 = 1$ and for any $n = 1, 2, \ldots$,

$$a_n(\psi) = \left(\prod_{k=1}^{n} \psi(k)\right)^{-1}.$$

In [31], the author introduces the following power series

$$\mathcal{I}_\psi(z) = \sum_{n=0}^{\infty} a_n(\psi) z^n \tag{2.6}$$

and shows by means of classical criteria that the mapping $z \mapsto \mathcal{I}_\psi(z)$ is an entire function. Note that the condition $\psi'(0^+) \geq 0$ implies that all of the coefficients in the definition of $\mathcal{I}_\psi(z)$ are strictly positive. We refer to [31] for interesting analytical properties enjoyed by these power series and also for connections with well-known special functions, such as, for instance, the modified Bessel functions and several generalizations of the Mittag-Leffler function. Next, let G_ρ be a random variable having the Gamma distribution with parameter $\rho > 0$, that is its distribution is given by $g(dt) = \frac{e^{-t}t^{\rho-1}}{\Gamma(\rho)}dt, t > 0$, with Γ the Euler gamma function. Then, in [30], the author suggested the following generalization

$$\begin{aligned} \mathcal{I}_\psi(\rho; z) &= \mathbb{E}\left[\mathcal{I}_\psi\left(G_\rho z\right)\right] \\ &= \frac{1}{\Gamma(\rho)} \int_0^\infty e^{-t} t^{\rho-1} \mathcal{I}_\psi\left(tz\right) dt. \end{aligned}$$

By means of the integral representation of the Gamma function $\Gamma(\rho) = \int_0^\infty e^{-t}t^{\rho-1}dt$, $\mathfrak{Re}(\rho) > 0$, see e.g. [25, Chap. 1], and an argument of dominated convergence, one obtains the following power series representation

$$\mathcal{I}_\psi(\rho; z) = \frac{1}{\Gamma(\rho)} \sum_{n=0}^{\infty} a_n(\psi) \Gamma(\rho + n) z^n \tag{2.7}$$

which is easily seen to be valid for any $|z| < R$, where we recall that R is defined in (2.3). Moreover, for any $|z| < R$, the mapping $\rho \mapsto \mathcal{I}_\psi(\rho; z)$ is a meromorphic function defined for all complex numbers ρ except at the poles of the Gamma function, that is at the points $\rho = 0, -1, \ldots$ However, they are removable singularities. Indeed, for any $|z| < R$ and any integer $N \in \mathbb{N}$, one has, by means of the recurrence relation $\Gamma(z+1) = z\Gamma(z)$, $\mathcal{I}_\psi(0; z) = 1$ and

$$\mathcal{I}_\psi(-N; z) = \sum_{n=0}^{N} (-1)^n \frac{\Gamma(N+1)}{\Gamma(N+1-n)} a_n(\psi) z^n.$$

Thus, by uniqueness of the analytical extension for any $|z| < R$, $\mathcal{I}_\psi(\rho; z)$ is an entire function in ρ. Note also that for $\rho = 0, -1, \ldots$, as a polynomial, $\mathcal{I}_\psi(-\rho; z)$ is an entire function in z. In the following, we recall a result from [32] which summarizes the above claims and provide an analytical continuation of $\mathcal{I}_\psi(\rho; z)$ in the case $R = \bar{\delta}$, that is when ξ is with paths of bounded variation.

Proposition 2.2 (Patie [32]). *(1) If $R = \infty$, then $\mathcal{I}_\psi(\rho; z)$ is an entire function in both arguments z and ρ.*

(2) If $R = \bar{\delta}$, then $\mathcal{I}_\psi(\rho; z)$ is analytic in the disc $|z| < \bar{\delta}$ and for any fixed $\rho = 0, -1, \ldots$, $\mathcal{I}_\psi(\rho; z)$, as a polynomial, is an entire function. Moreover, for any $\rho \in \mathbb{C}$, $\mathcal{I}_\psi(\rho; z)$ admits, in the half-plane $\mathfrak{Re}(z) < \frac{\bar{\delta}}{2}$, the following power series representation

$$\mathcal{I}_\psi(\rho; z) = \left(1 - \frac{z}{\bar{\delta}}\right)^{-\rho} \sum_{n=0}^{\infty} \mathcal{I}_\psi(-n; \bar{\delta}) \frac{\Gamma(\rho + n)}{\Gamma(\rho) n!} \left(\frac{z}{z - \bar{\delta}}\right)^n. \tag{2.8}$$

Finally, for any fixed $\mathfrak{Re}(z) < \frac{\bar{\delta}}{2}$, $\mathcal{I}_\psi(\rho; z)$ is an entire function in the argument ρ.

We mention that a representation as a contour integral of the function $\mathcal{I}_\psi(\rho; z)$ is given in the appendix below. Next, we write, for any $q > 0$,

$$\gamma = \phi(q)$$

and we set

$$\psi_\gamma(u) \quad = \quad \psi(u + \gamma) - q, \quad u, q \geq 0.$$

ψ_γ is well known to be the Laplace exponent of the so-called Esscher transform of ξ. Thus, it is again the Laplace exponent of a spectrally negative Lévy process. Moreover, we have $\psi'_\gamma(0^+) = \psi'(\gamma) = \frac{1}{\phi'(q)} > 0$ since ϕ is the Laplace exponent of a subordinator and hence it is an increasing function. We are now ready to state the following result which provides an expression for the law of $\Sigma_{\mathbf{e}_q}$.

Theorem 2.3. *([33],[32, Theorem 2.1]) Let $q > 0$. Then, there exists a constant $C_\gamma > 0$ such that*

$$\mathcal{I}_{\psi_\gamma}(\gamma; -x) \quad \sim \quad \frac{x^{-\gamma}}{C_\gamma} \quad \text{as } x \to \infty. \tag{2.9}$$

($f(x) \sim g(x)$ as $x \to a$ means that $\lim_{x \to a} \frac{f(x)}{g(x)} = 1$ for any $a \in [0, \infty]$.) Moreover, the law of $\Sigma_{\mathbf{e}_q}$ under $\mathbb{P}$ is absolutely continuous with a density, denoted by s_γ, given by

$$s_\gamma(t) \quad = \quad \gamma C_\gamma t^{-\gamma - 1} \mathcal{I}_{\psi_\gamma}\left(1 + \gamma; -t^{-1}\right), \; t > 0. \tag{2.10}$$

Remark 2.4. If we assume that $\psi'(0^+) < 0$, which is equivalent, from the strong law of large numbers for Lévy processes, to $\lim_{t \to \infty} \xi_t = -\infty$ a.s., then we have $\lim_{q \to 0} \phi(q) = \theta > 0$, where we recall that $\psi(\theta) = 0$. Under this condition, the perpetual exponential functional $\Sigma_\infty = \int_0^\infty e^{\xi_s} ds$ is well defined and its density, denoted by s_θ, is obtained as follows

$$s_\theta(t) = \lim_{q \to 0} s_{\phi(q)}(t), \; t > 0.$$

The expression of $s_\theta(t)$ can be found in [32, Theorem 2.1] and generalizes a result of Dufresne [12] obtained in the case of the Brownian motion with a negative drift.

The proof of the theorem is rather technical but the main steps can be described as follows. First, we use the Lamperti mapping which allows to connect the law of the exponential functional $\Sigma_{\mathbf{e}_q}$ to the law of the absorption time of a positive self-similar Markov process generalizing the Bessel processes. Then, by means of the self-similarity property, we show that the law of this latter stopping time is related to the probability that the absorption time of an associated transient Ornstein-Uhlenbeck process is finite, which turns out to be a quantity much easier to compute. Let us mention that such devices hold in the framework of two-sided Lévy processes. Finally, we derive an expression for this probability by combining complex analysis techniques with fluctuation identities for positive self-similar Markov processes obtained recently in [30] and [31]. The extension of this part of the proof to more general Lévy processes seems difficult. Indeed, assuming that the process has two-sided jumps but admits all positive exponential moments which implies the existence of a Laplace exponent ψ, then it is a difficult matter, if true, to show that the mapping $\mathcal{I}_{\psi_\gamma}\left(1+\gamma; -t^{-1}\right)$ is non-negative valued for any $t > 0$ which is a necessary condition for the expression (2.10) to be a density.

3. A generalized Geman-Yor formula

According to the Proposition 2.1, the pricing of Asian option in the framework of Lévy processes amounts to computing the first moment of the random variable $(\Sigma_t - K)_+$. As already discussed in the previous section, this is a difficult task and instead we compute, for any $K > 0$, the following functional

$$\mathbb{E}[(\Sigma_{\mathbf{e}_q} - K)_+]$$

where we recall that $\mathbf{e}_q$ is an exponentially distributed random variable of parameter $q > 0$ which is taken independent of ξ. We now state the generalization of the Geman-Yor formula to spectrally negative Lévy processes.

Theorem 3.1. *For any $K > 0$ and $q > \psi(1)$, we have*

$$\mathbb{E}[(\Sigma_{\mathbf{e}_q} - K)_+] = \frac{C_\gamma}{\gamma - 1} K^{1-\gamma} \mathcal{I}_{\psi_\gamma}(\gamma - 1; -K^{-1}). \tag{3.1}$$

Proof. Let us consider, first, the Mellin transform of the positive random variable $\left(\Sigma_{\mathbf{e}_q} - K\right)_+$ which is defined, for $\kappa \in i\mathbb{R}$, the imaginary line, by

$$\mathcal{M}(\kappa) = \mathbb{E}[(\Sigma_{\mathbf{e}_q} - K)_+^{-\kappa}].$$

It is plain, if both quantities exist, that

$$\mathbb{E}[(\Sigma_{\mathbf{e}_q} - K)_+] = \mathcal{M}(-1).$$

Next, let us write $\gamma = \phi(q)$ and for any integer N

$$\mathcal{M}^N(\kappa) = \int_0^\infty (t - K)_+^{-\kappa} s_\gamma^N(t) dt$$

where $s_\gamma^N(t) = \gamma C_\gamma t^{-\gamma-1} \mathcal{I}^N_{\psi_\gamma}(\gamma; -t^{-1})$ and $\mathcal{I}^N_{\psi_\gamma}(\gamma; z)$ is the power series $\mathcal{I}_{\psi_\gamma}(\gamma; z)$ truncated at the order N. Now, we split the proof of the identity (3.1) into two parts.

First, we consider the case when $R = \infty$. From (2.7), we have, for any integer N,

$$\begin{aligned}\mathcal{M}^N(\kappa) &= \int_K^\infty (t-K)^{-\kappa} s_\gamma^N(t)dt \\ &= \frac{C_\gamma}{\Gamma(\gamma)} \sum_{n=0}^{N} (-1)^n a_n(\psi_\gamma)\Gamma(\gamma+1+n) \int_K^\infty \left(1-\frac{K}{t}\right)^{-\kappa} t^{-n-\kappa-\gamma-1}dt\end{aligned}$$

where we have used the recurrence formula of the Gamma function $\Gamma(z+1) = z\Gamma(z), \mathfrak{Re}(z) > 0$. Next, performing the change of variable $v = \frac{K}{t}$, we get

$$\begin{aligned}\mathcal{M}^N(\kappa) &= \frac{C_\gamma}{\Gamma(\gamma)} \sum_{n=0}^{N} (-1)^n a_n(\psi_\gamma)\Gamma(\gamma+1+n) K^{-n-\kappa-\gamma} \int_0^1 (1-v)^{-\kappa} v^{\gamma+\kappa+n-1}dv \\ &= \frac{C_\gamma \Gamma(1-\kappa)}{\Gamma(\gamma)} K^{-\kappa-\gamma} \sum_{n=0}^{N} a_n(\psi_\gamma)(-K)^{-n}\Gamma(\gamma+\kappa+n) \qquad (3.2)\end{aligned}$$

where the last line follows from the integral representation of the Beta function, see e.g. [20, Formula 3.191(1)],

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 (1-v)^{x-1}v^{y-1}dv, \ \mathfrak{Re}(x), \mathfrak{Re}(y) > 0.$$

By the principle of analytical continuation, we deduce that the identity (3.2) is valid in the strip $\mathcal{S}_\gamma = \{\kappa \in \mathbb{C};\ -\gamma < \mathfrak{Re}(\kappa) < 1\}$. Next, we have, for any $\kappa \in \mathcal{S}_\gamma$,

$$\lim_{N\to\infty} \mathcal{M}^N(\kappa) = \frac{C_\gamma \Gamma(\gamma+\kappa)\Gamma(1-\kappa)}{\Gamma(\gamma)} K^{-\kappa-\gamma} \mathcal{I}_{\psi_\gamma}\left(\gamma+\kappa; -K^{-1}\right).$$

The function on the right-hand side of the previous equality being holomorphic on the positive half-plane, we deduce by an argument of dominated convergence, see e.g. [29, Chap. 2, Theorem 8.1] that, for any $\mathfrak{Re}(K) > 0$ and $\kappa \in \mathcal{S}_\gamma$,

$$\mathcal{M}(\kappa) = \frac{C_\gamma \Gamma(\gamma+\kappa)\Gamma(1-\kappa)}{\Gamma(\gamma)} K^{-\kappa-\gamma} \mathcal{I}_{\psi_\gamma}\left(\gamma+\kappa; -K^{-1}\right).$$

Moreover, since ϕ is increasing on $\mathbb{R}^+$, our assumption leads to the condition $\gamma > 1$. Hence, by resorting again to the principle of analytical continuation and using the recurrence relation of the Gamma function we obtain

$$\mathbb{E}[(\Sigma_{\mathbf{e}_q} - K)_+] = \frac{C_\gamma}{(\gamma-1)} K^{1-\gamma} \mathcal{I}_{\psi_\gamma}\left(\gamma-1; -K^{-1}\right)$$

which proves our claim in the case $R = \infty$.

Next, assuming that $R = \bar{\delta} < \infty$ where we recall that $\bar{\delta}$ is defined in (2.4) and keeping the same notation as above, we have from (2.8)

$$\begin{aligned}
\mathcal{M}^N(\kappa) &= \int_K^\infty (t-K)^{-\kappa} s_\gamma^N(t)dt \\
&= C_\gamma \sum_{n=0}^N \mathcal{I}_{\psi_\gamma}(-n;\bar{\delta}) \frac{\Gamma(\gamma+1+n)}{n!\Gamma(\gamma)} \\
&\quad \int_K^\infty t^{-(\gamma+\kappa+n+1)} \left(1-\frac{K}{t}\right)^{-\kappa} \left(1+\frac{1}{\bar{\delta}t}\right)^{-(\gamma+1+n)} dt\bar{\delta}^{-n} \\
&= C_\gamma K^{-\gamma-\kappa} \sum_{n=0}^N \mathcal{I}_{\psi_\gamma}(-n;\bar{\delta}) \frac{\Gamma(\gamma+1+n)}{n!\Gamma(\gamma)} \\
&\quad \int_0^1 v^{\gamma+\kappa+n-1} (1-v)^{-\kappa} \left(1+\frac{v}{\bar{\delta}K}\right)^{-(\gamma+1+n)} dv(\bar{\delta}K)^{-n}
\end{aligned}$$

where we have performed the change of variable $v = \frac{u}{K}$. Next, by means of the following identity, which is found in [20, Formula 3.197(4)],

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}(1+a)^{-x} = \int_0^1 (1-v)^{x-1} v^{y-1} (1+av)^{-x-y} dv,\ \mathfrak{Re}(x), \mathfrak{Re}(y) > 0, a > -1,$$

we deduce that, for any $K > 0$ and $\kappa \in \mathcal{S}_\gamma$,

$$\begin{aligned}
\mathcal{M}^N(\kappa) &= \frac{C_\gamma K^{-\gamma-\kappa}\Gamma(1-\kappa)}{\Gamma(\gamma)} \sum_{n=0}^N \mathcal{I}_{\psi_\gamma}(-n;\bar{\delta}) \frac{\Gamma(\gamma+\kappa+n)}{n!} (\bar{\delta}K)^{-n} \left(1+\frac{1}{\bar{\delta}K}\right)^{-(\kappa+\gamma+n)} \\
&= \frac{C_\gamma K^{-\gamma-\kappa}\Gamma(1-\kappa)}{\Gamma(\gamma)} \left(1+\frac{v}{\bar{\delta}K}\right)^{-\gamma-1} \sum_{n=0}^N \mathcal{I}_{\psi_\gamma}(-n;\bar{\delta}) \frac{\Gamma(\gamma+\kappa+n)}{n!} \left(1+\bar{\delta}K\right)^{-n} \\
&= \frac{C_\gamma K^{-\gamma-\kappa}\Gamma(1-\kappa)\Gamma(\gamma+\kappa)}{\Gamma(\gamma)} \mathcal{I}_{\psi_\gamma}^N\left(\gamma+\kappa; -K^{-1}\right).
\end{aligned}$$

Hence, we get by dominated convergence for any $K > 0$ and $\kappa \in \mathcal{S}_\gamma$,

$$\mathcal{M}(\kappa) = \frac{C_\gamma \Gamma(\gamma+\kappa)\Gamma(1-\kappa)}{\Gamma(\gamma)} K^{-\kappa-\gamma} \mathcal{I}_{\psi_\gamma}\left(\gamma+\kappa; -K^{-1}\right).$$

The proof of the Theorem is then completed by following a line of reasoning similar to the previous case. □

Remark 3.2. (1) As observed by Geman and Yor [19] in the Black-Scholes model, one can also compute easily the value of the Asian call option under the Lévy model in the case the strike K is non positive. Indeed, we have

$$\begin{aligned}
C_0(0,T) &= e^{-rT}\left(S_0 \int_0^T \mathbb{E}[e^{\xi_s}]ds - K\right) \\
&= \frac{1}{r}\left(1-e^{-rT}\right) S_0 - e^{-rT}K
\end{aligned}$$

where we recall that $\psi(1) = r$.

(2) By means of the symmetry relationship, established by Henderson and Wojakowski [22] in the Black-Scholes model, see also Eberlein and Papapantoleon [15] for its extension to the Lévy processes markets, between floating-strike and fixed-strike Asian options for assets driven, one could also derive from the previous result the price of the floating-strike Asian put option.

4. **Examples**

4.1. **The Black-Scholes model revisited.** We first consider the case when S follows the Black-Scholes dynamics. That is, under the unique risk-neutral probability measure $\mathbb{P}$, ξ is given, for any $t \geq 0$, by

$$\xi_t = \sigma B_t + \delta t$$

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion, $\sigma > 0$ and $\delta = r - \frac{\sigma^2}{2}$. It is plain that

$$\psi(u) = \frac{\sigma^2}{2}u^2 + \delta u,\ u \geq 0,$$

and $\psi(1) = r$. Next, we observe that, for any $q > 0$, $\phi(q) = \frac{\sqrt{2}}{\sigma}\left(\sqrt{q + \frac{\delta^2}{2\sigma^2}} - \frac{\delta}{\sqrt{2}\sigma}\right)$. Thus,

$$\psi_\gamma(u) = \frac{\sigma^2}{2}u^2 + \left(\sigma^2\gamma + \delta\right)u,\ u \geq 0.$$

Moreover, setting $b = 2\gamma + \frac{2}{\sigma^2}\delta$, we have, for any $n \geq 1$,

$$\begin{aligned} a_n(\psi_\gamma)^{-1} &= \prod_{k=1}^{n} \psi_\gamma(k) \\ &= \frac{\sigma^{2n}}{2^n} n! \prod_{k=1}^{n} k + b \\ &= \frac{\sigma^{2n}\Gamma(n+b+1)}{2^n\Gamma(b+1)} n!. \end{aligned}$$

Since $R = \infty$, we have, for any $z, \rho \in \mathbb{C}$,

$$\begin{aligned} \mathcal{I}_{\psi_\gamma}(\rho; z) &= \frac{\Gamma(b+1)}{\Gamma(\rho)} \sum_{n=0}^{\infty} \frac{\Gamma(\rho+n)}{n!\Gamma(n+b+1)} \left(-\frac{2z}{\sigma^2}\right)^n \\ &= \Phi\left(\rho, b+1; -\frac{2z}{\sigma^2}\right) \end{aligned}$$

where Φ stands for the confluent hypergeometric function. We refer to Lebedev [25, Chap. 9] for useful properties of this function. Next, using the following asymptotic

$$\Phi(\rho, b+1; -x) \sim \frac{\Gamma(b+1)}{\Gamma(b+1-\rho)} x^{-\rho} \quad \text{as } x \to \infty,$$

we get, from (2.9), that

$$C_\gamma = \frac{\Gamma(b+1-\gamma)}{\Gamma(b+1)}\left(\frac{2}{\sigma^2}\right)^\gamma.$$

An application of Theorem 3.1 yields, for any $q > \frac{\sigma^2}{2} + \delta$,

$$\mathbb{E}[(\Sigma_{\mathbf{e}_q} - K)_+] \;=\; \frac{\Gamma(b+1-\gamma)}{2^\gamma \Gamma(b+1)} \frac{1}{\gamma - 1} K^{1-\gamma} \Phi\left(\gamma - 1, b+1; -\frac{2}{K\sigma^2}\right).$$

Next, using the following integral representation of the confluent hypergeometric function

$$\Phi(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt, \ \mathfrak{Re}(b) > \mathfrak{Re}(a) > 0,$$

we deduce that

$$\begin{aligned}
\mathbb{E}[(\Sigma_{\mathbf{e}_q} - K)_+] &= \left(\frac{2}{\sigma^2}\right)^\gamma \frac{K^{1-\gamma}}{\Gamma(\gamma)(b+1-\gamma)} \int_0^1 e^{-\frac{2}{K\sigma^2}u} u^{\gamma-2} (1-u)^{b-\gamma+1} du \\
&= \frac{2}{\sigma^2} \frac{1}{\Gamma(\gamma)(b+1-\gamma)} \int_0^{\frac{2}{K\sigma^2}} e^{-x} x^{\gamma-2} \left(1 - \frac{K\sigma^2}{2} x\right)^{b-\gamma+1} dx
\end{aligned}$$

where we have performed the change of variable $x = \frac{2}{K\sigma^2}u$. By choosing $\sigma = 2$ and $\delta = 2\nu$, we recover the formula obtained by Geman and Yor, see [19, Formula (3.10)].

4.2. **The completely asymmetric tempered stable processes.** We now consider an example where the dynamics of the asset price is governed by a pure jump process. More specifically, we assume that ξ is a spectrally negative tempered α-stable Lévy process with $1 < \alpha < 2$. Its Laplace exponent admits the following simple form

$$\psi(u) = (\sigma u + \beta)^\alpha - \beta^\alpha, \ u \geq 0,$$

where the parameters σ and β are positive constants. We assume that σ and β are chosen such that $\psi(1) = r$, that is $\sigma = (r + \beta^\alpha)^{\frac{1}{\alpha}} - \beta$. These parametric Lévy processes are specific instances of the a family of truncated Lévy processes constructed by Boyarchenko and Levendorskii [5]. Note that if $\beta = 0$ then ψ boils down to the Laplace exponent of a spectrally negative α-stable Lévy process. Moreover, in the limit case $\alpha = 2$, we recover the Black-Scholes model with variance $2\sigma^2$ and drift $2\sigma\beta$. The Lévy measure of ξ is absolutely continuous with a density, v, given by

$$v(y) = C \frac{e^{\beta y}}{|y|^{\alpha+1}}, \quad y < 0,$$

for some constant $C > 0$. The inverse function of ψ is $\phi(q) = \frac{1}{\sigma}(q + \beta^\alpha)^{\frac{1}{\alpha}} - \beta$, $q > 0$, and

$$\psi_\gamma(u) = \sigma^\alpha \left(\left(u + d^{\frac{1}{\alpha}}\right)^\alpha - d\right)$$

where $d = \frac{\beta^\alpha + q}{\sigma^\alpha}$. Note that $R = \infty$. Then,

$$a_n(\psi_\gamma)^{-1} = \sigma^{\alpha n} \prod_{k=1}^n \left(\left(k + d^{\frac{1}{\alpha}}\right)^\alpha - d\right), \quad a_0(\psi_\gamma) = 1.$$

Such an expression motivates us to introduce a generalization of the Pochhammer symbol which is defined as $(z)_\gamma = \frac{\Gamma(z+\gamma)}{\Gamma(z)}, \mathfrak{Re}(z), \mathfrak{Re}(\gamma) > 0$. We define, for $n \in \mathbb{N}$, $\alpha > 0$ and $z \in \mathbb{C}, \mathfrak{Re}(z) \geq 0$, by

$$(z)_{n,\alpha} = \prod_{k=1}^{n} \left((k+z)^{\alpha} - z^{\alpha}\right) \text{ and } (z)_{0,\alpha} = 1.$$

Note the identities $(\beta)_{n,1} = (\beta)_n$, $(0)_{n,\alpha} = (1)_n^{\alpha}$, $(\beta)_{n,2} = (1)_n(2\beta+1)_n$. Using this notation, we get the following power series for any $z, \rho \in \mathbb{C}$,

$$\mathcal{I}_{\psi_\gamma}(\rho; z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\rho)}{(d^{\frac{1}{\alpha}})_{n,\alpha}\Gamma(\rho)} \left(-\frac{z}{\sigma^{\alpha}}\right)^n,$$

which can be expressed in terms of the confluent hypergeometric function in the case $\alpha = 2$. Finally, we obtain, with C_γ given in (A.2),

$$\mathbb{E}[(\Sigma_{\mathbf{e}_q} - K)_+] = \frac{C_\gamma}{\gamma - 1} K^{1-\gamma} \mathcal{I}_{\psi_\gamma}\left(\gamma - 1; -K^{-1}\right).$$

4.3. **The compound Poisson process with drift.** Finally, we consider as the last example a pure jump process but with finite activity. More precisely, we assume that the dynamics of ξ is given, for any $t \geq 0$, by

$$\xi_t = \delta t - \sum_{i=0}^{N_t} X_i$$

where $\delta > 0$, $N = (N_t)_{t\geq 0}$ is a Poisson process of parameter $p > 0$ and the random variables $X_0, X_1, \ldots$ are i.i.d. with common distribution the exponential law of parameter $e > 0$. The laplace exponent of ξ admits the following form

$$\psi(u) = u\frac{\delta u + \delta e - p}{u + e}, \quad u \geq 0.$$

It is easily seen that the condition $\delta = r + \frac{p}{1+e}$ gives $\psi(1) = r$. Moreover, a straightforward computation yields

$$\phi(q) = \frac{1}{2\delta}\left(\sqrt{4e\delta q + (\delta e - p - q)^2} + q + p - \delta e\right), \quad q \geq 0,$$

and

$$\psi_\gamma(u) = \frac{\delta}{a} u \frac{u+b}{u+a}$$

where $a = \gamma + e$ and $b = a - \frac{pe}{\delta}$. Thus,

$$a_n(\psi_\gamma) = \left(\frac{a}{\delta}\right)^n \frac{\Gamma(n+a+1)\Gamma(b+1)}{\Gamma(n+b+1)\Gamma(n+1)\Gamma(a+1)}, \quad a_0(\psi_\gamma) = 1,$$

and, for $|z| < \frac{\delta}{a}$ and $\rho \in \mathbb{C}$, we have

$$\begin{aligned} \mathcal{I}_{\psi_\gamma}(\rho; z) &= \frac{\Gamma(b+1)}{\Gamma(\rho)\Gamma(a+1)} \sum_{n=0}^{\infty} \frac{\Gamma(\rho+n)\Gamma(n+a+1)}{\Gamma(n+b+1)n!} \left(-\frac{az}{\delta}\right)^n \\ &= {}_2F_1\left(\rho, a+1; b+1; -\frac{az}{\delta}\right) \end{aligned}$$

where ${}_2F_1$ stands for the hypergeometric function, see Lebedev [25, Chap. 9] for a detailed account on this function. Next, recalling the remarkable identity

$$ {}_2F_1(-n,1+a;p;1) = \frac{\Gamma(b+1)\Gamma(n+1-p-a)}{\Gamma(b+1+n)\Gamma(1-p-a)}, $$

we recover from (2.8) the well-known Euler transformation

$$ {}_2F_1(\rho,1+a;b+1;z) = (1-z)^{-\rho}{}_2F_1\left(\rho,1-p-a;p;\frac{z}{z-1}\right), \quad |\arg(1-z)| < \pi, $$

which provides an analytical continuation of the hypergeometric function into the half plane $\mathfrak{Re}(z) < \frac{1}{2}$. Then, using the following asymptotic

$$ {}_2F_1(\rho,1+a;b+1;-x) \sim \frac{\Gamma(b+1)\Gamma(a+1-\rho)}{\Gamma(b+1-\rho)\Gamma(a+1)}x^{-\rho} \quad \text{as } x\to\infty, $$

one obtains that

$$ \begin{aligned} C_\gamma &= \frac{\Gamma(b+1-\gamma)\Gamma(a+1)}{\Gamma(b+1)\Gamma(a+1-\gamma)} \\ &= \frac{\Gamma\left(\frac{e(\delta-p)}{\delta}+1\right)\Gamma\left(\gamma+e+1\right)}{\Gamma\left(\gamma+\frac{e(\delta-p)}{\delta}+1\right)\Gamma(e+1)}. \end{aligned} $$

Finally, we get

$$ \mathbb{E}[(\Sigma_{\mathbf{e}_q}-K)_+] = \frac{C_\gamma}{(\gamma-1)}K^{1-\gamma}{}_2F_1\left(\gamma-1,\gamma+e+1;\gamma+\frac{e(\delta-p)}{\delta}+1;-\frac{a}{\delta K}\right). $$

Appendix A. Some additional formulae

We start the appendix by providing a contour integral representation of the function $\mathcal{I}_{\psi_\gamma}$ which is found in [32, Proposition 2.4, Proposition 3.1]. To this end, we recall that $\gamma = \phi(q)$ and let us observe that, in the case $R=\infty$, as $0 < \psi'_\gamma(0^+) < \infty$, we have, for any $u > 0$

$$ \begin{aligned} \psi_\gamma(u) &= \hat{\delta}_\gamma u + \frac{\sigma^2}{2}u^2 + \int_{-\infty}^0 (e^{uy}-1-uy)e^{\gamma y}m(dy) \\ &= u^2\bar{\varphi}_\gamma(u) \end{aligned} $$

where $\hat{\delta}_\gamma = \delta+\sigma\gamma+\int_{-\infty}^0\left(e^{\gamma y}-\mathbb{I}_{\{|y|<1\}}\right)y\,m(dy)$ and

$$ \bar{\varphi}_\gamma(u) = \frac{\hat{\delta}_\gamma}{u}+\frac{\sigma^2}{2}+\int_0^\infty e^{-uy}\int_{-\infty}^{-y}\int_{-\infty}^{-s}e^{\gamma v}m(dv)dsdy. $$

Thus, one may define the function

$$ \begin{aligned} a_s(\psi_\gamma) &= \frac{1}{\Gamma^2(s+1)}a_s(\bar{\varphi}_\gamma) \\ &= \frac{1}{\Gamma^2(s+1)}\prod_{k=1}^{\infty}\frac{\bar{\varphi}_\gamma(k+s+1)}{\bar{\varphi}_\gamma(k)} \end{aligned} $$

and observe the identity

$$a_{s+1}(\bar{\varphi}_\gamma) = \frac{1}{\bar{\varphi}_\gamma(s+1)} a_s(\bar{\varphi}_\gamma)$$

with $a_0(\bar{\varphi}_\gamma) = 1$. Hence $a_s(\bar{\varphi}_\gamma)$ is a meromorphic function in $F_{-\gamma} = \{z \in \mathbb{C};\ \mathfrak{Re}(z) > -\gamma - 1\}$ with simple poles at the points $z_k = -k-1$ for $k = 0, 1, \ldots$ and $z_k > -\gamma - 1$.

Proposition A.1. *Let us assume that $R = \bar{\delta}$. Then, for any $\rho \neq 0, -1, \ldots$, $\mathcal{I}_{\psi_\gamma}(\rho; .)$ admits an analytical continuation in the entire complex plane cut along the positive real axis given by*

$$\mathcal{I}_{\psi_\gamma}(\rho; z) = \frac{1}{2i\pi\Gamma(\rho)} \int_{-i\infty}^{i\infty} a_s(\varphi_\gamma)\Gamma(s+\rho)\Gamma(-s) z^s ds, \quad |\arg(z)| < \pi, \tag{A.1}$$

where the contour is indented to ensure that all poles (resp. nonnegative poles) of $\Gamma(s+\rho)$ (resp. $\Gamma(-s)$) lie to the left (resp. right) of the intended imaginary axis and for any $\mathfrak{Re}(s) > -1$

$$a_s(\varphi_\gamma) = \prod_{k=1}^{\infty} \frac{\varphi_\gamma(k+s+1)}{\varphi_\gamma(k)}$$

with $\varphi_\gamma(s) = \bar{\delta}_\gamma - \widehat{v}_\gamma(s)$ and $\widehat{v}_\gamma(s) = \int_0^\infty e^{-sr} \int_{-\infty}^{-r} e^{\gamma v} m(dv) dr$.

Otherwise, if $R = \infty$, using the same contour as above, we have

$$\mathcal{I}_{\psi_\gamma}(\rho; z) = \frac{1}{2i\pi\Gamma(\rho)} \int_{-i\infty}^{i\infty} a_s(\bar{\varphi}_\gamma) \frac{\Gamma(s+\rho)}{\Gamma(s+1)} \Gamma(-s) z^s ds$$

which is valid in the sector $|\arg(z)| < \pi/2$.

We now provide some representations of the constant appearing in the asymptotic (2.9) in terms of the Laplace exponent ψ.

Proposition A.2. *If $R = \bar{\delta}$ then*

$$C_\gamma = a_{-\gamma}(\varphi_\gamma).$$

Otherwise, we have, writing $\psi(u) = u\varphi(u)$,

$$C_\gamma = \begin{cases} \psi_\gamma'(0^+) & \text{if } \gamma = 1 \\ \psi_\gamma'(0^+) \left(\prod_{k=1}^n \varphi(k)\right)^{-1} & \text{if } \gamma = n+1,\ n = 1, 2 \ldots \\ \frac{1}{\Gamma(1-\gamma)} a_{-\gamma}(\bar{\varphi}_\gamma) & \text{otherwise.} \end{cases} \tag{A.2}$$

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