Probabilistic Bisection Search for Stochastic Root-Finding

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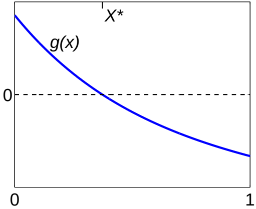
Stochastic Root-Finding Problem



- Consider a function $g : [0,1] \to \mathbb{R}$.
- Assumption: There exists a unique $X^* \in [0,1]$ such that
    - $g(x) > 0$ for $x < X^*$,
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Goal: Find $X^* \in [0,1]$.

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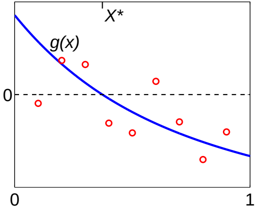

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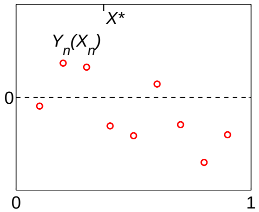

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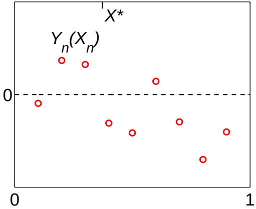

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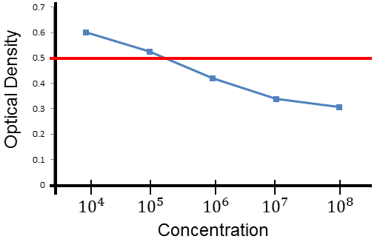
  Decisions:
  - Where to place samples X_n for $n = 0, 1, 2, \ldots$
  - How to estimate X^* after n iterations.

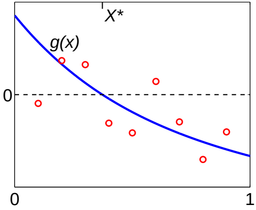
Applications

- Simulation optimization:
  - $g(x)$ as a gradient
- Sequential statistics:



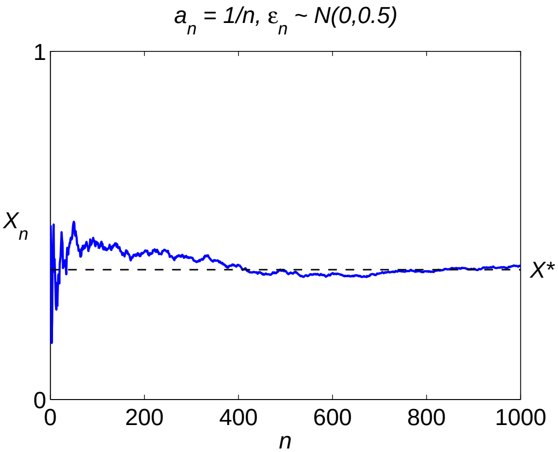
- Finance:
  - Pricing American options
  - Estimating risk measures

Stochastic Approximation (Robbins and Monro, 1951)

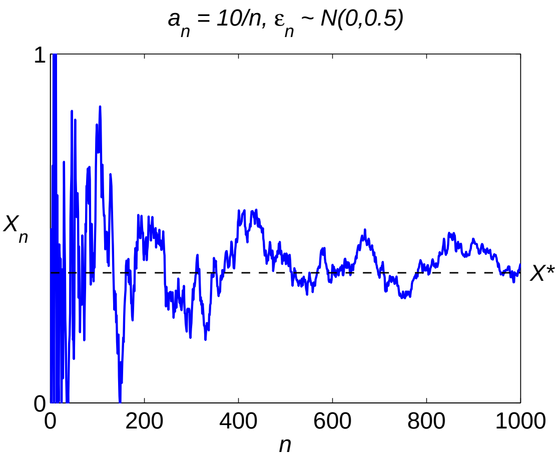


1. Choose an initial estimate $X_0 \in [0,1]$;
2. Determine a tuning sequence $(a_n)_n \geq 0$, $\sum_{n=0}^{\infty} a_n^2 < \infty$, and $\sum_{n=0}^{\infty} a_n = \infty$.
   (Example: $a_n = d/n$ for $d > 0$.)
3. $X_{n+1} = \Pi_{[0,1]}(X_n + a_n Y_n(X_n))$, where $\Pi_{[0,1]}$ is the projection to $[0,1]$.

A "Good" Tuning Sequence



A "Bad" Tuning Sequence – too Large



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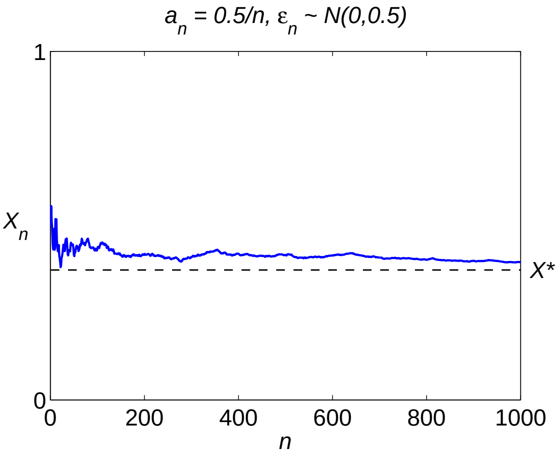
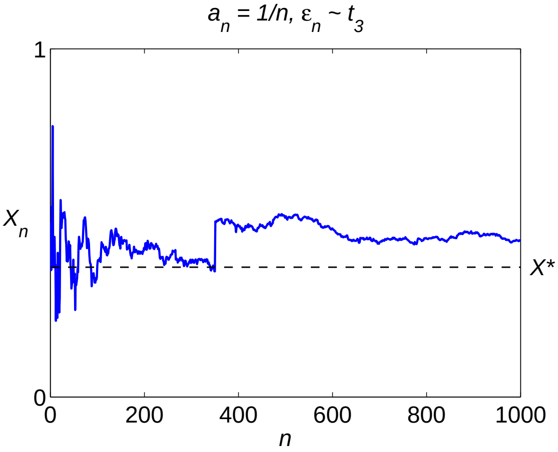
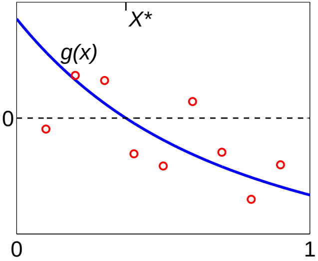


Lack of Robustness – ε_n with Heavy-Tails



A Different Approach

What about a bisection algorithm?



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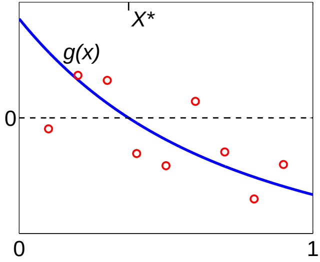
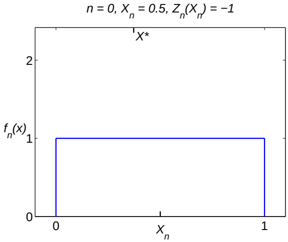


- Deterministic bisection algorithm will fail almost surely.
- Need to account for the noise.

The Probabilistic Bisection Algorithm

The Probabilistic Bisection Algorithm (Horstein, 1963)

- Input: $Z_n(X_n) := \text{sign}(Y_n(X_n))$.
- Assume a prior density f_0 on $[0, 1]$.



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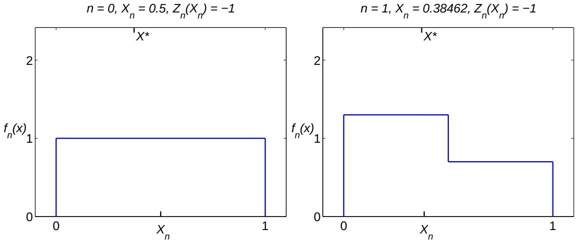

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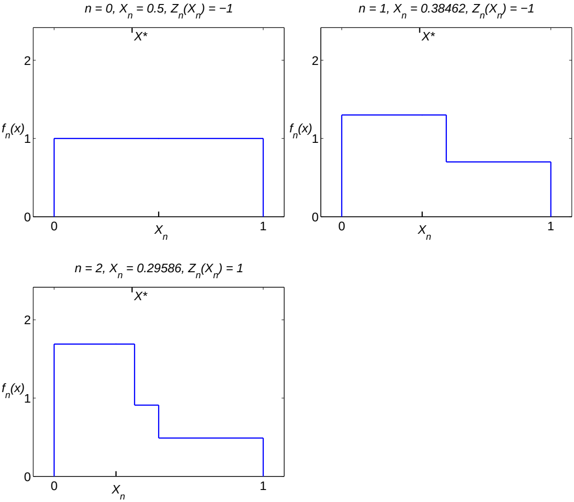

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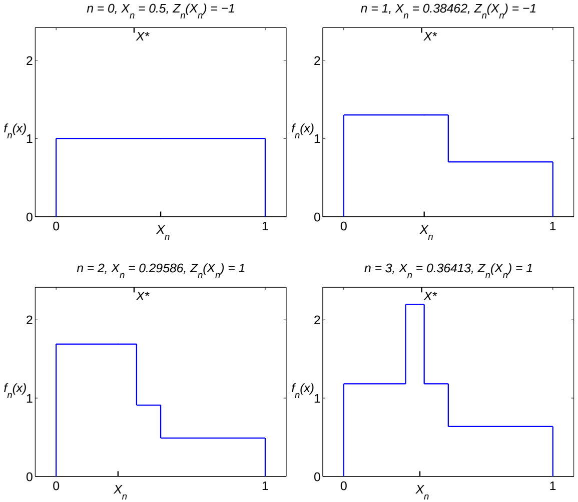
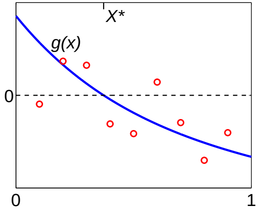


Stochastic Root-Finding Revisited



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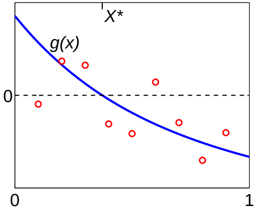

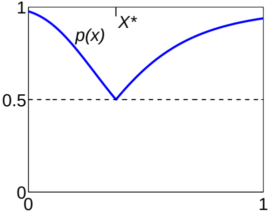

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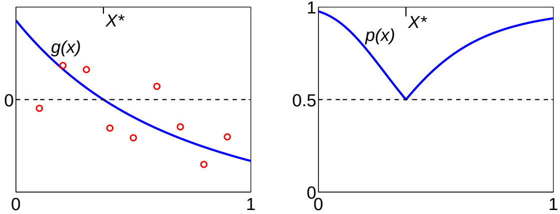

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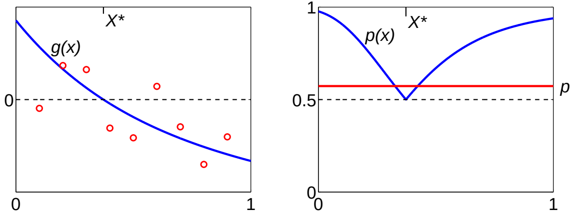

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  - $p(\cdot)$ is constant.

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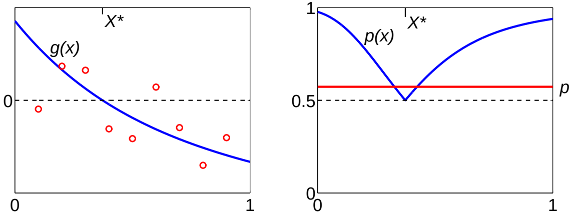

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- Stylized Setting:
  - $p(\cdot)$ is constant.
  - $p(\cdot)$ is known.

Stylized Setting

- $g(x)$ is a step function with a jump at X^*, for example, in edge detection applications (Castro and Nowak, 2008).

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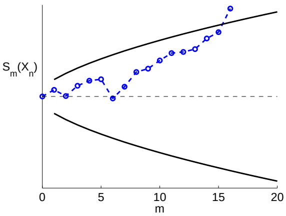
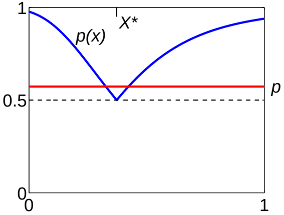
- Sample sequentially at point X_n and use $S_m(X_n) = \sum_{i=1}^m Y_{n,i}(X_n)$ to construct an α-*level test of power 1* (Siegmund, 1985):

$$N_n = \inf\left\{ m : |S_m| \geq [(m+1)(\log(m+1) + 2\log(1/\alpha))]^{1/2} \right\}.$$

Then $\mathbb{P}_{X_n = X^*}\{N_n < \infty\} \leq \alpha$, $\mathbb{P}_{X_n \neq X^*}\{N_n < \infty\} = 1$, and

$$\mathbb{P}_{X_n < X^*}\{S_{N_n}(X_n) > 0\} \geq 1 - \alpha/2 = p_c,$$
$$\mathbb{P}_{X_n > X^*}\{S_{N_n}(X_n) < 0\} \geq 1 - \alpha/2 = p_c.$$





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1. Place a prior density f_0 on the root X^*, f_0 has domain $[0, 1]$.
   Example: $U(0, 1)$.
2. For n=0,1,2, ...
   (a) Measure at the median $X_n := F_n^{-1}(1/2)$.
   (b) Update the posterior density:

$$\text{if } Z_n(X_n) = +1, \qquad f_{n+1}(x) = \begin{cases} 2p_c \cdot f_n(x), & \text{if } x > X_n, \\ 2q_c \cdot f_n(x), & \text{if } x \leq X_n, \end{cases}$$

$$\text{if } Z_n(X_n) = -1, \qquad f_{n+1}(x) = \begin{cases} 2q_c \cdot f_n(x), & \text{if } x > X_n, \\ 2p_c \cdot f_n(x), & \text{if } x \leq X_n. \end{cases}$$

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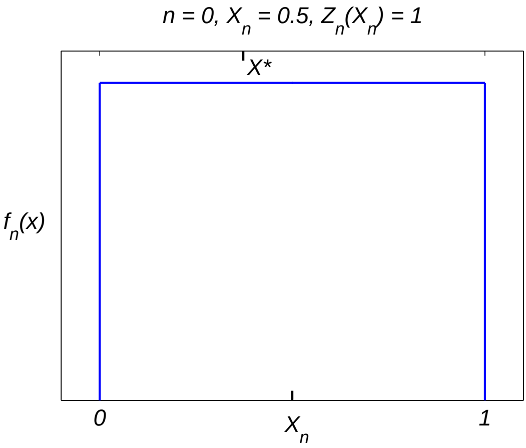

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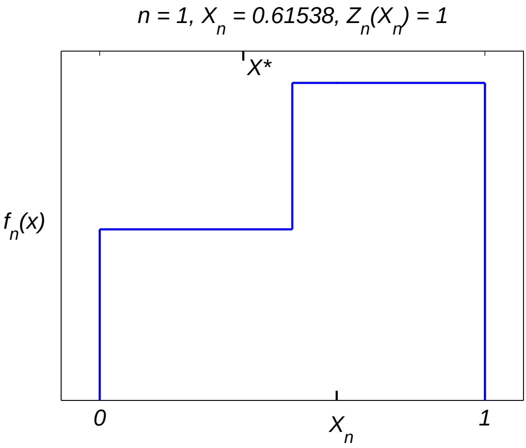

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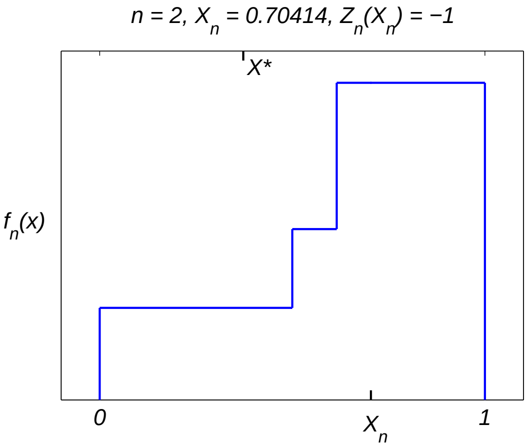

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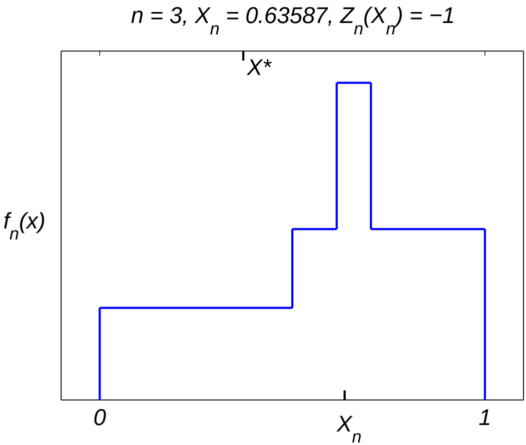

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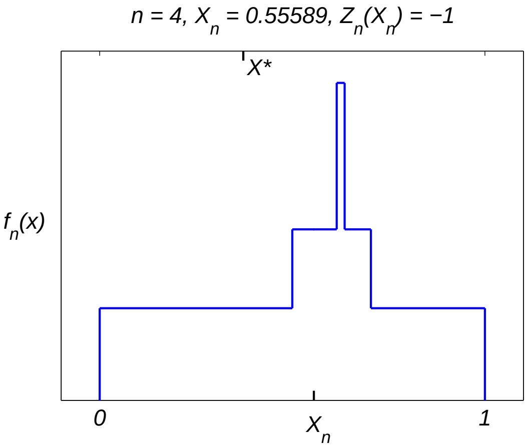

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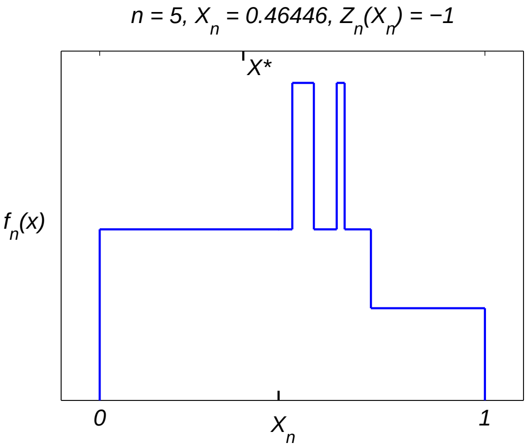

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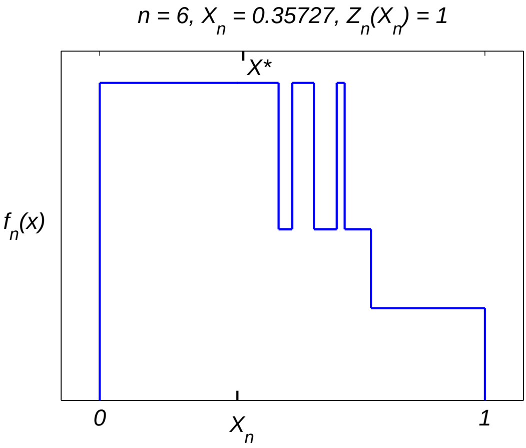

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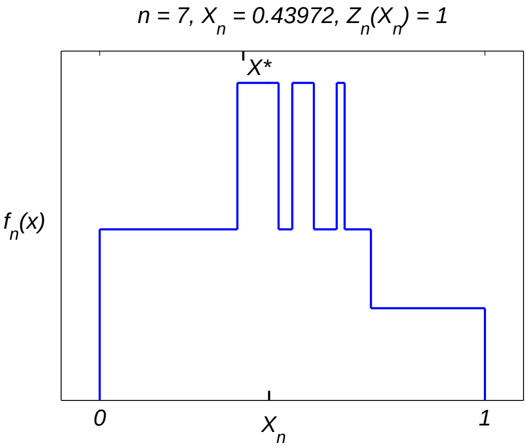

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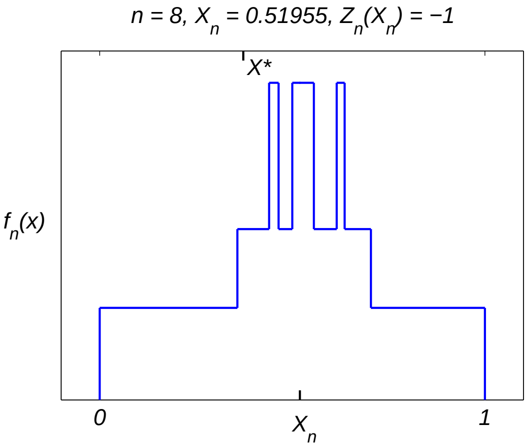

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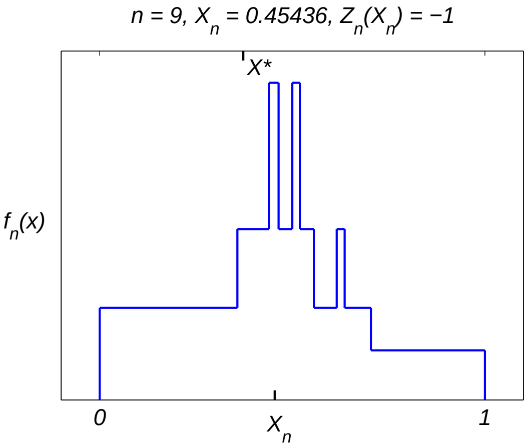

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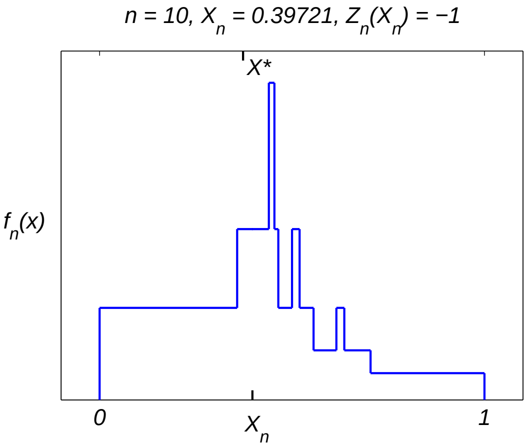

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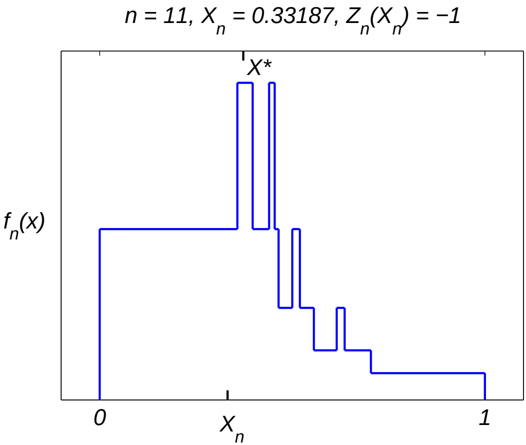

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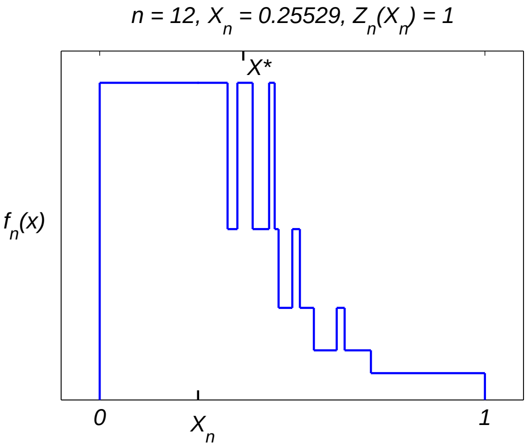

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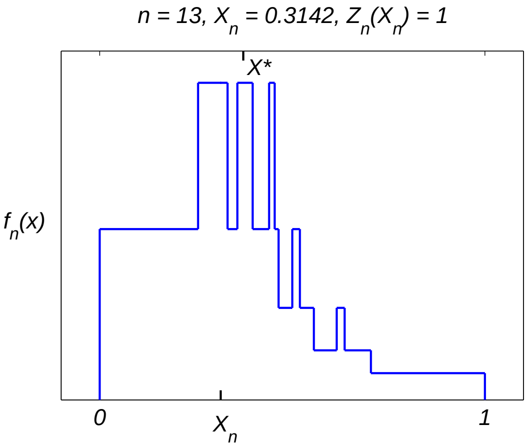

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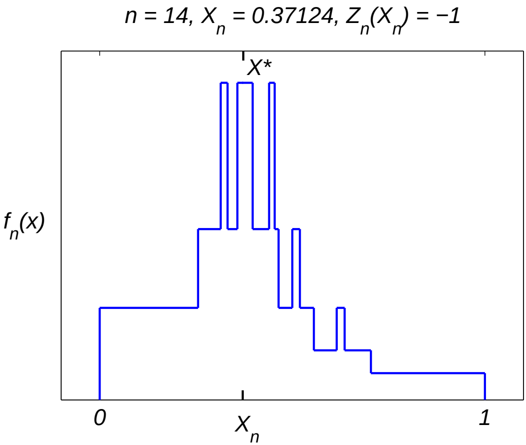

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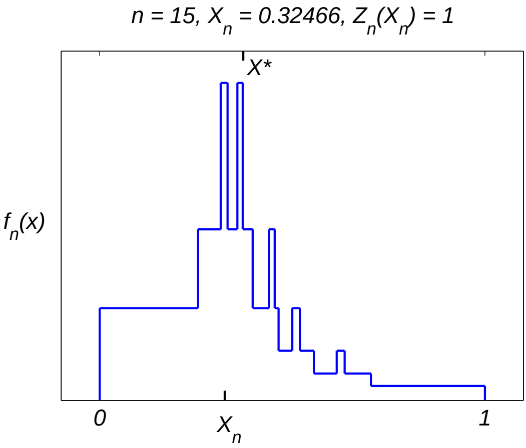

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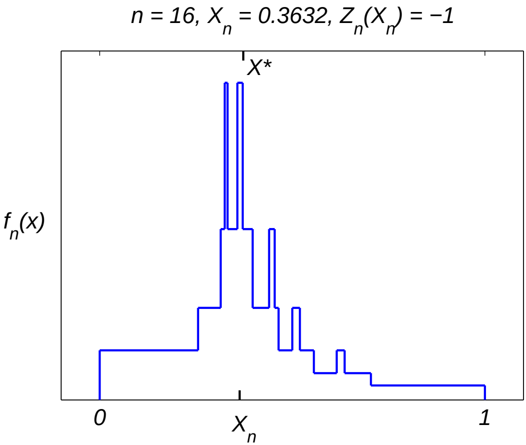

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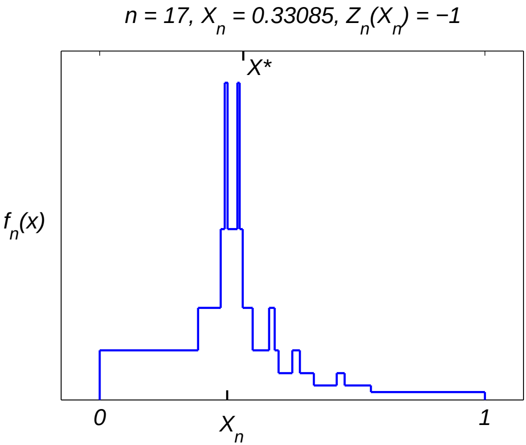

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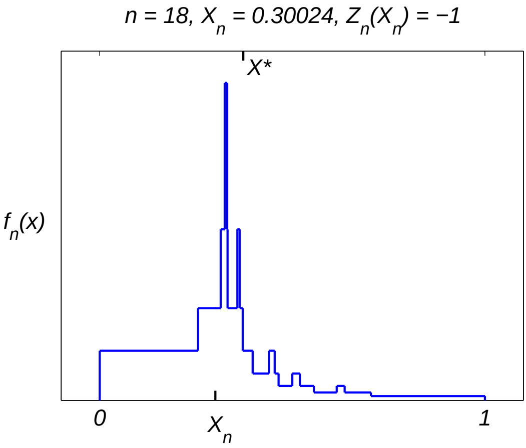

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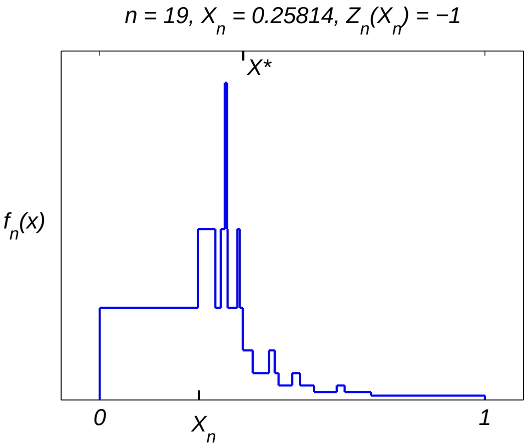

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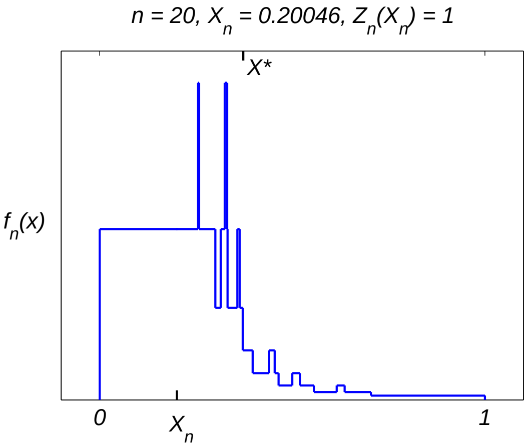

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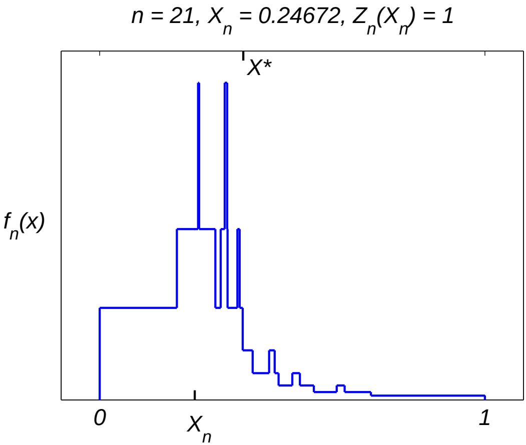

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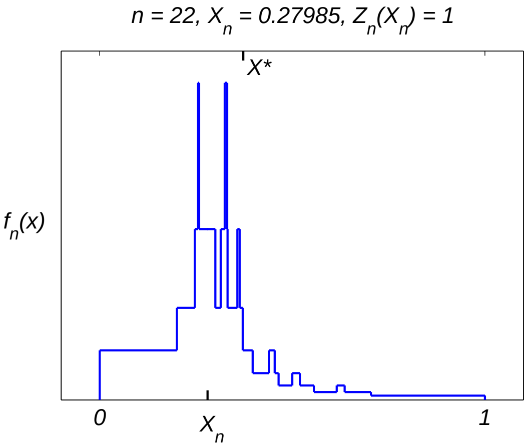

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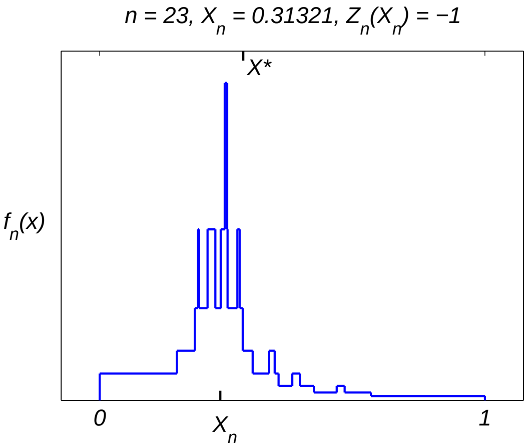

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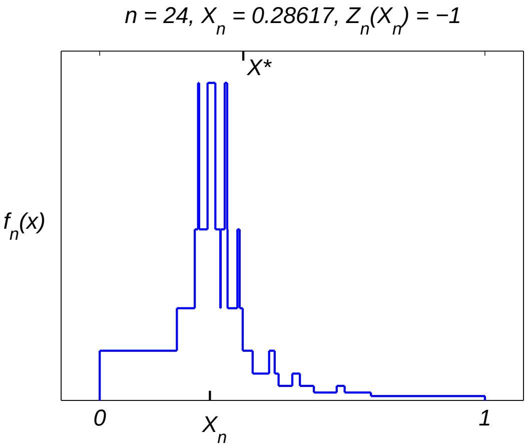

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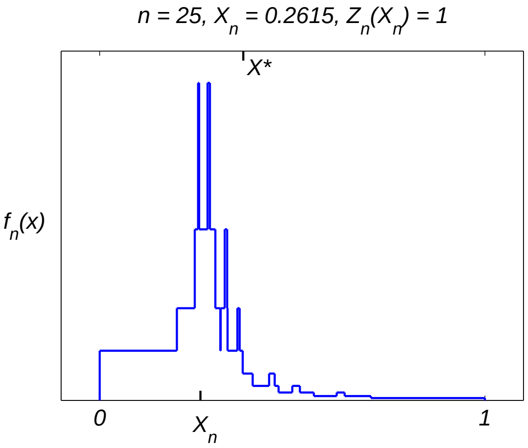

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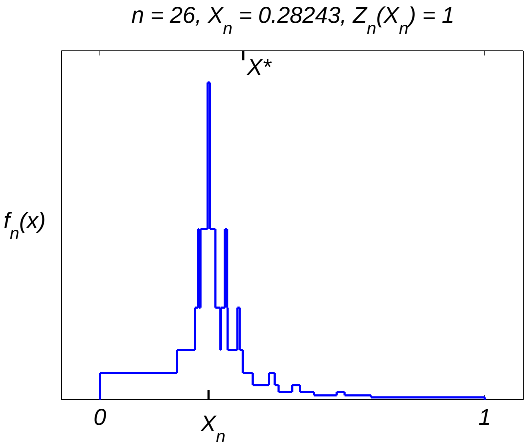

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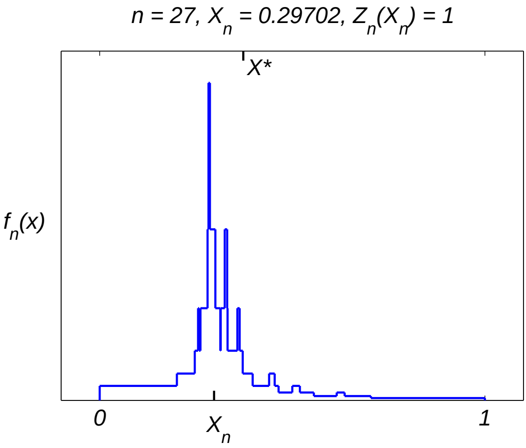

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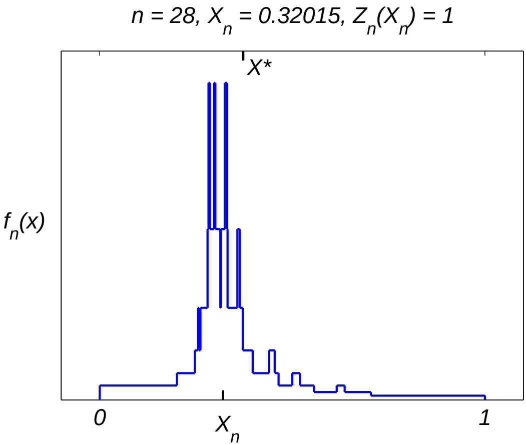

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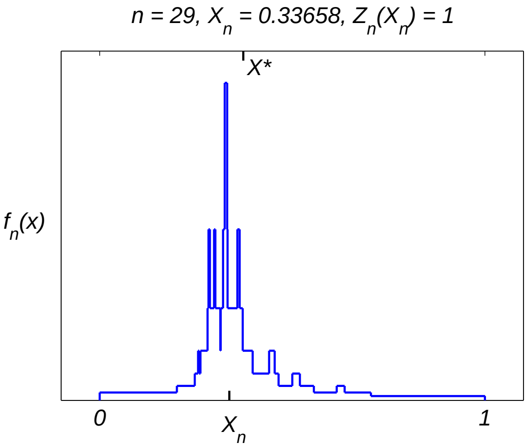

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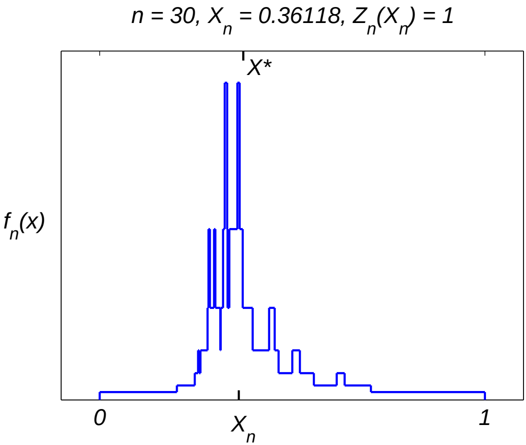

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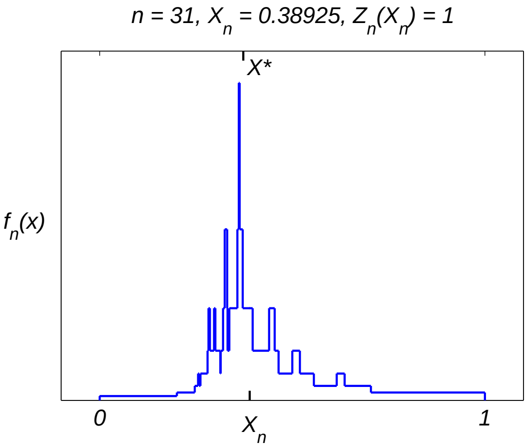

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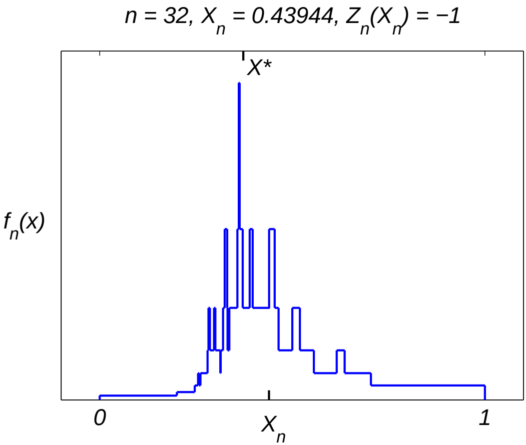

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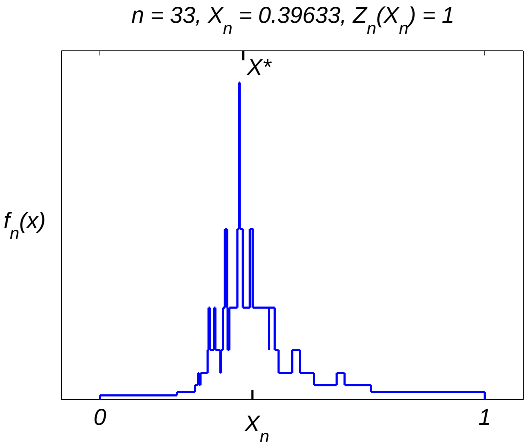

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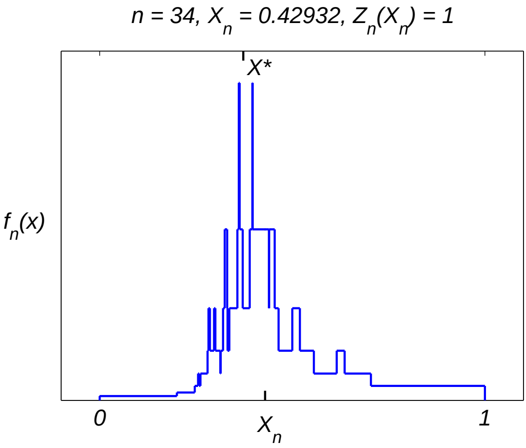

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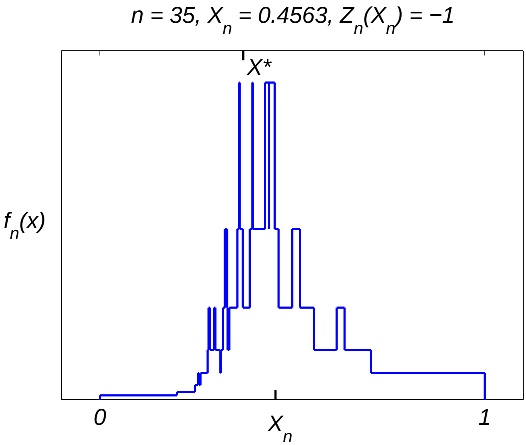

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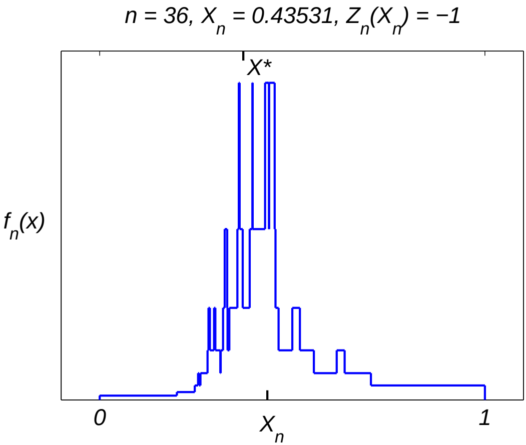

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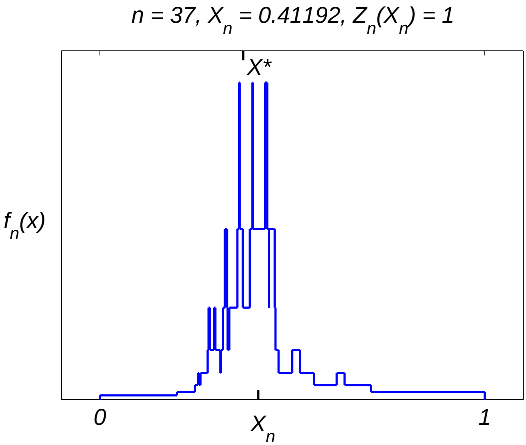

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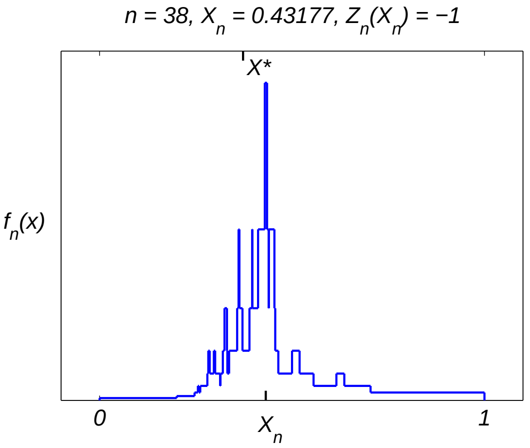

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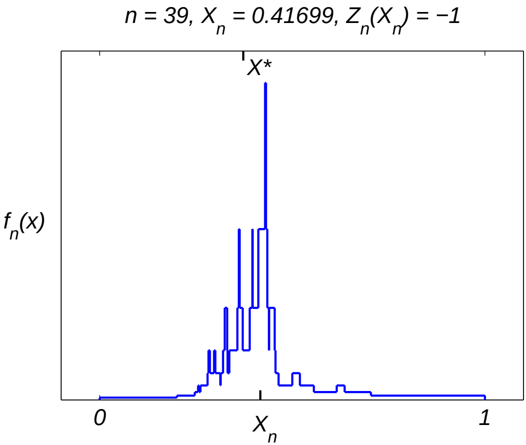

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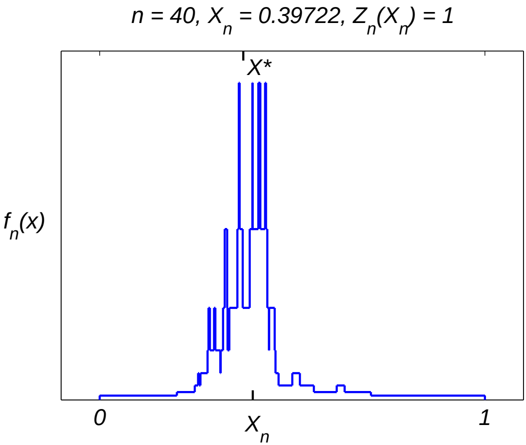

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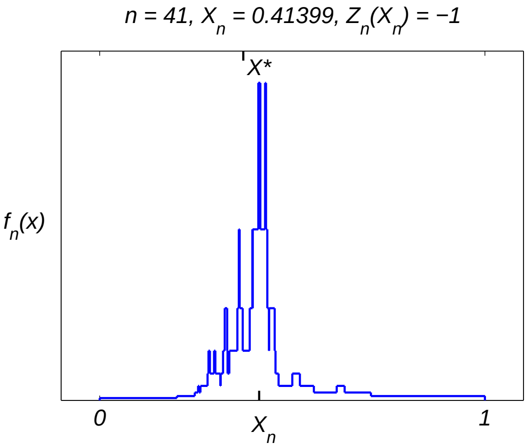

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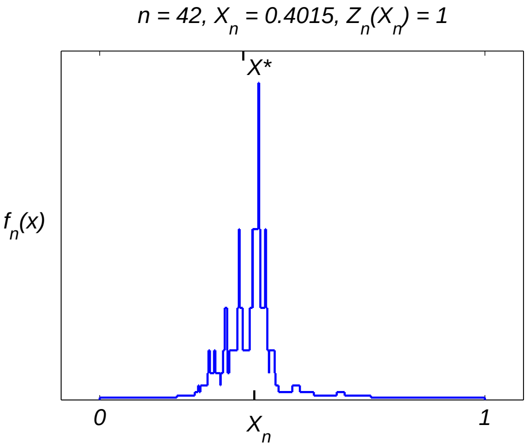

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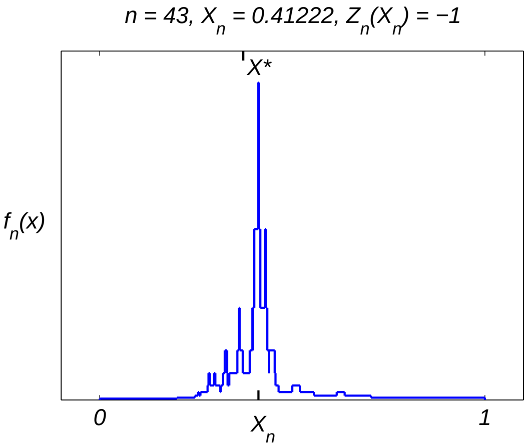

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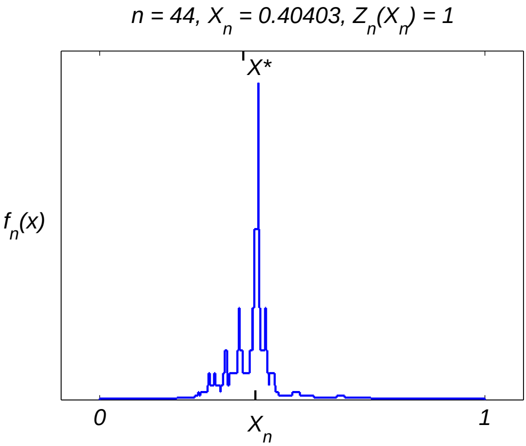

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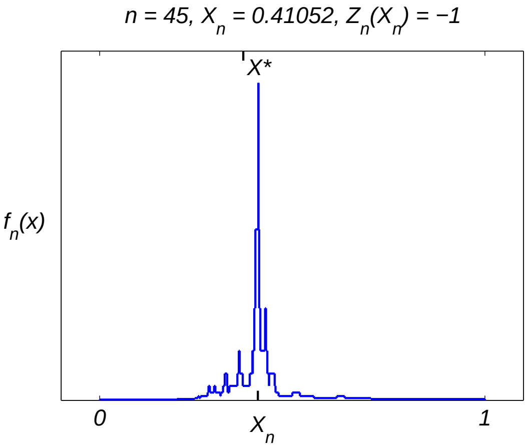

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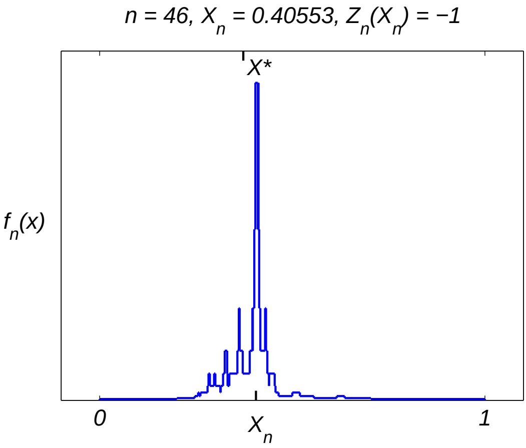

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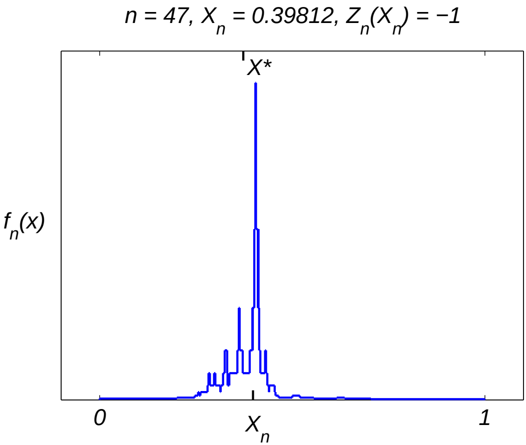

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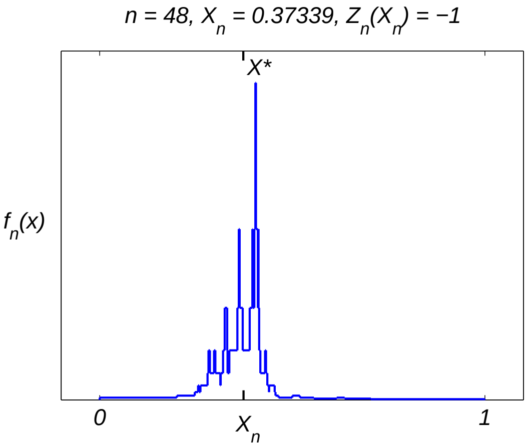

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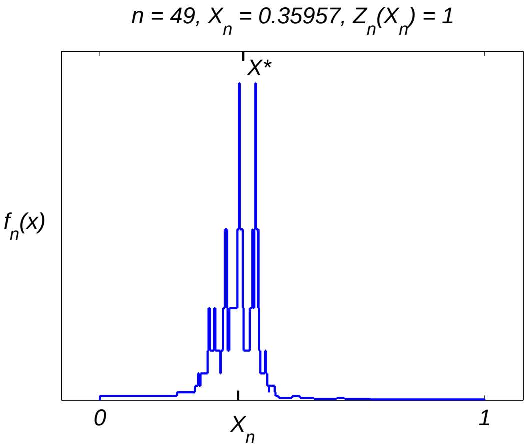

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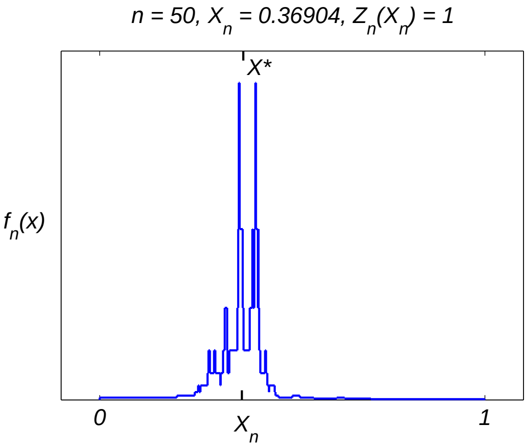

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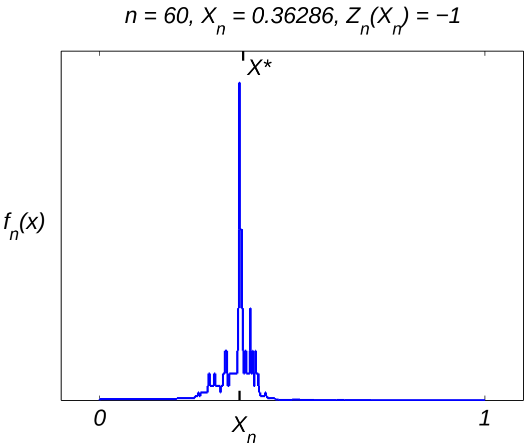

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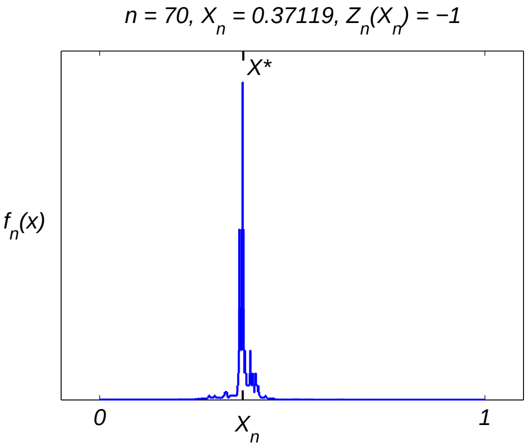

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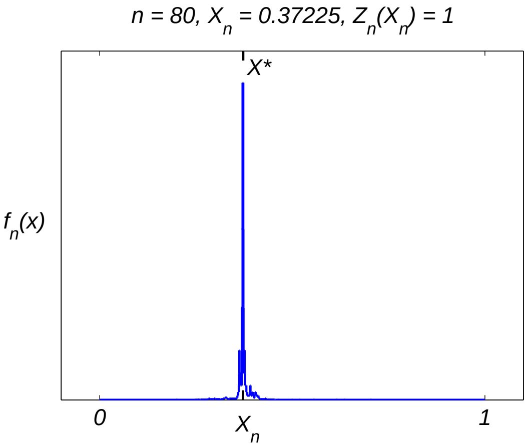

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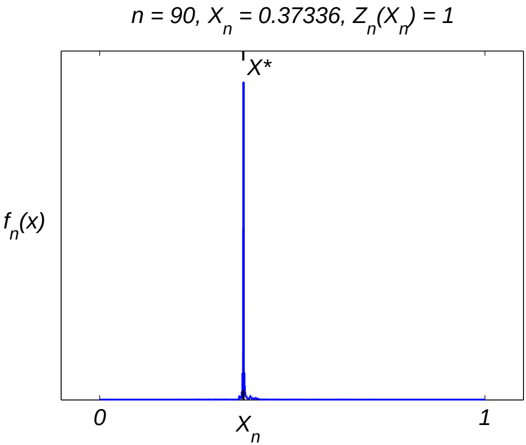

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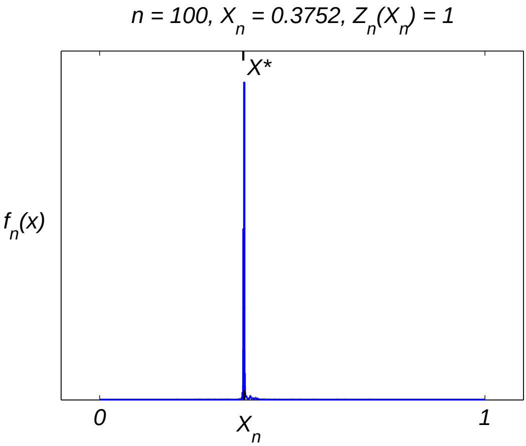

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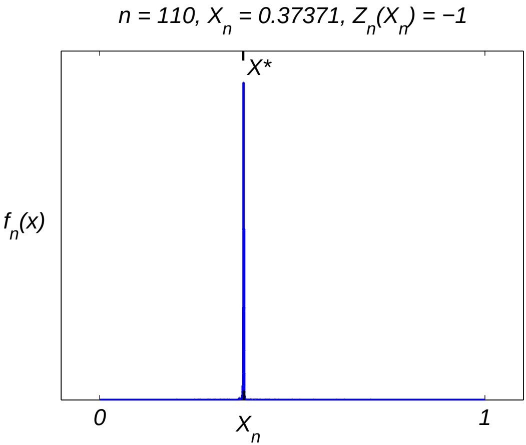

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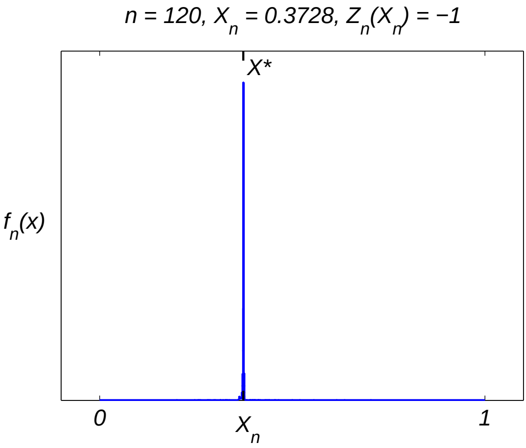

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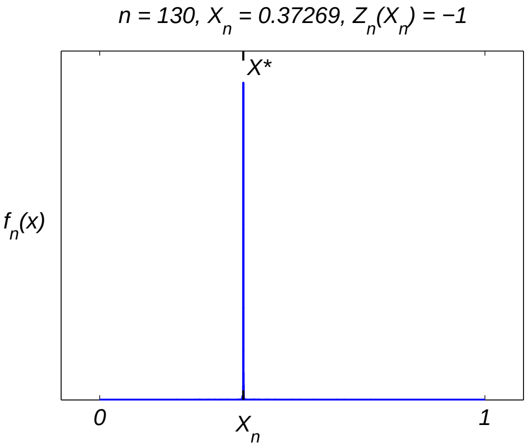

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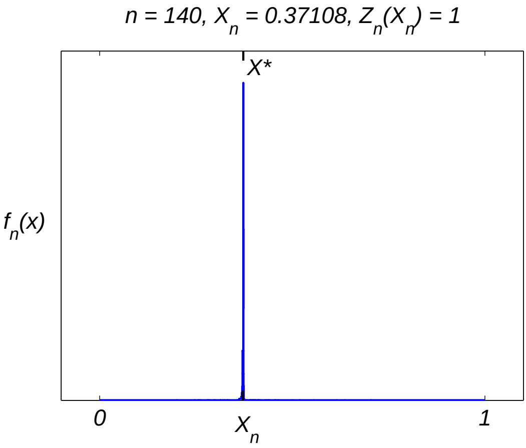

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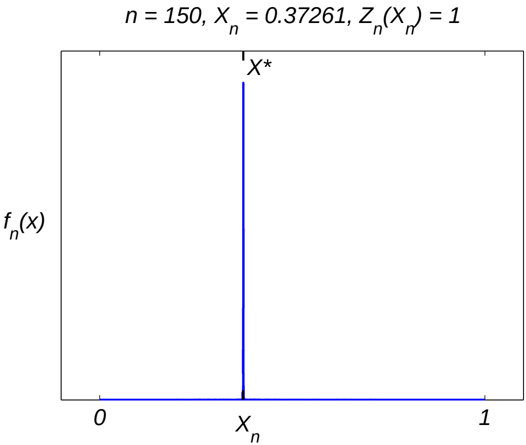
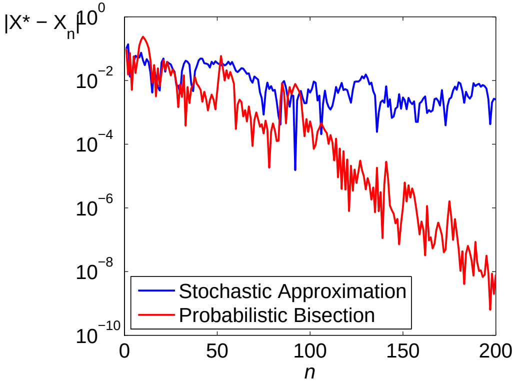
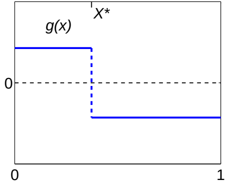


Comparison to Stochastic Approximation





Literature Review: Probabilistic Bisection Algorithm

- First introduced in Horstein (1963).
- Discretized version: Burnashev and Zigangirov (1974).
- Feige et al. (1997), Karp and Kleinberg (2007), Ben-Or and Hassidim (2008), Nowak (2008), Nowak (2009), ...
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> *"The probabilistic bisection algorithm seems to work extremely well in practice, but it is hard to analyze and there are few theoretical guarantees for it, especially pertaining error rates of convergence."*

Algorithm Analysis

Consistency

Setting for probabilistic bisection with power 1 tests:

- $X^* \in [0, 1]$ fixed and unknown.
- $X_n \neq X^*$ for any finite $n \in \mathbb{N}$.
- $p(X_n) \geq p_c$ for all $n \in \mathbb{N}$.
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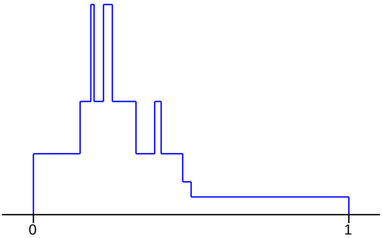
Theorem

$X_n \to X^*$ almost surely as $n \to \infty$.

Analysis of Posterior Density



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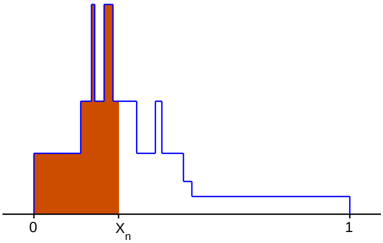

- If $Z_n = +1$:

$$f_{n+1}(x) = 2q_c \cdot f_n(x), \quad x < X_n,$$
$$f_{n+1}(x) = 2p_c \cdot f_n(x), \quad x \geq X_n,$$

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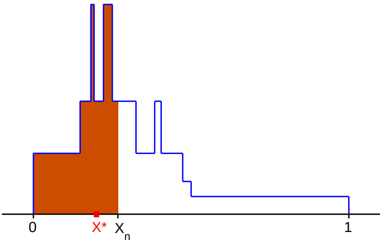

Case I: If $X^* < X_n : \mathbb{P}(Z_n = +1) = p(X_n) \geq p_c$

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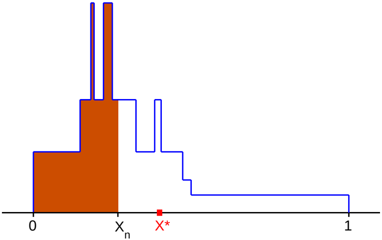

Case II: If $X^* > X_n : \mathbb{P}(Z_n = +1) = 1 - p(X_n) \leq p_c$

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Analysis of Posterior Density cont.

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Analysis of Posterior Density cont.

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- The process $f_n(X^*)$ behaves almost like a geometric random walk **independently of** $(X_n)_n$. The goal is then to locate this geometric random walk efficiently!

Confidence Intervals for X^*

- Notation: $\mu = p_c \ln 2p_c + q_c \ln 2q_c$.
- For $\alpha \in (0,1)$, define

$$b_n = n\mu - n^{1/2}(-0.5 \ln \alpha)^{1/2}(\ln 2p_c - \ln 2q_c).$$

- Define

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Proof:

Application of Hoeffding's inequality.

□

Size of Confidence Interval

Theorem

Choose $p_c \geq 0.85$, $\alpha \in (0,1)$. *For* $0 < r < \mu - q_c \ln 2p_c$ *there exists a* $N(p_c, r, \alpha) \in \mathbb{N}$, *such that*

$$\mathbb{P}(|J_n| \leq e^{-rn}, X^* \in J_n) \geq 1 - \alpha,$$

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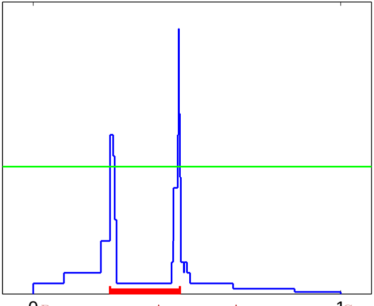
Proof Idea:



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Define $\hat{X}_n$ to be any point in J_n, then there exists $r > 0$ such that

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- And we have true confidence intervals for X^*.
- But n is the number of measurement points, what about total wall-clock time?

Wall-Clock Time

At each iteration of the Probabilistic Bisection Algorithm:

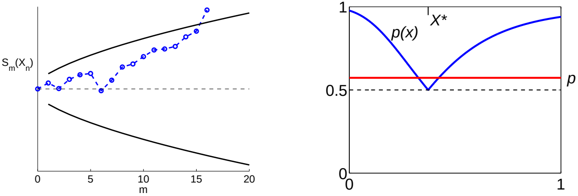
- Sample sequentially at point X_n and observe $S_m(X_n) = \sum_{i=1}^m Y_{n,i}(X_n)$, until

$$N_n = \inf\left\{m : |S_m| \geq [(m+1)(\log(m+1) + 2\log(1/\alpha))]^{1/2}\right\},$$

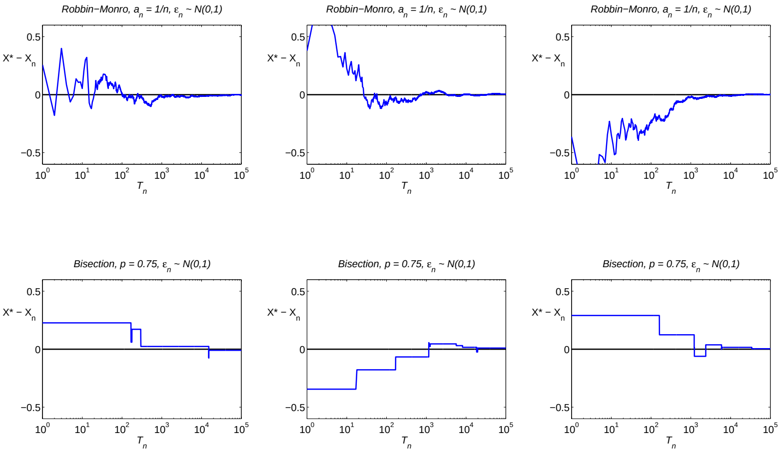
then $\mathbb{P}_{X_n = X^*}\{N_n < \infty\} \leq \alpha$, $\mathbb{P}_{X_n \neq X^*}\{N_n < \infty\} = 1$, and

$$\mathbb{P}_{X_n < X^*}\{S_{N_n}(X_n) > 0\} \geq 1 - \alpha/2 = p_c,$$
$$\mathbb{P}_{X_n > X^*}\{S_{N_n}(X_n) < 0\} \geq 1 - \alpha/2 = p_c.$$

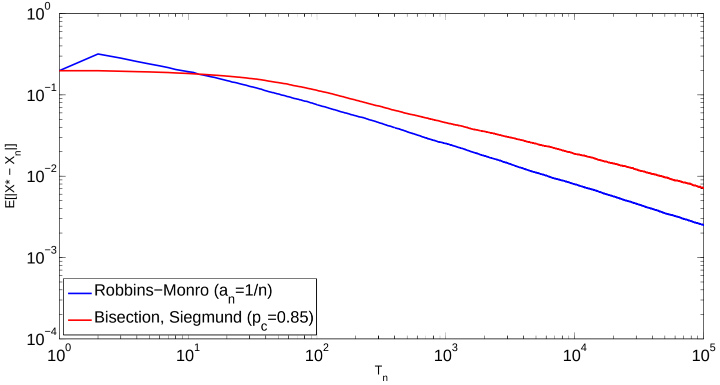
- Wall-clock time: $T_n = \sum_{i=1}^n N_n$.



Sample Paths



Numerical Comparison



Rate of Convergence in Wall-Clock Time?

- Farrell (1964):

$$\mathbb{E}_{g(x)}[N] \sim (1/g(x))^2 \log\log(1/|g(x)|) \text{ as } g(x) \to 0,$$

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- If

$$g(x) \to 0 \text{ as } x \to X^*,$$

and if we use X_n as the best estimate of X^* then the Probabilistic Bisection Algorithm with power one tests is **asymptotically slower** than Stochastic Approximation.

Conjecture

- X_n might not be the best estimate for X^* when we use power one tests.
- Intuitively, observations where we spend more time should also be closer to X^*, hence an estimator of the form

$$\tilde{X}_n = \frac{1}{T_n} \sum_{i=1}^{n} N_i X_i$$

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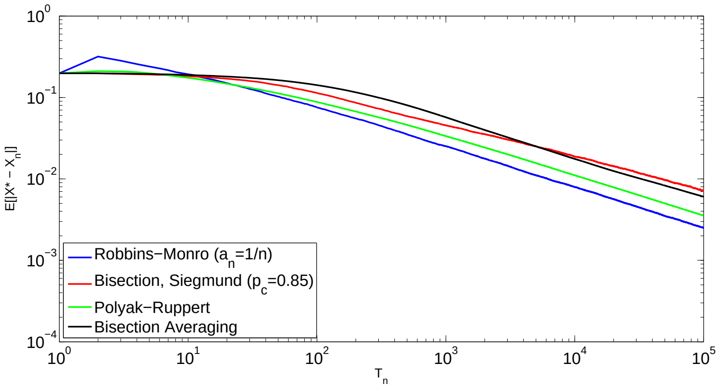
- **Sufficient Condition:** $|X_n - X^*| = O(e^{-rn})$ for some $r > 0$.

Numerical Comparison Cont.



Conclusions

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Positive:

- Provides true confidence interval of the root X^*.
- Works extremely well if there is a jump at $g(X^*)$ (geometric rate of convergence).
- Only one tuning parameter.

Drawbacks:

- Seems to be asymptotically slower than Stochastic Approximation (but by not much).
- Higher computational cost.

Future Research:

- Robustness of algorithm.
- Use parallel computing (very little switching of $(X_n)_n$).
- Extension to higher dimensions.

THANK YOU!

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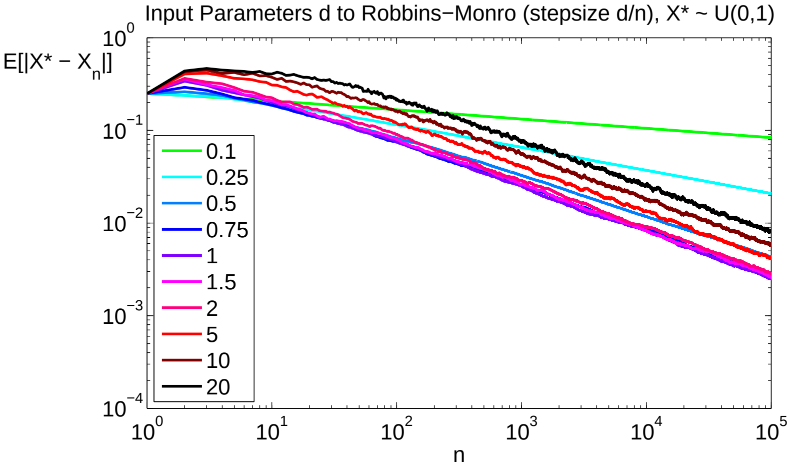
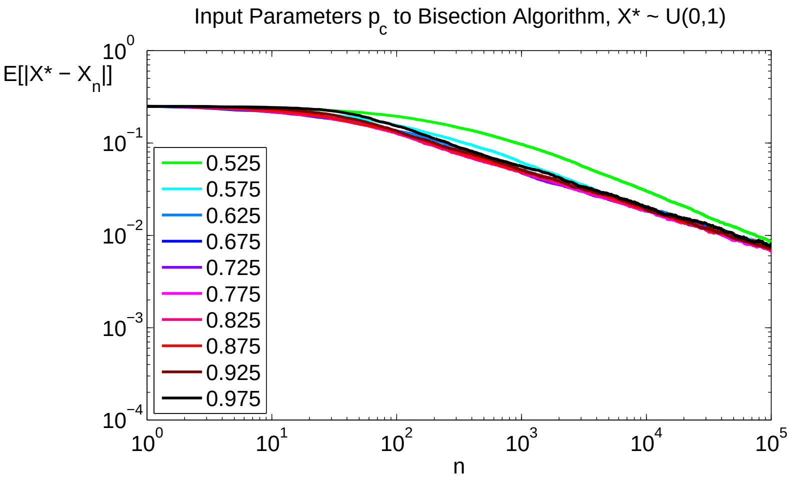
Tuning Parameters

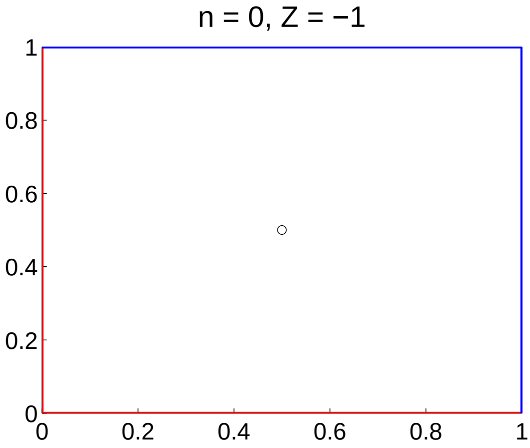


Tuning Parameters



Higher Dimensions

Boundary detection:



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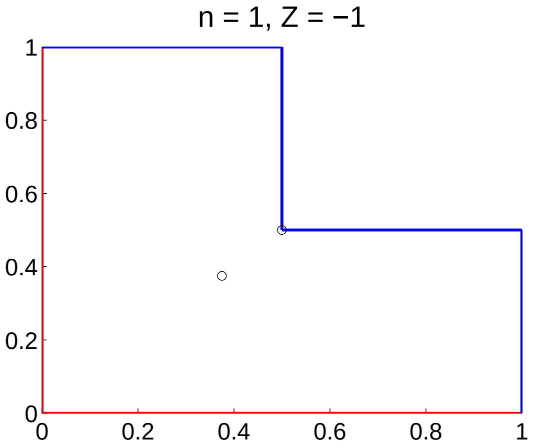

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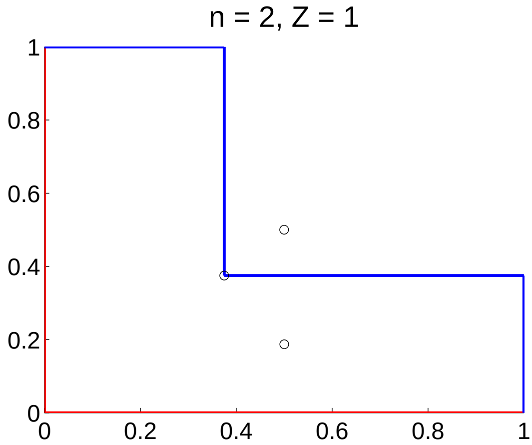

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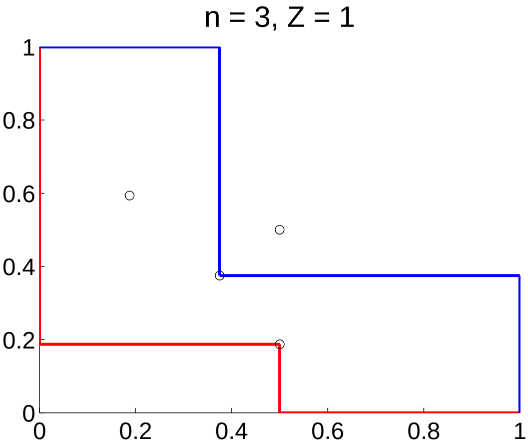

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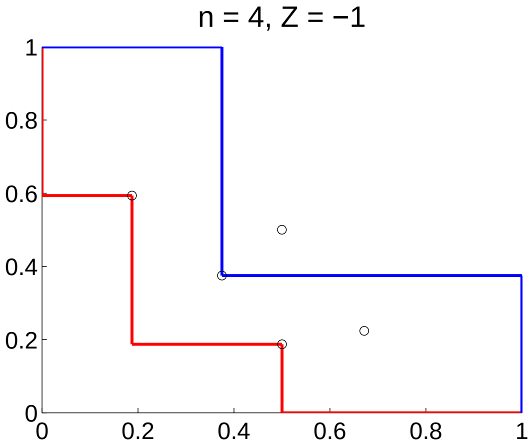

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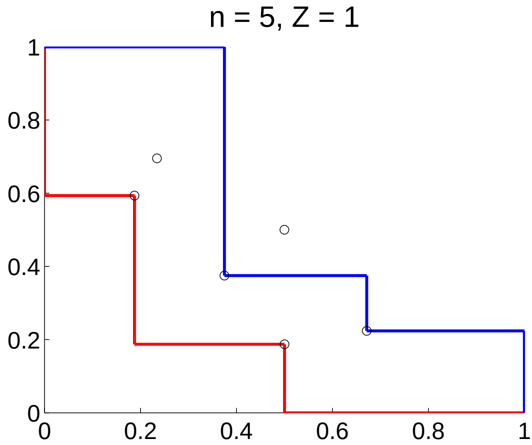

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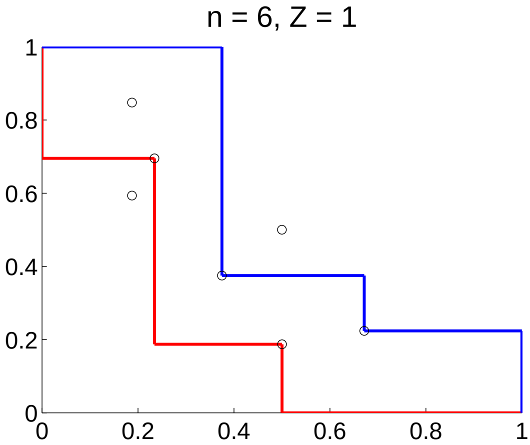

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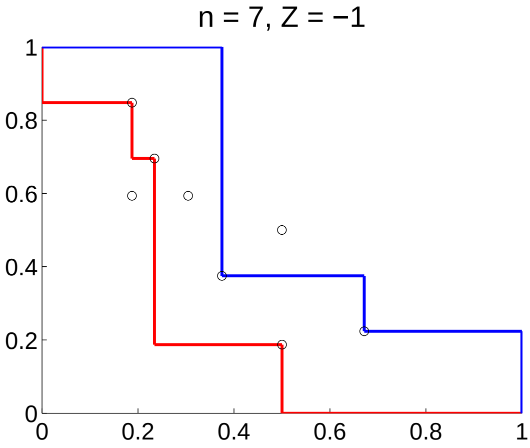

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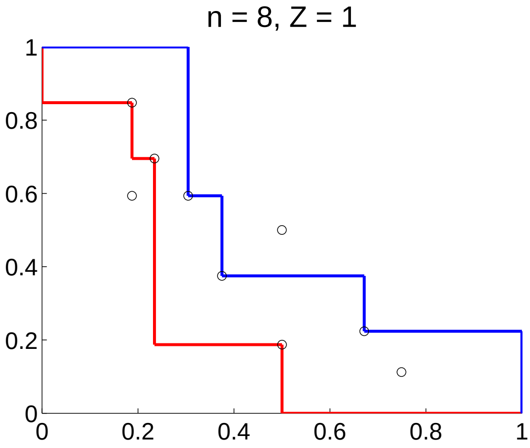

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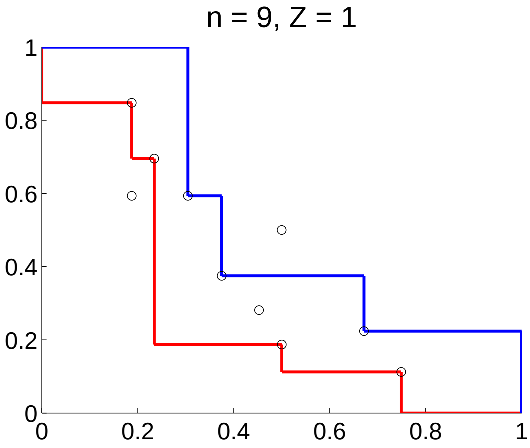

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