Bayesian Multiple Target Localization

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Here’s the program

1. Study a Bayesian active learning problem.
2. Use our results to find faces in images like these:
This is our Bayesian active learning problem: Localize targets by counting

- There are \( k \) targets in some set \( \Omega = [0, 1] \).
- Their (unknown) locations are \( \theta_1, \ldots, \theta_k \in \Omega \), drawn iid from a prior density.
- For \( n = 1, \ldots, N \)
  - We choose \( A_n \subseteq \Omega \)
  - We observe \( X_n \).
    \( X_n \)’s distribution depends on the number of targets in \( A_n \).
- **Goal**: choose the questions to reduce uncertainty in the target locations, as measured by the expected entropy of the posterior.
The dyadic policy constructs $A_n$ recursively:

- We partition $\Omega$ into $2^n$ regions.
- $A_n$ is the union of half of these regions.
- To obtain the partition for $A_{n+1}$, we take the partition for $A_n$ and split each region into two sub-regions with equal mass under the prior (but not necessarily equal size, if the prior is not uniform).

![Diagram showing the dyadic policy]

The diagram illustrates the progression of the dyadic policy for $n=1$, $n=2$, and $n=3$. The regions are partitioned recursively, with each subsequent level dividing the previous regions in half.
The dyadic policy can be easily parallelized

- The dyadic policy’s questions $A_n$ do not depend on the answers $X_n$.
- It is non-adaptive.
- This makes it easy to parallelize.
- We compare it with OPT, the value of an optimal adaptive policy.
The dyadic policy is a constant factor approximation to the optimal adaptive policy

**Theorem**

\[
\frac{\text{DYADIC}}{\text{OPT}} \geq \frac{D_k}{C_k}.
\]

- \(\text{DYADIC} = H(p_0) - E^{\text{DYADIC}}[H(p_N)]\) is the expected entropy reduction under the dyadic policy.
- \(\text{OPT} = H(p_0) - \inf_\pi E^\pi[H(p_N)]\) is the expected entropy reduction under an optimal adaptive policy.
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  is the expected entropy reduction under the dyadic policy.
- \(\text{OPT} = H(p_0) - \inf_\pi E^\pi[H(p_N)]\)
  is the expected entropy reduction under an optimal adaptive policy.
- \(C_k\) is the channel capacity,
  \[
  C_k = \sup_q H\left(\sum_{z=0}^k q(z)f(\cdot|z)\right) - \sum_{z=0}^k q(z)H(f(\cdot|z))
  \]
- \(D_k\) is this explicit expression,
  \[
  D_k = H\left(\frac{1}{2^k}\sum_{z=0}^k \binom{k}{z} f(\cdot|z)\right) - \frac{1}{2^k}\sum_{z=0}^k \binom{k}{z} H(f(\cdot|z))
  \]
The dyadic policy is a 2-approximation to OPT in the noise-free case

When observations are **noise-free**:  
- \( D_k = H(\text{Bin}(k, \frac{1}{2})) \geq \frac{1}{2} \) and \( C_k = \log(k + 1) \), implying  
  \[
  \frac{\text{DYADIC}}{\text{OPT}} \geq \frac{H(\text{Bin}(k, \frac{1}{2}))}{\log(k + 1)} \geq \frac{1}{2}
  \]

- If \( k = 1 \) (there is a single target), then \( \frac{\text{DYADIC}}{\text{OPT}} \geq 1 \) and the dyadic policy is optimal among adaptive policies.

- Additional result not in the paper:  
  In the noise-free case, for \( k \geq 1 \),  
  the dyadic policy is optimal among non-adaptive policies.
Localizing $k$ objects simultaneously is much faster than localizing them one at a time.

- Plot shows:
  \[
  \frac{\text{(\# of questions to reduce total entropy by } k \times 20 \text{ bits)}/k}{k}
  \]
- Benchmark 1 localizes targets one-at-a-time using bisection.
- Benchmark 2 is sequential bifurcation (Bettonvil and Kleijnen 1997).
- Green and blue lines come from our bounds.
Proof Sketch

**Theorem**

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- Expected entropy reduction of the optimal: \( \text{OPT} \leq C_k N \).
Proof Sketch

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- Expected entropy reduction of the dyadic: \( \text{DYADIC} = D_k N \).
Proof Sketch

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- Expected entropy reduction of the optimal: \( \text{OPT} \leq C_k N \).
- Expected entropy reduction of the dyadic: \( \text{DYADIC} = D_k N \).
  - The answers to the dyadic questions are **independent**.
  - The entropy reduction under the dyadic is the sum of the entropy reduction due to each question.
  - The number of targets in \( A_n \) is Binomial\( (k, \frac{1}{2}) \), allowing us to compute the entropy reduction from this question.
The dyadic policy can be used to localize multiple object instances in computer vision

Let's build a tool that can find faces in images like this:
The standard approach applies a high accuracy classifier at each pixel in the image.

We have access to a high-accuracy classifier.

- input is the image, and a pixel within the image.
- output is whether or not there is a face centered at that pixel.
- we use a boosted collection of 500 stumps (features are windowed sums of oriented gradients)

Standard practice is to run the high-accuracy classifier at each pixel within the image to find all of the faces.

- But, this is slow.
We use the dyadic policy to localize multiple object instances

Instead, we do this:

- **(Screening)** Use the dyadic policy, together with a fast low-accuracy classifier, to compute a posterior distribution on object instance locations.

- **(Refinement)** Using the posterior to prioritize pixels, run a slow high-accuracy classifier on a (hopefully small) number of pixels to confirm the object instance locations.

Fewer calls to the high-accuracy classifier vs. the standard approach makes it faster.
Screening Step 1:
Run a fast low-accuracy classifier at all pixels

Step 1: Run a fast low-accuracy classifier at each pixel in the image.

- Our low-accuracy classifier is a boosted collection of 50 stumps.
- 50 stumps is 10% of the high-accuracy classifier.
Step 2: Compute the answers to the dyadic questions

- $X_n$ is the sum of the low-accuracy response over the region $A_n$.
- The regions $A_n$, $n = 1, \ldots, 8$, are
  - $X_n$ can be computed quickly using an integral image.
Screening Step 3: Compute the posterior

Under other policies, computing the posterior is slow (needs MCMC).

Under the dyadic policy, **computing the posterior is fast**:

- Let $C = \cap_{n=1}^{N} B_n$, where each $B_n$ is either $A_n$ or $\Omega \setminus A_n$.
- The expected number of targets in $C$ under the posterior is

  \[
  E[\text{number of targets in } C|X_{1:N}] = k \prod_{n=1}^{N} \left( \frac{e_n}{k} \right)^{s_n} \left( 1 - \frac{e_n}{k} \right)^{1-s_n},
  \]

  where

  - $s_n = 1\{B_n = A_n\}$.
  - $e_n = E[\text{number of targets in } A_n|X_n] = \sum_{j=0}^{k} j P(X_n=x_n|Z_n=j) P(Z_n=j) / P(X_n=x_n)$,
Following screening, we do refinement. We use one of these algorithms.

- **Posterior Rank (PR):** Rank pixels by the marginal probability of containing a target, and apply the high-accuracy classifier in this order.

- **Iterated Posterior Rank (IPR):** Like Iterated Rank, but when a target is found, we “mask” the target and recompute the posterior and pixel order. Masking is done without recomputing the answers to the dyadic questions:

\[
E[\text{number of new targets in } C|X_{1:N}] = \frac{k}{N} \prod_{n=1}^{N} \left( \frac{e'_n}{k} \right)^{s_n} \left( 1 - \frac{e'_n}{k} \right)^{1-s_n},
\]

where \(e_n\) has been replaced by \(e'_n\), the expected number of new targets in \(A_n\), given \(X_n\) and the locations of discovered targets.
Example: Seinfeld, Iterated Posterior Rank

Iteration 1
Example: Seinfeld, Iterated Posterior Rank

Iteration 2
Example: Seinfeld, Iterated Posterior Rank

Iteration 3
Example: Seinfeld, Iterated Posterior Rank

Iteration 4
The dyadic policy reduces the number of times we call the slow high-accuracy classifier on real images.

Results for 35 images from the MIT+CMU corpus.
- Dataset contains 130 faces (3.7 faces per image on average)
- Dataset contains $1.2 \times 10^7$ pixels
- We found all the faces by running the high-accuracy classifier on only 3% of the pixels using posterior rank.
- A naive approach would run it on more than 50% of the pixels.
The dyadic policy reduces the number of times we call the slow high-accuracy classifier on simulated images.

- IR = iterated rank (baseline that goes in arbitrary order)
- (I)PR = (iterated) posterior rank (our algorithms)
- EP = entropy pursuit
Future Work

- Practical improvements for computer vision
  - Unknown $k$ (easy)
  - Integration of better weak / strong classifiers (easy)
  - Likelihoods that depend on the size of the queried region (hard)
  - Separate multiple scales (hard)

- Other applications
  - Screening for important input factors to computational codes.
  - Group testing [our model generalizes the classical model]
  - Combinatorial chemistry
  - Heavy hitter detection in network analysis
Thank You!
Backup
Example: The Doors, Posterior Rank
Simulated Image: Iterated Posterior Rank
Simulated Image: Iterated Posterior Rank
Simulated Image: Iterated Posterior Rank
Simulated Image: Iterated Posterior Rank
Integral Images

- \text{Integral}(x, y) is the sum of the low-accuracy classifier’s responses at all pixels below and to the left of y.
- \text{Integral}(x, y) can be computed iteratively for all \( x \) and \( y \) by touching each pixel once.
- Sums over rectangles can be computed quickly from these values: sub within box = \text{integral}(\text{upper left}) - \text{integral}(\text{upper right}) - \text{integral}(\text{lower left}) + \text{integral}(\text{lower right})
The dyadic policy reduces the number of times we call the slow high-accuracy classifier on simulated images.