1. *Grading by Matrix Completion.* ORIE 4741 uses peer grading to determine scores on final projects. Each project has an underlying quality; some are good, some less good. Some students are fair graders, and report the project quality as their grade. Some are easy graders, and report a higher grade. Some are harsh graders, and report a lower grade. As a result, some students fear that peer grading is unfair: why should their grade suffer simply through the random chance of a harsh reviewer?

An ideal solution might be to have every student grade every project. However, this solution is rarely popular with reviewers. Instead, in this homework problem, we will explore whether we can predict the ratings that all other reviewers *would have* given had they reviewed all projects. We will try this technique both on a synthetic data and on the real peer-review rating scores for ORIE4741 projects in 2017.

Formally, let’s define our problem. There are $d$ students enrolled in ORIE 4741 who have formed $n$ project groups. Each student is responsible for grading $p$ projects. (In our class, $p=2$.) We’ll collect the grades into a grade matrix $A \in \mathbb{R}^{n \times d}$: $A_{ij}$ will represent the grade that student $j$ would assign to project $i$.

Of course, we cannot assign each student to grade every project. Instead, we make peer review assignments $\Omega = \{(i_1, j_1), \ldots, \}$. Here, $(i, j) \in \Omega$ if student $j$ is assigned to grade project $i$.

Unfortunately, this means that some projects are assigned harder graders than other projects. Our goal is to find a fair way to compute a project’s final grade. We consider two methods:

- **Averaging.** The grade $g_i$ for project $i$ is the average of the grades given by peer reviewers:

  $$g_i^{\text{avg}} = \frac{1}{\Omega_i} \sum_{j : (i, j) \in \Omega} A_{ij}$$

  where $\Omega_i = |\{j : (i, j) \in \Omega\}|$ is the number of graders project $i$ receives.
Matrix completion. We fit a low rank model to the grade matrix and use it to compute an estimate \( \hat{A} \) of the grade matrix by solving

$$
\minimize \sum_{(i,j) \in \Omega} \ell(A_{ij}, (X^T Y)_{ij}) + r(X) + r(Y),
$$

where \( X \in \mathbb{R}^{k \times n} \) and \( Y \in \mathbb{R}^{k \times d} \). We will try a few different losses \( \ell \) and regularizers \( r \) to see which work best.

We will then compute our estimate \( \hat{A} \) as

$$
\hat{A} = X^T Y.
$$

In other words, \( \hat{A} \) is the rank-\( k \) matrix that matches the observations best in the sense of the losses \( \ell \) and regularizers \( r \) we picked.

We compute the grade \( g_i \) for project \( i \) as the average of these estimated grades:

$$
g_{i}^{mc} = \frac{1}{d} \sum_{j=1}^{d} \hat{A}_{ij}
$$

In this problem, we will consider which of these two grading schemes, averaging or matrix completion, is better.

(a) Analytical problem. Consider \( n = 2 \) project groups and \( d = 4 \) peer graders. Suppose group 1 did well on their project and deserves a grade of 6; whereas group 2 deserves a grade of 3. Graders 1 and 2 are easy graders, and graders 3 and 4 are harsh.

Each project is graded by three graders. The grades given are

$$
A = \begin{bmatrix}
\times & 8 & 4 & 4 \\
4 & 4 & 2 & \times
\end{bmatrix}.
$$

Here, an \( \times \) in the \((i,j)\)th entry means the \( j \)th student was not responsible for grading the \( i \)th project.

Use both methods, averaging and matrix completion, to compute grades for the two groups. Here, you should be able to compute the results of both methods by hand (on paper). Explain how you computed \( \hat{A} \).

Compare your results. Which grading method would you say is more fair?

(b) A more realistic example. Work on the Jupyter notebook at