Announcements

▶ section this week: clustering
▶ hw5 out, due Thursday Nov 19 9:30am
▶ project midterm report peer reviews due Sunday Nov 15 11:59pm
▶ quiz Thursday 6:15pm - Friday 11:59pm; set a reminder!

(All times ET)
Outline

Missing data

Unsupervised learning
Low rank models
Principal Components Analysis
Generalized Low Rank Models
Imputing missing data
Multidimensional losses
More about regularizers
Clustering
Missing data

examples:
Missing data

eamples:

▶ weather data: missing data due to
Missing data

examples:

- weather data: missing data due to sensor failures
examples:

- weather data: missing data due to sensor failures
- survey data: missing data due to
Missing data

examples:

- weather data: missing data due to sensor failures
- survey data: missing data due to non-response
Missing data

examples:

- weather data: missing data due to sensor failures
- survey data: missing data due to non-response
- purchase/click/like data: missing data due to
Missing data

examples:

► weather data: missing data due to sensor failures
► survey data: missing data due to non-response
► purchase/click/like data: missing data due to lack of purchase/click/like
Missing data

examples:

- weather data: missing data due to sensor failures
- survey data: missing data due to non-response
- purchase/click/like data: missing data due to lack of purchase/click/like
- drug trial: missing data due to
Missing data

examples:

- weather data: missing data due to sensor failures
- survey data: missing data due to non-response
- purchase/click/like data: missing data due to lack of purchase/click/like
- drug trial: missing data due to subjects leaving trial
### Data table: survey data

<table>
<thead>
<tr>
<th>age</th>
<th>gender</th>
<th>state</th>
<th>income</th>
<th>education</th>
</tr>
</thead>
<tbody>
<tr>
<td>29</td>
<td>F</td>
<td>CT</td>
<td>$53,000</td>
<td>college</td>
</tr>
<tr>
<td>57</td>
<td>?</td>
<td>NY</td>
<td>$19,000</td>
<td>high school</td>
</tr>
<tr>
<td>?</td>
<td>M</td>
<td>CA</td>
<td>$102,000</td>
<td>masters</td>
</tr>
<tr>
<td>41</td>
<td>F</td>
<td>NV</td>
<td>$23,000</td>
<td>?</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
How to cope with missing data?

strategy 1:

► drop rows or columns with missing data
How to cope with missing data?

strategy 1:

▶ drop rows or columns with missing data

how well would this work for

▶ weather data
▶ survey data
▶ purchase/click/like data
▶ drug trial
How to cope with missing data?

strategy 2:

▶ fill in missing entries with row or column mean
How to cope with missing data?

strategy 2:

▶ fill in missing entries with row or column mean

how well would this work for

▶ weather data
▶ survey data
▶ purchase/click/like data
▶ drug trial
How to cope with missing data?

strategy 3:

▶ use other columns to predict missing entries
How to cope with missing data?

strategy 3:

▶ use other columns to **predict** missing entries

how well would this work for

▶ weather data
▶ survey data
▶ purchase/click/like data
▶ drug trial
How to cope with missing data?

strategy 3:

- use other columns to \textit{predict} missing entries

how well would this work for

- weather data
- survey data
- purchase/click/like data
- drug trial

problem: what if \textbf{all columns} have (some) missing data?
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Unsupervised learning

strategy 4:

▶ simultaneously learn regression coefficients and covariates to predict every entry in data well

this is such a weird idea that we will need new terminology:

▶ we no longer can divide the data into inputs and outputs, or features and labels, or covariates and responses
▶ all we have are some features for each example
▶ this setting is called unsupervised
Data table

$n$ examples (patients, respondents, households, assets)
$d$ features (tests, questions, sensors, times)

$$
\begin{bmatrix}
Y
\end{bmatrix}
= 
\begin{bmatrix}
Y_{11} & \cdots & Y_{1d} \\
\vdots & \ddots & \vdots \\
Y_{n1} & \cdots & Y_{nd}
\end{bmatrix}
$$

- $i$th row of $Y$ is feature vector for $i$th example
- $j$th column of $Y$ gives values for $j$th feature across all examples
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### Low rank model

**given:** \( n \times d \) data table \( Y \), \( r \leq n, d \)

**find:** \( X \in \mathbb{R}^{n \times r}, \ W \in \mathbb{R}^{r \times d} \) for which

\[
\begin{bmatrix}
X
\end{bmatrix}
\begin{bmatrix}
W
\end{bmatrix}
\approx
\begin{bmatrix}
Y
\end{bmatrix}
\]

i.e., \( x_i^T w_j \approx Y_{ij} \), where

\[
\begin{bmatrix}
X
\end{bmatrix}
= \begin{bmatrix}
\overline{x_1^T} \\
\vdots \\
\overline{x_n^T}
\end{bmatrix}
\begin{bmatrix}
W
\end{bmatrix}
= \begin{bmatrix}
w_1 & \cdots & w_d
\end{bmatrix}
\]

**interpretation:**

- \( r = \text{Rank}(XW) \) is the rank of the model
- \( X \) and \( W \) are (compressed) representation of \( Y \)
- \( x_i \in \mathbb{R}^r \) is a point associated with example \( i \)
- \( w_j \in \mathbb{R}^r \) is a point associated with feature \( j \)
- inner product \( x_i^T w_j \) approximates \( Y_{ij} \)
Exact low rank fitting

**simplest case:** suppose \( Y \in \mathbb{R}^{n \times d} \) has no missing entries

**Q:** what is the smallest \( r \) so that

\[
Y = XW
\]

for \( X \in \mathbb{R}^{n \times r} \), \( W \in \mathbb{R}^{r \times d} \)?

(\( XW \) is called a **factorization** of \( Y \))
Exact low rank fitting

simplest case: suppose $Y \in \mathbb{R}^{n \times d}$ has no missing entries

**Q:** what is the smallest $r$ so that

$$Y = XW$$

for $X \in \mathbb{R}^{n \times r}$, $W \in \mathbb{R}^{r \times d}$?

$(XW$ is called a **factorization** of $Y$)

**A:** $r = \text{Rank}(Y)$!
Exact low rank fitting

**Theorem:** for $Y \in \mathbb{R}^{n \times d}$,

$$\text{Rank}(Y) = \min\{r : Y = XW, \quad X \in \mathbb{R}^{n \times r}, \quad W \in \mathbb{R}^{r \times d}\}$$
Exact low rank fitting

**Theorem:** For \( Y \in \mathbb{R}^{n \times d} \),

\[
\text{Rank}(Y) = \min\{ r : Y = XW, \quad X \in \mathbb{R}^{n \times r}, \ W \in \mathbb{R}^{r \times d} \}
\]

**Proof:** 1) We can find \( X \) and \( W \) with \( Y = XW \) and \( r = \text{Rank}(Y) \):

- Suppose \( Y = U\Sigma V^T \) is the skinny SVD of \( Y \)
- Then \( \text{Rank}(Y) = \) number of columns of \( U \) and of \( V \)
- Let \( X = U, \ W = \Sigma V^T \)
- Then \( Y = XW \)
**Exact low rank fitting**

**Theorem:** for $Y \in \mathbb{R}^{n \times d}$,

$$\text{Rank}(Y) = \min \{ r : Y = XW, \quad X \in \mathbb{R}^{n \times r}, \ W \in \mathbb{R}^{r \times d} \}$$

**Proof:** 1) we can find $X$ and $W$ with $Y = XW$ and $r = \text{Rank}(Y)$:

- suppose $Y = U\Sigma V^T$ is the skinny SVD of $Y$
- then $\text{Rank}(Y) = \text{number of columns of } U \text{ and of } V$
- let $X = U$, $W = \Sigma V^T$
- then $Y = XW$

2) for any $X$ and $W$ st $Y = XW$, $\text{Rank}(Y) \leq r$:

- $\text{Rank}(Y) = \text{Rank}(XW) \leq \min(\text{Rank}(X), \text{Rank}(W)) \leq r$
Exact low rank fitting

**theorem:** for \( Y \in \mathbb{R}^{n \times d} \),

\[
\text{Rank}(Y) = \min \{ r : Y = XW, \quad X \in \mathbb{R}^{n \times r}, \ W \in \mathbb{R}^{r \times d} \}
\]

**proof:** 1) we can find \( X \) and \( W \) with \( Y = XW \) and \( r = \text{Rank}(Y) \):

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- \( \text{Rank}(Y) = \text{Rank}(XW) \leq \min(\text{Rank}(X), \text{Rank}(W)) \leq r \)

so \( \text{Rank}(Y) \) is the smallest \( r \) st \( Y = XW \)
if we’re willing to represent $Y$ approximately, can we use a smaller rank $r$?
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Low rank models

**Principal Components Analysis**

Generalized Low Rank Models
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Clustering
**Principal components analysis (PCA)**

**Principal components analysis (PCA):** Given $Y \in \mathbb{R}^{n \times d}$, solve

$$\text{minimize} \quad \| Y - XW \|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} (Y_{ij} - x_i^T w_j)^2$$

with $X \in \mathbb{R}^{n \times r}$, $W \in \mathbb{R}^{r \times d}$

- a very old problem (Pearson 1901, Hotelling 1933)
- least squares low rank fitting
Principal components analysis (PCA)

Principal components analysis (PCA): Given $Y \in \mathbb{R}^{n \times d}$, solve

$$\text{minimize } \| Y - XW \|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} (Y_{ij} - x_i^T w_j)^2$$

with $X \in \mathbb{R}^{n \times r}$, $W \in \mathbb{R}^{r \times d}$

- a very old problem (Pearson 1901, Hotelling 1933)
- least squares low rank fitting

notice: objective depends only on product $XW$, so if $(X, W)$ is a solution, so is $(\tilde{X}, \tilde{W}) = (XT, T^{-1}W)$ for any invertible matrix $T \in \mathbb{R}^{r \times r}$:

$$\tilde{X}\tilde{W} = XTT^{-1}W = XW.$$
**Principal components analysis (PCA)**

**Principal components analysis (PCA):** Given \( Y \in \mathbb{R}^{n \times d} \), solve

\[
\text{minimize} \quad \| Y - XW \|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} (Y_{ij} - x_i^T w_j)^2
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with \( X \in \mathbb{R}^{n \times r} \), \( W \in \mathbb{R}^{r \times d} \)

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\[
\tilde{X} \tilde{W} = XTT^{-1}W = XW.
\]

make sure **interpretation** of solution is invariant under \( T \)
PCA finds best covariates

example with $d = 2$, $r = 1$

**regression:** fix $X = Y_{:,1}$ (first column of $Y$), solve

$$\text{minimize} \quad \| Y - XW \|_F^2 \quad \text{wrt variable } W \in \mathbb{R}^{1 \times 2}$$
PCA finds best covariates

Example with $d = 2$, $r = 1$

**PCA:** solve

$$\text{minimize} \quad \| Y - XW \|_F^2 \quad \text{wrt variables } X \in \mathbb{R}^{n \times 1}, W \in \mathbb{R}^{1 \times 2}$$
On lines and planes of best fit

[Pearson 1901]
Low rank models for gait analysis

<table>
<thead>
<tr>
<th>time</th>
<th>forehead (x)</th>
<th>forehead (y)</th>
<th>···</th>
<th>right toe (y)</th>
<th>right toe (z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td>1.4</td>
<td>2.7</td>
<td>···</td>
<td>-0.5</td>
<td>-0.1</td>
</tr>
<tr>
<td>$t_2$</td>
<td>2.7</td>
<td>3.5</td>
<td>···</td>
<td>1.3</td>
<td>0.9</td>
</tr>
<tr>
<td>$t_3$</td>
<td>3.3</td>
<td>-.9</td>
<td>···</td>
<td>4.2</td>
<td>1.8</td>
</tr>
<tr>
<td></td>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
</tbody>
</table>

- rows of $W$ are principal stances
- rows of $X$ decompose stance into combination of principal stances
Interpreting principal components

columns of $Y$ (features) (height of point over time)
Interpreting principal components

columns of $Y$ (features) (depth of point over time)
Interpreting principal components

row of \( W \)
(archetypical example)
(principal stance)
Interpreting principal components

columns of $X$ (archetypical features) (principal timeseries)
Interpreting principal components

column of $XW$ (red) (predicted feature)
column of $Y$ (blue) (observed feature)
Principal components analysis (PCA): Given $Y \in \mathbb{R}^{n \times d}$, solve

$$\minimize \quad \| Y - XW \|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{d} (Y_{ij} - x_i^T w_j)^2$$

with $X \in \mathbb{R}^{n \times r}$, $W \in \mathbb{R}^{r \times d}$

how should we solve this problem?
Principal components analysis (PCA): Given \( Y \in \mathbb{R}^{n \times d} \), solve

\[
\text{minimize} \quad \| Y - XW \|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{d} (Y_{ij} - x_i^T w_j)^2
\]

with \( X \in \mathbb{R}^{n \times r}, \ W \in \mathbb{R}^{r \times d} \)

how should we solve this problem?

- idea 1: use the SVD
- idea 2: alternating minimization over \( X \) and \( W \)
The Frobenius norm

the **Frobenius norm**

\[ \| A \|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{d} A_{ij}^2} \]

some useful identities:

- \( \| A \|_F = \| \text{vec}(A) \| \)
- \( \| A \|_F = \| A^T \|_F \)
- \( \| A \|_F^2 = \text{tr}(A^T A) \)
- if \( U \) is orthogonal (i.e., \( U^T U = I \)), then \( \| UA \|_F = \| A \|_F \)
The Frobenius norm

the Frobenius norm

$$\| A \|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^d A_{ij}^2}$$

some useful identities:

- $\| A \|_F = \| \text{vec}(A) \|$
- $\| A \|_F = \| A^T \|_F$
- $\| A \|_F^2 = \text{tr}(A^T A)$
- if $U$ is orthogonal (i.e., $U^T U = I$), then $\| UA \|_F = \| A \|_F$

proof:

$$\| UA \|_F^2 = \text{tr}((UA)^T UA) = \text{tr}(A^T U^T UA) = \text{tr}(A^T A) = \| A \|_F^2$$
PCA: solution via the SVD

**PCA:** with $X \in \mathbb{R}^{n \times r}$, $W \in \mathbb{R}^{r \times d}$, solve

$$\text{minimize} \quad \| Y - XW \|^2_F = \sum_{i=1}^m \sum_{j=1}^n (Y_{ij} - x_i^T w_j)^2$$

**Eckart-Young-Mirsky theorem:** if

$$Y = U\Sigma V^T = \sum_{i=1}^{\text{Rank}(Y)} \sigma_i u_i v_i^T$$

is the SVD of $Y$, then

$$X = U_r, \quad W = \Sigma_r V_r^T$$

is a solution to PCA, where

$$\Sigma_r = \text{diag}(\sigma_1, \ldots, \sigma_r), \quad U_r = [u_1 \cdots u_r], \quad V_r = [v_1 \cdots v_r].$$
PCA: solution via the SVD

PCA: with $X \in \mathbb{R}^{n \times r}$, $W \in \mathbb{R}^{r \times d}$, solve

$$\text{minimize} \quad \| Y - XW \|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} (Y_{ij} - x_i^T w_j)^2$$

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$$X = U_r, \quad W = \Sigma_r V_r^T$$

is a solution to PCA, where

$$\Sigma_r = \text{diag}(\sigma_1, \ldots, \sigma_r), \quad U_r = [u_1 \cdots u_r], \quad V_r = [v_1 \cdots v_r].$$

with this $X$ and $W$,

$$\| Y - XW \|_F^2 = \| U \Sigma V^T - U_r \Sigma_r V_r^T \|_F^2 = \sum_{i=r+1}^{\text{Rank}(Y)} \sigma_i^2$$
Proof of Eckart-Young-Mirsky theorem I

proof step 1: reduce to diagonal.
if \( Y = U\Sigma V^T \) is the full SVD, then

\[
U^T U = UU^T = I \text{ and } V^T V = VV^T = I,
\]

so

\[
\| Y - XW \|_F^2 = \| U\Sigma V^T - XW \|_F^2
\]
\[
= \| U^T U\Sigma V^T V - U^T XWV \|_F^2
\]
\[
= \| \Sigma - U^T XWV \|_F^2
\]
\[
= \| \Sigma - Z \|_F^2
\]

where \( Z = U^T XWV \) is a rank \( r \) matrix.
we want to show

\[
\sum_{i=r+1} \sigma_i \leq \| \Sigma - Z \|_F^2
\]

for any rank \( r \) matrix \( Z \).
Proof of Eckart-Young-Mirsky theorem II

**proof step 2: eigenvalue interlacing.**
let’s use Weyl’s theorem for eigenvalues:
for any matrices $A, B \in \mathbb{R}^{n \times d}$,

$$
\sigma_{i+j-1}(A + B) \leq \sigma_i(A) + \sigma_j(B), \quad 1 \leq i, j \leq n.
$$

set $A = \Sigma - Z$, $B = Z$, $j = r + 1$ to get

$$
\sigma_{i+r}(\Sigma) \leq \sigma_i(\Sigma - Z) + \sigma_{r+1}(Z), \quad 1 \leq i \leq n - r
$$

$$
\sigma_{i+r} \leq \sigma_i(\Sigma - Z), \quad 1 \leq i \leq n - r,
$$

using $\text{Rank}(Z) \leq r$. square and sum from $i = 1$ to $\text{Rank}(Y) - r$:

$$
\|\Sigma - \Sigma_r\|_F^2 = \sum_{i=r+1}^{\text{Rank}(Y)} \sigma_i^2 \leq \sum_{i=1}^{\text{Rank}(Y)-r} \sigma_i^2(\Sigma - Z) \leq \|\Sigma - Z\|_F^2.
$$
PCA: solution via AM

\[
\text{minimize } \| Y - XW \|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} (Y_{ij} - x_i^T w_j)^2
\]

**Alternating Minimization (AM):** fix \( W^0 \). for \( t = 1, \ldots, \)

- \( X^t = \text{argmin}_X \| Y - XW^{t-1} \|_F^2 \)
- \( W^t = \text{argmin}_W \| Y - X^t W \|_F^2 \)

**properties:**

- objective decreases at each iteration
- objective bounded below, so the procedure converges
- (it is true but we won’t prove that) with probability 1 over choices of \( W^0 \), AM converges to an optimal solution
PCA: AM subproblem is separable

how would you solve the AM subproblem

\[ W^t = \arg\min_W \| Y - X^t W \|_F^2 = \arg\min_W \sum_{j=1}^{d} \|y_j - X^t w_j\|^2 \]

where \( Y = [y_1 \cdots y_d] \), \( W = [w_1 \cdots w_d] \)?
PCA: AM subproblem is separable

how would you solve the AM subproblem

\[
W^t = \arg\min_W \|Y - X^t W\|_F^2 = \arg\min_W \sum_{j=1}^d \|y_j - X^t w_j\|^2
\]

where \( Y = [y_1 \cdots y_d], \ W = [w_1 \cdots w_d] \)?

- problem separates over columns of \( W \):

\[
w_j^t = \arg\min_w \|y_j - X^t w\|^2
\]

- for each column of \( W \), it's just a least squares problem!

\[
w_j = ((X^t)^T X^t)^{-1}(X^t)^T y_j
\]
PCA: solution via AM

\[
\text{minimize } \| Y - XW \|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{d} (Y_{ij} - x_i^T w_j)^2
\]

**Alternating Minimization (AM):** fix $W^0$. for $t = 1, \ldots,$

- for $i = 1, \ldots, n,$
  \[
  x_i^t = Y_i: (W^{t-1})^T (W^{t-1}(W^{t-1})^T)^{-1}
  \]

- for $j = 1, \ldots, d,$
  \[
  w_j^t = ((X^T X^t)^{-1}(X^t)^T y_j
  \]
PCA: solution via AM

\[ \text{minimize} \quad \| Y - XW \|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{d} (Y_{ij} - x_i^T w_j)^2 \]

computational tricks:

- cache gram matrix \( G = (X^t)^T X^t \)
- parallelize over \( j \)

**Alternating Minimization (AM):** fix \( W^0 \). for \( t = 1, \ldots, \)

- cache factorization of \( G = W^{t-1}(W^{t-1})^T \)
- in parallel, for \( i = 1, \ldots, n, \)

\[ x_i^t = Y_i : (W^{t-1})^T (W^{t-1}(W^{t-1})^T)^{-1} \]

- cache factorization of \( G = (X^t)^T X^t \)
- in parallel, for \( j = 1, \ldots, d, \)

\[ w_j^t = ((X^t)^T X^t)^{-1}(X^t)^T y_j \]
PCA: solution via AM

minimize \( \| Y - XW \|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} (Y_{ij} - x_i^T w_j)^2 \)

complexity?

**Alternating Minimization (AM):** fix \( W^0 \). for \( t = 1, \ldots, \).
PCA: solution via AM

\[
\text{minimize} \quad \| Y - XW \|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} (Y_{ij} - x_i^T w_j)^2
\]

complexity?

**Alternating Minimization (AM):** fix \( W^0 \). for \( t = 1, \ldots, \)

- cache factorization of \( G = W^{t-1}(W^{t-1})^T \quad (O(dr^2 + r^3)) \)
PCA: solution via AM

\[ \text{minimize } \| Y - XW \|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} (Y_{ij} - x_i^T w_j)^2 \]

complexity?

**Alternating Minimization (AM):** fix \( W^0 \). for \( t = 1, \ldots, \)

\[ \begin{align*}
\quad & \text{cache factorization of } G = W^{t-1}(W^{t-1})^T \quad (O(dr^2 + r^3)) \\
\quad & \text{in parallel, for } i = 1, \ldots, n, \\
& x_i^t = (W^{t-1}(W^{t-1})^T)^{-1} W^{t-1} Y_i^T \quad (O(dr + r^2))
\end{align*} \]
PCA: solution via AM

minimize \( \| Y - XW \|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} (Y_{ij} - x_i^T w_j)^2 \)

complexity?

**Alternating Minimization (AM):** fix \( W^0 \). for \( t = 1, \ldots, \)

- cache factorization of \( G = W^{t-1}(W^{t-1})^T \) \((O(dr^2 + r^3))\)
- in parallel, for \( i = 1, \ldots, n, \)

\[
x_i^t = (W^{t-1}(W^{t-1})^T)^{-1} W^{t-1} Y_i^T
\]

- cache factorization of \( G = (X^t)^T X^t \) \((O(nr^2 + r^3))\)
PCA: solution via AM

minimize  \[ \| Y - XW \|_F^2 = \sum_{i=1}^m \sum_{j=1}^n (Y_{ij} - x_i^T w_j)^2 \]

complexity?

**Alternating Minimization (AM):** fix \( W^0 \). for \( t = 1, \ldots, \)

- cache factorization of \( G = W^{t-1}(W^{t-1})^T \) \( (O(dr^2 + r^3)) \)
- in parallel, for \( i = 1, \ldots, n, \)
  \[ x_i^t = (W^{t-1}(W^{t-1})^T)^{-1} W^{t-1} Y_i^T \]

- cache factorization of \( G = (X^t)^TX^t \) \( (O(nr^2 + r^3)) \)
- in parallel, for \( j = 1, \ldots, d, \)
  \[ w_j^t = ((X^t)^TX^t)^{-1}(X^t)^T y_j \]
Outline

Missing data
Unsupervised learning
Low rank models
Principal Components Analysis

Generalized Low Rank Models

Imputing missing data
Multidimensional losses
More about regularizers
Clustering
now suppose we observe $Y_{ij}$ only for $(i, j) \in \Omega \subset \{1, \ldots, n\} \times \{1, \ldots, d\}$
now suppose we observe $Y_{ij}$ only for $(i, j) \in \Omega \subset \{1, \ldots, n\} \times \{1, \ldots, d\}$

Matrix completion:

$$\text{minimize } \sum_{(i,j) \in \Omega} (Y_{ij} - x_i^T w_j)^2 + \lambda \sum_{i=1}^n \|x_i\|_2^2 + \lambda \sum_{j=1}^d \|w_j\|_2^2$$

two regimes:

- **some entries missing**: don’t waste data; “borrow strength” from entries that are **not** missing
- **most entries missing**: matrix completion still works!
Huber PCA

\[
\text{minimize } \sum_{(i,j) \in \Omega} \text{huber}(Y_{ij} - x_i^T w_j) + \sum_{i=1}^{n} \|x_i\|_2^2 + \sum_{j=1}^{d} \|w_j\|_2^2
\]
Huber PCA

huber loss with corrupted data (asymmetric noise)

relative mse

fraction of corrupted entries

0.00 0.05 0.10 0.15 0.20 0.25 0.30

0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

glrm

pca
Generalized low rank models

\[
\text{minimize} \quad \sum_{(i,j) \in \Omega} \ell_j(Y_{ij}, x_i^T w_j) + \sum_{i=1}^n r_i(x_i) + \sum_{j=1}^d \tilde{r}_j(w_j)
\]

- observe only \((i,j) \in \Omega\) (other entries are missing)
- loss functions \(\ell_j\) for each column
  - assume \(Y_{ij} \in \mathcal{Y}_j\) for every \((i,j) \in \Omega\)
  - \(\ell_j : \mathcal{Y}_j \times \mathbb{R} \to \mathbb{R}\)
  - e.g., different losses for reals, booleans, categoricals, ordinals, . . .
- regularizers \(r : \mathbb{R}^{1 \times r} \to \mathbb{R}, \tilde{r} : \mathbb{R}^r \to \mathbb{R}\)
Losses

minimize \( \sum_{(i,j) \in \Omega} L_j(x_i, y_j, A_{ij}) + \sum_{i=1}^m r_x(x_i) + \sum_{j=1}^n r_y(y_j) \)

choose loss \( L : \mathbb{R} \times \mathcal{F} \to \mathbb{R} \) adapted to data type \( \mathcal{F} \):

<table>
<thead>
<tr>
<th>data type</th>
<th>loss</th>
<th>( L(u, a) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>real</td>
<td>QuadLoss</td>
<td>((u - a)^2)</td>
</tr>
<tr>
<td>real</td>
<td>L1Loss</td>
<td>(</td>
</tr>
<tr>
<td>real</td>
<td>HuberLoss</td>
<td>( \text{huber}(u - a) )</td>
</tr>
<tr>
<td>boolean</td>
<td>HingeLoss</td>
<td>((1 - ua)_+)</td>
</tr>
<tr>
<td>boolean</td>
<td>LogisticLoss</td>
<td>(\log(1 + \exp(-au)))</td>
</tr>
<tr>
<td>ordinal</td>
<td>BvSLoss</td>
<td>( \sum_{a' = 1}^d (1 - u(a \geq a'))<em>+ + \sum</em>{a' = 1}^{a-1} (1 - u + a')<em>+ + \sum</em>{a' = a+1}^d (1 + u - a')_+ )</td>
</tr>
<tr>
<td>ordinal</td>
<td>OrdinalHingeLoss</td>
<td>( \sum_{a' = 1}^d (1 - u(a \geq a'))<em>+ + \sum</em>{a' = 1}^{a-1} (1 - u + a')<em>+ + \sum</em>{a' = a+1}^d (1 + u - a')_+ )</td>
</tr>
<tr>
<td>categorical</td>
<td>OvALoss</td>
<td>((1 - u_a)<em>+ + \sum</em>{a' \neq a} (1 + u_{a'})_+ )</td>
</tr>
<tr>
<td>categorical</td>
<td>MultinomialLoss</td>
<td>( \frac{\exp(u_a)}{\sum_{a' = 1}^d \exp(u_{a'})} )</td>
</tr>
</tbody>
</table>
Regularizers

minimize \( \sum_{(i,j) \in \Omega} L_j(x_i y_j, A_{ij}) + \sum_{i=1}^{m} r_x(x_i) + \sum_{j=1}^{n} r_y(y_j) \)

choose regularizers \( r_x, r_y \) to impose structure:

<table>
<thead>
<tr>
<th>structure</th>
<th>( r_x )</th>
<th>( r_y )</th>
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</thead>
<tbody>
<tr>
<td>small</td>
<td>QuadReg</td>
<td>QuadReg</td>
</tr>
<tr>
<td>sparse</td>
<td>OneReg</td>
<td>OneReg</td>
</tr>
<tr>
<td>nonnegative</td>
<td>NonNegConstraint</td>
<td>NonNegConstraint</td>
</tr>
<tr>
<td>clustered</td>
<td>UnitOneSparseConstraint</td>
<td>ZeroReg</td>
</tr>
</tbody>
</table>
Impute missing data

impute most likely true data $\hat{A}_{ij}$

$$\hat{A}_{ij} = \arg\min_a L_j(x_i y_j, a)$$

► implicit constraint: $\hat{A}_{ij} \in F_j$

► when $L_j$ is quadratic, $\ell_1$, or Huber loss, then $\hat{A}_{ij} = x_i y_j$

► if $F \neq \mathbb{R}$, $\arg\min_a L_j(x_i y_j, a) \neq x_i y_j$

► e.g., for hinge loss $L(u, a) = (1 - ua)_+$, $\hat{A}_{ij} = \text{sign}(x_i y_j)$
Impute heterogeneous data

mixed data types

remove entries
Impute heterogeneous data

- Mixed data types
- Remove entries
- QPCA rank 10 recovery
- Error

GLRM rank 10 recovery

Error
Impute heterogeneous data

mixed data types

remove entries

qPCA rank 10 recovery

error

gLRM rank 10 recovery

error
**Julia implementation: demo**

**example:** fit rank 5 GLRM in 2 lines of code:

```julia
    glrm = GLRM(A, 5, datatypes)
    X,Y = fit!(glrm)
```
Validate model

\[
\text{minimize } \sum_{(i,j) \in \Omega} L_{ij}(A_{ij}, x_i y_j) + \lambda \left( \sum_{i=1}^{m} r_x(x_i) + \sum_{j=1}^{n} r_y(y_j) \right)
\]

How to choose model parameters \((k, \lambda)\)?
Validate model

minimize $\sum_{(i,j) \in \Omega} L_{ij}(A_{ij}, x_i y_j) + \lambda \left( \sum_{i=1}^{m} r_x(x_i) + \sum_{j=1}^{n} r_y(y_j) \right)$

How to choose model parameters $(k, \lambda)$?
Leave out 10% of entries, and use model to predict them
Outline

Missing data
Unsupervised learning
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Principal Components Analysis
Generalized Low Rank Models
Imputing missing data
Multidimensional losses
More about regularizers
Clustering
Hospitalizations are low rank

hospitalization data set

demographics

diagnoses

procedures

comorbidities

[Schuler et al., 2016]
Impute censored data

market segmentation

<table>
<thead>
<tr>
<th>customer</th>
<th>apples</th>
<th>oranges</th>
<th>pears</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>yes</td>
<td>?</td>
<td>yes</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>yes</td>
<td>yes</td>
<td>?</td>
<td>...</td>
</tr>
<tr>
<td>3</td>
<td>?</td>
<td>?</td>
<td>yes</td>
<td>...</td>
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<td>...</td>
<td>...</td>
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<td></td>
</tr>
</tbody>
</table>

- rows of $W$ are purchasing patterns for market segments
- rows of $X$ classify customers into market segment(s)
- imputation: recommend new products, target advertising campaign
Impute censored data

synthetic data:

- generate rank-5 matrix of probabilities, $p \in \mathbb{R}^{300 \times 300}$

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<th>pears</th>
<th>...</th>
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</thead>
<tbody>
<tr>
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<td>...</td>
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<td>2</td>
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<tr>
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</table>
## Impute censored data

**synthetic data:**

- entry \((i,j)\) is + with probability \(p_{ij}\)

<table>
<thead>
<tr>
<th>customer</th>
<th>apples</th>
<th>oranges</th>
<th>pears</th>
<th>⋯</th>
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<tbody>
<tr>
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</table>
Impute censored data

synthetic data:

- but we only observe +s...

<table>
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<tr>
<th></th>
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</table>
**Impute censored data**

synthetic data:

- ... and we only observe 10% of the +s

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</table>

... ... ... ... can we predict 10 more +s?
Impute censored data

synthetic data:

- ... and we only observe 10% of the +s

<table>
<thead>
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<th>pears</th>
<th>⋮</th>
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<td>⋮</td>
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<td>⋮</td>
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</tbody>
</table>

can we predict 10 more +s?
Impute censored data

![Graph showing the relationship between probability of +1 and regularization parameter.](image)

- **Y-axis**: Probability of +1
- **X-axis**: Regularization parameter

**Legend**
- Green triangles: Precision@10

Precision@10 increases with the regularization parameter.
Outline

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Multi-dimensional loss

- approximate using vectors $x_i W_j \in \mathbb{R}^{1 \times d_j}$ instead of numbers
- need $\ell_j : \mathbb{R}^{1 \times d_j} \times \mathcal{Y}_j \to \mathbb{R}$

$$\text{minimize} \quad \sum_{(i,j) \in \Omega} \ell_j(x_i; W_j, Y_{ij}) + \sum_{i=1}^m r_i(x_i) + \sum_{j=1}^n \tilde{r}_j(W_j)$$

- useful for approximating categorical variables
  - columns of $W_j$ represent different labels of categorical variable
- gives more flexible/accurate models for ordinal variables
Multivariate categorical loss

- choose any loss function for multiclass classification to penalize $x_i Y$
  - e.g., one-vs-all (elementwise hinge loss) [Rifkin 2004]

$$
\ell(z, y) = (1 - z_y)_+ + \sum_{y' \neq y} (1 + z_{y'})_+
$$

\[
\begin{bmatrix}
\text{CA} & \text{NV} & \cdots & \text{PA} & \text{NY} \\
T & F & \cdots & F & F \\
F & F & \cdots & T & F \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\approx
\begin{bmatrix}
\overline{x_1} \\
\vdots \\
\overline{x_m}
\end{bmatrix}
\]
Multivariate ordinal loss

- automatically detect which labels are more similar
- fit positions of data ($X$) and separating hyperplanes ($W$) simultaneously
Scaling losses

Analogue of standardization for GLRMs:

\[ \mu_j = \arg\min_{\mu} \sum_{i:\,(i,j)\in\Omega} \ell_j(\mu, Y_{ij}) \]

\[ \sigma_j^2 = \frac{1}{n_j - 1} \sum_{i:\,(i,j)\in\Omega} \ell_j(\mu_j, Y_{ij}) \]

- \( n_j \) is number of observations in column \( j \)
- \( \mu_j \) generalizes column mean
- \( \sigma_j^2 \) generalizes column variance

To fit a standardized GLRM, solve

\[ \text{minimize} \quad \sum_{(i,j)\in\Omega} \frac{\ell_j(Y_{ij}, x_i W_j + \mu_j)}{\sigma_j^2} + \sum_{i=1}^{\text{r}} r_i(x_i) + \sum_{j=1}^{\text{d}} \tilde{r}_j(W_j) \]
Scaling losses

Analogue of standardization for GLRMs:

\[
\mu_j = \arg\min_\mu \sum_{i:(i,j) \in \Omega} \ell_j(\mu, Y_{ij})
\]

\[
\sigma_j^2 = \frac{1}{n_j - 1} \sum_{i:(i,j) \in \Omega} \ell_j(\mu_j, Y_{ij})
\]

- \( n_j \) is number of observations in column \( j \)
- \( \mu_j \) generalizes column mean
- \( \sigma_j^2 \) generalizes column variance

To fit a standardized GLRM, solve

\[
\text{minimize } \sum_{(i,j) \in \Omega} \ell_j(Y_{ij}, x_i W_j + \mu_j) / \sigma_j^2 + \sum_{i=1}^n r_i(x_i) + \sum_{j=1}^d \tilde{r}_j(W_j)
\]

can be put in standard form: add an offset by modifying \( r \)!
American community survey

2013 ACS:

- 3M respondents, 87 economic/demographic survey questions
  - income
  - cost of utilities (water, gas, electric)
  - weeks worked per year
  - hours worked per week
  - home ownership
  - looking for work
  - use foodstamps
  - education level
  - state of residence
  - ...

- 1/3 of responses missing
Application: exploratory data analysis

<table>
<thead>
<tr>
<th>age</th>
<th>gender</th>
<th>state</th>
<th>...</th>
</tr>
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<tbody>
<tr>
<td>29</td>
<td>F</td>
<td>CT</td>
<td>...</td>
</tr>
<tr>
<td>57</td>
<td>?</td>
<td>NY</td>
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<td>CA</td>
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</tbody>
</table>

$\begin{bmatrix} w_1 & \cdots & w_d \end{bmatrix}$

$\begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix}$

cluster respondents?
Application: exploratory data analysis

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\[ X \approx \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix} \]  

- cluster respondents? **cluster rows of** \( X \)
Application: exploratory data analysis

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\[
\begin{bmatrix}
    w_1 & \cdots & w_d
\end{bmatrix}
\]

\[
\begin{bmatrix}
    x_1^T \\
    \vdots \\
    x_n^T
\end{bmatrix}
\]

- cluster respondents? **cluster rows of** \( X \)
- demographic profiles?
**Application: exploratory data analysis**

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\[
\begin{bmatrix}
\mathbf{w}_1 & \cdots & \mathbf{w}_d
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathbf{x}_1^T \\
\vdots \\
\mathbf{x}_n^T
\end{bmatrix}
\]

- cluster respondents? **cluster rows of** \( X \)
- demographic profiles? **rows of** \( W \)
Application: exploratory data analysis

<table>
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</tbody>
</table>

$\begin{bmatrix} w_1 & \cdots & w_d \end{bmatrix}$

$\begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix}$

- cluster respondents? **cluster rows of** $X$
- demographic profiles? **rows of** $W$
- which features are similar?
Application: exploratory data analysis

\[
\begin{bmatrix}
W_1 & \cdots & W_d
\end{bmatrix}
\]

<table>
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<tr>
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<th>\cdots</th>
</tr>
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<td>M</td>
<td>CA</td>
<td>\cdots</td>
</tr>
<tr>
<td>41</td>
<td>F</td>
<td>NV</td>
<td>\cdots</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
</tbody>
</table>

- cluster respondents? **cluster rows of** $X$
- demographic profiles? **rows of** $W$
- which features are similar? **cluster columns of** $W$
Application: exploratory data analysis

<table>
<thead>
<tr>
<th>age</th>
<th>gender</th>
<th>state</th>
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</tr>
</thead>
<tbody>
<tr>
<td>29</td>
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<td>CT</td>
<td></td>
</tr>
<tr>
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</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{bmatrix}
  w_1 & \cdots & w_d
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \ldots
\end{bmatrix}
\]

- cluster respondents? **cluster rows of** \( X \)
- demographic profiles? **rows of** \( W \)
- which features are similar? **cluster columns of** \( W \)
- impute missing entries?
Application: exploratory data analysis

\[
\begin{bmatrix}
  w_1 & \cdots & w_d \\
\end{bmatrix}
\]

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<th>state</th>
<th>\ldots</th>
</tr>
</thead>
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<td>\ldots</td>
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<td>F</td>
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<td>\ldots</td>
</tr>
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<td>\vdots</td>
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</tr>
</tbody>
</table>

- cluster respondents? **cluster rows of** \( X \)
- demographic profiles? **rows of** \( W \)
- which features are similar? **cluster columns of** \( W \)
- impute missing entries? \( \arg\min_{y \in Y_j} \ell_j(y, x_i^T w_j) \)
Fitting a GLRM to the ACS

- construct a rank 10 GLRM with loss functions respecting data types
  - huber for real values
  - hinge loss for booleans
  - ordinal hinge loss for ordinals
  - one-vs-all hinge loss for categoricals
- scale losses and regularizers
- fit the GLRM

in 2 lines of code:

```python
glrm, labels = GLRM(Y, 10, scale = true)
X,W = fit!(glrm)
```
American community survey

most similar features (in demography space):

- Alaska: Montana, North Dakota
- California: Illinois, cost of water
- Colorado: Oregon, Idaho
- Ohio: Indiana, Michigan
- Pennsylvania: Massachusetts, New Jersey
- Virginia: Maryland, Connecticut
- Hours worked: weeks worked, education
Low rank models for dimensionality reduction

U.S. Wage & Hour Division (WHD) compliance actions:

<table>
<thead>
<tr>
<th>company</th>
<th>zip</th>
<th>violations</th>
<th>⋮</th>
</tr>
</thead>
<tbody>
<tr>
<td>Holiday Inn</td>
<td>14850</td>
<td>109</td>
<td>⋮</td>
</tr>
<tr>
<td>Moosewood Restaurant</td>
<td>14850</td>
<td>0</td>
<td>⋮</td>
</tr>
<tr>
<td>Cornell Orchards</td>
<td>14850</td>
<td>0</td>
<td>⋮</td>
</tr>
<tr>
<td>Lakeside Nursing Home</td>
<td>14850</td>
<td>53</td>
<td>⋮</td>
</tr>
<tr>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
</tr>
</tbody>
</table>

- 208,806 rows (cases) × 252 columns (violation info)
- 32,989 zip codes...

---

1labor law violation demo: https://github.com/h2oai/h2o-3/blob/master/h2o-r/demos/rdemo.census.labor.violations.large.R
Low rank models for dimensionality reduction

ACS demographic data:

<table>
<thead>
<tr>
<th>zip</th>
<th>unemployment</th>
<th>mean income</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>94305</td>
<td>12%</td>
<td>$47,000</td>
<td>...</td>
</tr>
<tr>
<td>06511</td>
<td>19%</td>
<td>$32,000</td>
<td>...</td>
</tr>
<tr>
<td>60647</td>
<td>23%</td>
<td>$23,000</td>
<td>...</td>
</tr>
<tr>
<td>94121</td>
<td>4%</td>
<td>$178,000</td>
<td>...</td>
</tr>
</tbody>
</table>

- 32,989 rows (zip codes) $\times$ 150 columns (demographic info)
- GLRM embeds zip codes into (low dimensional) demography space
Low rank models for dimensionality reduction

Zip code features:
Low rank models for dimensionality reduction

build 3 sets of features to predict violations:

▶ categorical: expand zip code to categorical variable
▶ concatenate: join tables on zip
▶ GLRM: replace zip code by low dimensional zip code features

fit a supervised (deep learning) model:

<table>
<thead>
<tr>
<th>method</th>
<th>train error</th>
<th>test error</th>
<th>runtime</th>
</tr>
</thead>
<tbody>
<tr>
<td>categorical</td>
<td>0.2091690</td>
<td>0.2173612</td>
<td>23.7600000</td>
</tr>
<tr>
<td>concatenate</td>
<td>0.2258872</td>
<td>0.2515906</td>
<td>4.4700000</td>
</tr>
<tr>
<td>GLRM</td>
<td>0.1790884</td>
<td>0.1933637</td>
<td>4.3600000</td>
</tr>
</tbody>
</table>
recap: why use GLRMNs?

use GLRMNs to

- fill in missing data
- embed data points into low dimensional space
- reduce dimensionality of large categorical features
- design recommender systems
Outline

Missing data
Unsupervised learning
Low rank models
Principal Components Analysis
Generalized Low Rank Models
Imputing missing data
Multidimensional losses
More about regularizers
Clustering
## Factor model of sector returns

<table>
<thead>
<tr>
<th>ticker</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAPL</td>
<td>0.05</td>
<td>-0.21</td>
<td>...</td>
</tr>
<tr>
<td>KRX</td>
<td>0.07</td>
<td>-0.18</td>
<td>...</td>
</tr>
<tr>
<td>GOOG</td>
<td>-0.11</td>
<td>0.24</td>
<td>...</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

- rows of $Y$ are sector return time series
- rows of $X$ are sector exposures
Low rank models for power

electricity usage profiles

<table>
<thead>
<tr>
<th>household</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$\cdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.4</td>
<td>0.5</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>2.7</td>
<td>1.3</td>
<td>0.9</td>
</tr>
<tr>
<td>3</td>
<td>3.3</td>
<td>4.2</td>
<td>1.8</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>

- rows of $Y$ are electricity usage profiles
- rows of $X$ decompose household power usage into distinct usage profiles
Regularizers

\[
\text{minimize} \quad \sum_{(i,j) \in \Omega} \ell_j(Y_{ij}, x_i^T w_j) + \sum_{i=1}^n r_i(x_i) + \sum_{j=1}^d \tilde{r}_j(w_j)
\]

choose regularizers \( r, \tilde{r} \) to impose structure:

<table>
<thead>
<tr>
<th>structure</th>
<th>( r(x) )</th>
<th>( \tilde{r}(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>small</td>
<td>( |x|_2^2 )</td>
<td>( |y|_2^2 )</td>
</tr>
<tr>
<td>sparse</td>
<td>( |x|_1 )</td>
<td>( |y|_1 )</td>
</tr>
<tr>
<td>nonnegative</td>
<td>( 1(x \geq 0) )</td>
<td>( 1(y \geq 0) )</td>
</tr>
</tbody>
</table>
Nonnegative matrix factorization

\[
\text{minimize} \quad \sum_{(i,j) \in \Omega} (Y_{ij} - x_i^T w_j)^2 + \sum_{i=1}^{n} 1_+(x_i) + \sum_{j=1}^{d} 1_+(w_j)
\]

regularizer is indicator of nonnegative orthant

\[
1_+(x) = \begin{cases} 
0 & x \geq 0 \\
\infty & \text{otherwise}
\end{cases}
\]
Nonnegative matrix factorization

\[
\text{minimize} \quad \sum_{(i,j) \in \Omega} (Y_{ij} - x_i^T w_j)^2 + \sum_{i=1}^n 1_+ (x_i) + \sum_{j=1}^d 1_+ (w_j)
\]

regularizer is indicator of nonnegative orthant

\[
1_+ (x) = \begin{cases}
0 & x \geq 0 \\
\infty & \text{otherwise}
\end{cases}
\]

subproblems are nonnegative least squares problems:

\[
x_i^{t+1} = \arg\min_{x > 0} \sum_{j: (i,j) \in \Omega} (Y_{ij} - x^T w_j^t)^2 \tag{1}
\]

\[
w_j^{t+1} = \arg\min_{w > 0} \sum_{i: (i,j) \in \Omega} (Y_{ij} - (x_i^{t+1})^T w)^2 \tag{2}
\]
Outline

Missing data

Unsupervised learning

Low rank models

Principal Components Analysis

Generalized Low Rank Models

Imputing missing data

Multidimensional losses

More about regularizers

Clustering
Clustering

A clustering algorithm groups data points into clusters.

Examples:

- **Medical diagnosis.** Cluster patients with similar medical histories.
- **Topic model.** Cluster documents with similar patterns of word usage.
- **Market segmentation.** Cluster customers with similar purchase patterns.
the *k*-means problem:

- given data points \( y_i \in \mathbb{R}^d, \ i = 1, \ldots, n \)
- find \( k \) centers \( w_l \in \mathbb{R}^d, \ l = 1, \ldots, k \)
- and assignments \( c_i \in \{1, \ldots, k\}, \ i = 1, \ldots, n \)
- to minimize

\[
\sum_{i=1}^{n} \|y_i - w_{c_i}\|^2
\]
Lloyd’s algorithm for $k$-means

**Lloyd’s algorithm** (aka the $k$-means algorithm): to minimize

$$\sum_{i=1}^{n} \|y_i - w_{c_i}\|^2,$$

repeat

1. assign points to centers

$$c_i = \arg\min_{l=1,\ldots,k} \|y_i - w_l\|^2, \quad i = 1, \ldots, n$$

2. update centers: let $C_l = \{i : c_i = l\}$ be points assigned to cluster $l$, and set

$$w_l = \frac{1}{|C_l|} \sum_{i \in C_l} y_i, \quad l = 1, \ldots, k$$

visualizing the algorithm:

http://stanford.edu/class/ee103/visualizations/kmeans/kmeans.html
Lloyd’s algorithm for \( k \)-means

**Lloyd’s algorithm** (aka the \( k \)-means algorithm): to minimize

\[
\sum_{i=1}^{n} \| y_i - w_{c_i} \|^2,
\]

repeat

1. assign points to centers

\[ c_i = \arg \min_{l=1,\ldots,k} \| y_i - w_l \|^2, \quad i = 1, \ldots, n \]

2. update centers

\[ w_l = \frac{1}{|C_l|} \sum_{i \in C_l} y_i = \arg \min_{l=1,\ldots,k} \sum_{i : c_i = l} \| y_i - w \|^2, \quad l = 1, \ldots, k \]
Quadratic clustering

minimize \( \sum_{(i,j) \in \Omega} (Y_{ij} x_i^T w_j)^2 + \sum_{i=1}^{n} 1_1(x_i) \)

- \( 1_1 \) is the indicator function of a selection, i.e.,

\[
1_1(x) = \begin{cases} 
0 & x = e_l \text{ for some } l \in \{1, \ldots, k\} \\
\infty & \text{otherwise}
\end{cases}
\]

where \( e_l \) is the \( l \)th unit vector

alternating minimization reproduces \( k \)-means (but allows missing data)
Quadratic clustering

minimize $\sum_{(i,j) \in \Omega} (Y_{ij} x_i^T w_j)^2 + \sum_{i=1}^n 1_1(x_i)$

$1_1$ is the indicator function of a selection, i.e.,

$1_1(x) = \begin{cases} 0 & x = e_l \text{ for some } l \in \{1, \ldots, k\} \\ \infty & \text{otherwise} \end{cases}$

where $e_l$ is the $l$th unit vector

alternating minimization reproduces $k$-means
(but allows missing data)
Check AM reproduces $k$-means

let $w^l$ be $l$th row of $W$, $l = 1, \ldots, k$

\[
\sum_{(i,j) \in \{1,\ldots,n\} \times \{1,\ldots,d\}} (Y_{ij} - x_i^T w_j)^2 = \sum_{l=1}^{k} \sum_{i \in C_l} \sum_{j=1}^{d} (Y_{ij} - x_i^T w_j)^2
\]

\[
= \sum_{l=1}^{k} \sum_{i \in C_l} \sum_{j=1}^{d} (Y_{ij} - e_l w_j)^2
\]

\[
= \sum_{l=1}^{k} \sum_{i \in C_l} \sum_{j=1}^{d} (Y_{ij} - w_j^l)^2
\]

\[
= \sum_{l=1}^{k} \sum_{i \in C_l} \| Y_i - w^l \|^2
\]

$\triangleright$ to minimize over $W$: set $w^l$ to be the mean of $Y_i$ for $i \in C_l$

\[
w^l = \frac{1}{C_l} \sum_{i \in C_l} Y_i
\]
Check AM reproduces $k$-means

let $w^l$ be $l$th row of $W$, $l = 1, \ldots, k$

$$
\sum_{(i,j) \in \{1,\ldots,n\} \times \{1,\ldots,d\}} (Y_{ij} - x_i^T w_j)^2
= \sum_{l=1}^{k} \sum_{i \in C_l} \sum_{j=1}^{d} (Y_{ij} - x_i^T w_j)^2

= \sum_{l=1}^{k} \sum_{i \in C_l} \sum_{j=1}^{d} (Y_{ij} - e_l w_j)^2

= \sum_{l=1}^{k} \sum_{i \in C_l} \sum_{j=1}^{d} (Y_{ij} - w_l^l)^2

= \sum_{l=1}^{k} \sum_{i \in C_l} \|Y_i - w_l^l\|^2
$$

To minimize over $X$: set $x_i$ to be the unit vector $e_l$

$$
x_i = e_l \quad \text{where} \quad l = \arg\min l' \in 1, \ldots, k \|Y_i - w_l^l\|^2
$$
What’s a cluster?
Modifying $k$-means

different regularizers:

- clusters
- rays
- lines
- planes
- cones
Modifying $k$-means

different regularizers:

- clusters
- rays
- lines
- planes
- cones

different losses:

- $k$-means: $\ell(y, z) = (y - z)^2$
- $k$-medioids: $\ell(y, z) = |y - z|$
- $\ell(y, z) = \text{huber}(y - z)$
- ...
Fitting GLRM with alternating minimization

\[
\text{minimize} \quad \sum_{(i,j) \in \Omega} L_j(x_i w_j, Y_{ij}) + \sum_{i=1}^{m} r_i(x_i) + \sum_{j=1}^{n} \tilde{r}_j(w_j)
\]

repeat:

1. minimize objective over \(x_i\) (in parallel)
2. minimize objective over \(w_j\) (in parallel)

properties:

- subproblems easy to solve
- objective decreases at every step, so converges if losses and regularizers are bounded below
- (not guaranteed to find global solution, but) usually finds good model in practice
- naturally parallel, so scales to huge problems
Alternating updates

given $X^0, W^0$

for $t = 1, 2, \ldots$ do

  for $i = 1, \ldots, m$ do

    $x_i^t = \text{update}_{L,r}(x_i^{t-1}, W^{t-1}, Y)$

  for $j = 1, \ldots, n$ do

    $w_j^t = \text{update}_{L,\tilde{r}}(w_j^{(t-1)T}, X(t)^T, Y^T)$

  ▶ no need to exactly minimize

  ▶ choose fast, simple update rules
A simple, fast update rule

proximal gradient method: let

\[ g = \sum_{j: (i,j) \in \Omega} \nabla \ell_j(x_iw_j, Y_{ij})w_j \]

and update

\[ x_i^{t+1} = \text{prox}_{\alpha_t r}(x_i^t - \alpha_t g) \]

- **simple**: only requires ability to evaluate \( \nabla L \) and \( \text{prox}_r \)
- **stochastic variant**: use noisy estimate for \( g \)
- **time per iteration**: \( O(\frac{(n+d+|\Omega|)k}{p}) \) on \( p \) processors
Recap: GLRM}s

Generalized Low Rank Models are a **framework** that encompasses a bunch of unsupervised learning models.

many of these GLRMs have names:

<table>
<thead>
<tr>
<th>Model</th>
<th>(\ell(y, z))</th>
<th>(r(x))</th>
<th>(\tilde{r}(w))</th>
<th>reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>PCA</td>
<td>((y - z)^2)</td>
<td>0</td>
<td>0</td>
<td>[Pearson 1901]</td>
</tr>
<tr>
<td>NNMF</td>
<td>((y - z)^2)</td>
<td>(1_+(x))</td>
<td>(1_+(w))</td>
<td>[Lee 1999]</td>
</tr>
<tr>
<td>sparse PCA</td>
<td>((y - z)^2)</td>
<td>(|x|_1)</td>
<td>(|w|_1)</td>
<td>[D’Aspremont 2004]</td>
</tr>
<tr>
<td>sparse coding</td>
<td>((y - z)^2)</td>
<td>(|x|_1)</td>
<td>(|w|_2)</td>
<td>[Olshausen 1997]</td>
</tr>
<tr>
<td>k-means</td>
<td>((y - z)^2)</td>
<td>(1_1(x))</td>
<td>0</td>
<td>[Tropp 2004]</td>
</tr>
<tr>
<td>matrix completion</td>
<td>((y - z)^2)</td>
<td>(|x|_2)</td>
<td>(|w|_2)</td>
<td>[Keshavan 2010]</td>
</tr>
<tr>
<td>robust PCA</td>
<td>(</td>
<td>y - z</td>
<td>)</td>
<td>(|x|_2)</td>
</tr>
<tr>
<td>logistic PCA</td>
<td>(\log(1 + \exp(-yz)))</td>
<td>(|x|_2)</td>
<td>(|w|_2)</td>
<td>[Collins 2001]</td>
</tr>
<tr>
<td>boolean PCA</td>
<td>((1 - yz)_+)</td>
<td>(|x|_2)</td>
<td>(|w|_2)</td>
<td>[Srebro 2004]</td>
</tr>
</tbody>
</table>
Resources

- GLRM

- fitting GLRMS
  https://github.com/madeleineudell/LowRankModels.jl