Indifference-zone Ranking and Selection with 10,000 or More Alternatives

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We have $k$ alternative systems that can be simulated.

Each time we simulate alternative $x$, we observe

$$y \sim \text{Normal}(\theta_x, \sigma_x^2)$$

(independent across $x$ and time)

where $\theta_x$ is unknown.

We assume known constant $\sigma_x^2 = \sigma^2$ in this talk. Results can be generalized to known heterogeneous $\sigma_x^2$. Unknown $\sigma_x^2$ is ongoing work.

**Goal:** Use simulation efficiently to find $\arg \max_x \theta_x$. 
A **policy** is an adaptive rule for deciding which alternatives to sample.

Given a system configuration \( \theta = (\theta_1, \ldots, \theta_k) \), and a policy \( \pi \),

\[
\text{PCS}(\pi, \theta) \quad \text{(Probability of Correct Selection)}
\]

is the probability that \( \pi \) selects an alternative in \( \arg\max_x \theta_x \).
Definitions: Preference Zone, Indifference Zone, IZ Guarantee

- The **preference zone** is:

  \[ PZ(\delta) = \{ \theta \in \mathbb{R}^k : \theta[1] - \theta[2] \geq \delta \} , \]

  where \( \theta[1] \geq \theta[2] \geq \ldots \geq \theta[k] \), and \( \delta > 0 \) is fixed.

- The **indifference zone (IZ)** is the complement: \( \mathbb{R}^k \setminus PZ(\delta) \).

- We say a policy \( \pi \) has an **IZ guarantee** with parameters \( \delta \) and \( P^* \) if

  \[ PCS(\pi, \theta) \geq P^* \quad \text{for all } \theta \in PZ(\delta). \]

- **Goal**: Find a policy that satisfies the IZ guarantee, while taking as few samples as possible.
There are Many Procedures with an IZ Guarantee

Lots of previous work has constructed policies that satisfy the IZ guarantee.

- Fixed sample size policies: [Bechhofer, 1954]
- Two-stage policies: [Dudewicz and Dalal, 1975, Rinott, 1978]
- Fully sequential policies
Overdelivery Leads To Oversampling (usually)

- Let $\pi$ be a procedure with an IZ guarantee for a fixed $P^*$ and $\delta$.
- $P^*$ is the PCS that $\pi$ guarantees it will deliver.
- For any $\theta \in PZ(\delta)$,
  \[ PCS(\pi, \theta) - P^* \]
  is the overdelivery on PCS.
- Overdelivery on PCS is inefficient: we could have taken fewer samples and achieved the guaranteed PCS faster.
- For existing policies and large $k$, this overdelivery causes the number of samples to be significantly larger than needed.
Contribution 1: We Have Tight PCS Bounds

- Given $P^*$ and $\delta$, we construct a fully sequential policy $\pi$ for which
  1. We prove $\pi$ has the IZ guarantee:
     \[
     PCS(\pi, \theta) \geq P^*, \quad \text{for all } \theta \in PZ(\delta).
     \]
  2. We prove the lower bound $P^*$ on PCS is tight in continuous time:
     \[
     \inf_{\theta \in PZ(\delta)} PCS(\pi, \theta) = P^*
     \]

- This policy is inspired by a Bayesian analysis, and we call it the **Bayes-Inspired IZ (BIZ)** policy. (even though the theoretical results apply to non-Bayesian PCS).
When the number of alternatives is large, this policy samples much less than existing IZ policies.

For the largest problems, BIZ requires between 10 and 15 times fewer samples than KN.
Let $Y_{tx}$ be the sum of all observations from alternative $x$ by time $t$.

Let $Q$ be a probability measure concentrated on least-favorable configurations.

For $A \subseteq \{1, \ldots, k\}$, define

$$q_{tx}(A) = Q\{x = X_* \mid X_* \in A, (Y_{tx'})_{x' \in A}\}.$$ 

This can be interpreted as the posterior probability that $x$ is the best alternative, given that the best is in the set $A$.

One can show for $x \in A$:

$$q_{tx}(A) = \exp \left( \frac{\delta}{\sigma^2} Y_{tx} \right) \middle/ \sum_{x' \in A} \exp \left( \frac{\delta}{\sigma^2} Y_{tx'} \right).$$
Informal Definition of BIZ

Final selection

$P_n$

$q_{tx}(A_n)$

$c$

$\tau_1$ (eliminate $Z_1$)

$\tau_2$ (eliminate $Z_2$)
Formal Definition of BIZ in Discrete or Continuous Time

- **Parameters:** $\mathbb{T} \in \{\mathbb{R}_+, \mathbb{Z}_+\}$, $c \leq 1 - (P^*)^{1/(k-1)}$, $\delta > 0$, $P^* > 1/k$.
- **Initialization:** $\tau_0 = 0$, $A_0 = \{1, \ldots, k\}$, $P_0 = P^*$.
- **Elimination Time:**
  \[
  \tau_{n+1} = \inf \left\{ t \in \mathbb{T} \cap [\tau_n, \infty) : \min_{x \in A_n} q_{tx}(A_n) \leq c \text{ or } \max_{x \in A_n} q_{tx}(A_n) \geq P_n \right\}.
  \]
- **Eliminated Alternative and Contention Set:**
  \[
  Z_{n+1} = \arg \min_{x \in A_n} q_{\tau_{n+1}}(A_n), \quad A_{n+1} = A_n \setminus Z_{n+1}
  \]
- **Stopping Boundary:**
  \[
  P_{n+1} = P_n \left/ \left(1 - \min_{x \in A_n} q_{\tau_{n+1}}(A_n)\right)\right.
  \]
- **The selected alternative** is the single alternative in $A_{k-1}$.
Fix parameters $c \leq 1 - (P^*)^{1/(k-1)}$, $\delta > 0$, $P^* > 1/k$.

1. Let $A \leftarrow \{1, \ldots, k\}$, $t \leftarrow 0$, $P \leftarrow P^*$.
2. While $\max_{x \in A} q_{tx}(A) < P$
   
   2a. While $\min_{x \in A} q_{tx}(A) \leq c$
      
     - Let $x \in \arg\min_{x} q_{tx}(A)$.
     - Let $P \leftarrow P / (1 - q_{tx}(A))$.
     - Remove $x$ from $A$.

   2b. Sample from each $x \in A$ to obtain $Y_{t+1,x}$. Then increment $t$.

3. Select $\hat{x} \in \arg\max_{x \in A} Y_{tx}$ as our estimate of the best.

Recall:

$$q_{tx}(A) = \exp \left( \frac{\delta}{\sigma^2} Y_{tx} \right) \Bigg/ \sum_{x' \in A} \exp \left( \frac{\delta}{\sigma^2} Y_{tx'} \right).$$
Main Theorem: Tight PCS Bounds

Theorem

Fix any $\delta > 0$, $P^* \in (1/k, 1)$, $T \in \{\mathbb{Z}_+, \mathbb{R}_+\}$, $c \leq 1 - (P^*)^{1/(k-1)}$ and let $\pi$ be the corresponding BIZ policy. Then,

$$\text{PCS}(\pi, \theta) \geq P^* \quad \forall \theta \in \text{PZ}(\delta)$$

Moreover, if $T = \mathbb{R}_+$, then

$$\inf_{\theta \in \text{PZ}(\delta)} \text{PCS}(\pi, \theta) = P^*$$
We construct BIZ with Bayesian ideas . . .

. . . but BIZ is a non-Bayesian Method

- BIZ is motivated using Bayesian ideas, and manipulation of Bayesian PCS is critical in proofs of theoretical results.
- However, BIZ is a non-Bayesian method.
- You do not need to have a prior to use BIZ, and its IZ guarantee is non-Bayesian.
Choosing $c = 0$ recovers the sequential ranking procedure $P^*_B$ from [Bechhofer et al., 1968].

In this special case, this policy does not eliminate.

It was previously unknown that $P^*_B$ is exact in continuous-time.
**SC**  
\( P^* = 0.9 \)

**MDM**  
\( P^* = 0.9 \)

**RPI**  
\( P^* = 0.8 \)
For the largest problems, BIZ requires 10 to 15 times fewer samples than KN.
BIZ is a fully sequential IZ policy that delivers **exactly** $P^*$.
(in continuous time, and under the worst $\theta$ in the preference zone).

To my knowledge, this is the **first fully sequential elimination IZ policy** with this property for $k > 2$.

This method of Bayesian analysis with least-favorable priors is a general theoretical tool.
Thank You!
References I

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An improvement on Paulson’s sequential ranking procedure.

Fully sequential indifference-zone selection procedures with variance-dependent sampling.
A fully sequential procedure for indifference-zone selection in simulation.

Simple procedures for selecting the best simulated system when the number of alternatives is large.

A sequential procedure for selecting the population with the largest mean from k normal populations.

Sequential procedures for selecting the best one of k koopman-darmois populations.
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On two-stage selection procedures and related probability-inequalities.
In continuous time, BIZ can be extended easily to the case where sampling variances are heterogeneous, and the main theoretical results are still true (IZ guarantee with a tight bound).

In discrete time, the same technique can be applied, but the IZ guarantee no longer holds exactly. Ongoing work: does it hold in a limiting sense?
Numerical Comparisons: Slippage Configuration

- $\theta = [0, \ldots, 0, \delta]$, $P^* = 0.9$, $\sigma = 10$, $\delta = 1$, estimated with $\geq 10,000$ independent replications.
- BIZ uses $c = 1 - (P^*)^{1/(k-1)}$ (eliminate aggressively).
- BKS is $P^*_B$ ($c = 0$) from [Bechhofer et al., 1968].
Settings are the same as before: \( \theta = [-\delta, -2\delta, \ldots, -k\delta] \), \( P^* = 0.9 \), \( \sigma = 10 \), \( \delta = 1 \), estimated with \( \geq 10,000 \) independent replications.

BIZ uses \( c = 1 - (P^*)^{1/(k-1)} \) (eliminate aggressively).

BKS is \( P_B^* \) (\( c = 0 \)) from [Bechhofer et al., 1968].
$P^* = 0.8, \sigma = 10, \delta = 1$.

Each set of sampling means $\mu$ was generated randomly from an independent normal prior and then adjusted to lie in $PZ(\delta)$.

BIZ uses $c = 1 - (P^*)^{1/(k-1)}$ (eliminate aggressively). BKS is $P_B^*$ ($c = 0$) from [Bechhofer et al., 1968].
Images show continuation region for $k = 3$, in linear coordinates (left) and exponential coordinates (right).

BKS (BIZ with $c = 0$) stops when $Y_t$ exits the continuation region.
Numerical Comparisons: Random Problem Instances

- $k = 100$.
- $P^* = 0.8$.
- $\theta$ was generated randomly from an independent normal prior.
- It was then adjusted so that no alternative other than the best is within $\delta$ of the best, i.e., so that $\theta \in PZ(\delta)$. 
Initialization: \((\tau_0 = 0, A_0 = \{1, 2, 3, 4\})\)

<table>
<thead>
<tr>
<th>t</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
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<tbody>
<tr>
<td>t=1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>t=2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>
| t=3| 1 | 2 | 3 | 4 | (\tau_1 = 3, Z_1 = 2, A_1 = \{1, 3, 4\})
| t=4| 1 3 4  |
| t=5| 1 3 4  |
| t=6| 1 3 4  |
| t=7| 1 3 4  | (\tau_3 = 7, Z_3 = 1, A_3 = \{3\})

Selection: Select alternative 3 as the best.
Heterogeneous Sampling Variance: Continuous Time

- In general, sampling variances $\sigma_x^2$ are heterogeneous and unknown.
- In continuous time, this problem is easily addressed:
  1. The sampling variance $\sigma_x^2$ can be estimated perfectly given
     \[(Y_{tx} : 0 \leq t \leq \varepsilon)\] for any $\varepsilon > 0$.
  2. Replace $Y_{t,x} \sim \mathcal{N}(\theta_x, t\sigma_x^2)$ with $Y_{\sigma^2_{t,x}/\sigma_x^2} \sim \mathcal{N}(\theta_x, t)$ and we obtain
     a ranking and selection problem with common sampling variance 1.
  3. Use BIZ for common sampling variance 1 on the transformed $Y$ values.
- The IZ guarantee and the tightness of the bound remain true.
In discrete time, the problem is harder:

Idea: Replace $Y_{t,x}$ with $Y_{n_x(t),x}/\hat{\sigma}_x^2$ where $n_x(t)$ rounds up $\hat{\sigma}_x^2 t$.

The IZ guarantee no longer holds.

Ongoing work: Does the IZ guarantee hold in a limiting sense?
Proof Sketch

Let $CS$ be the event of correct selection.
For $\theta \in \mathbb{R}^d$, let $Q_\theta$ be a prior that is uniform on the permutations $\theta$. In particular, $Q = Q[\delta,0,...,0]$.

**Lemma (Symmetry)**

$PCS(\pi, \theta)$ is invariant to permutations of $\theta$.
Moreover, $PCS(\pi, \theta) = Q^\pi_\theta \{CS\}$.

**Lemma (Monotonicity)**

For $\theta \in PZ(\delta)$, $Q^\pi_\theta \{CS\} \geq Q^\pi[\delta,0,...,0] \{CS\}$.

**Lemma (Bayes PCS of Least-favorable Configuration)**

$Q^\pi[\delta,0,...,0] \{CS\} \geq P^*$, with equality if $\mathbb{T} = \mathbb{R}_+$. 

**BIZ Construction: Elimination**

BIZ is an **elimination** procedure.

- It defines a sequence of stopping times, $0 = \tau_0 \leq \tau_1 \cdots \leq \tau_{k-1} < \infty$.
- For $n < k$,
  - $\tau_n$ is the time that the $n^{\text{th}}$ alternative is eliminated.
  - $Z_n \in \arg\min_{x \in A_{n-1}} Y_{\tau_n,x}$ is the $n^{\text{th}}$ alternative eliminated.
  - $A_n = \{1, \ldots, k\} \setminus \{Z_m : m \leq n\}$ are the remaining alternatives.
- At time $\tau_{k-1}$, we stop sampling and select the single remaining alternative as best.

Elimination allows us to quickly eliminate very bad alternatives, reducing sampling effort.
In the original problem, we observe independent \( \mathcal{N}(\theta_x, \sigma_x^2) \) values from alternative \( x \).
- The sum of all observations up to the current time is a random walk.

In our continuous-time generalization, we let \( (Y_{tx} : t \in \mathbb{R}_+) \) be a Brownian motion with drift \( \theta_x \) and volatility \( \sigma_x = \sigma \).

Let \( T \in \{\mathbb{Z}_+, \mathbb{R}_+\} \). We restrict elimination and stopping decisions to be in \( T \).

When \( T = \mathbb{Z}_+ \), the resulting procedure can be implemented in discrete time.