Service Levels in \((Q, R)\) Systems

\(p\) depends on multiple factors:

- Impact of stockout on future sales
- Loss of goodwill
- Corporate mindset:
  - customer-focused vs. cost-focused
Service Levels in \((Q, R)\) Systems

- When we minimize cost in the \((Q, R)\) model, the parameter \(p\) controls how often we stock out.
- Often it is more intuitive and useful to specify a desired minimum service level instead of \(p\).
- Roughly speaking, “service level” is a measure of how often we satisfy demand from in-stock inventory.
Definition of Service Level

- There are multiple ways of defining the “service level” precisely. We will use the following definition, also called the “fill rate”
- The fill rate is the percentage of demand that is satisfied from stock.
- Let $\beta$ be the desired fill rate.
Shortage Rate

\[ n(R) = \text{expected number of units short per cycle} \]
\[ Q = \text{expected number of units demanded per cycle} \]

Over the long run, the fraction of demand that stocks out is

\[ \text{shortage rate} = \frac{n(R)}{Q} \]
The fill rate is 1 minus this amount:

\[ \text{fill rate} = 1 - \text{shortage rate} = 1 - \frac{n(R)}{Q} \]

We want this to be \( \geq \beta \).
What is the fill rate in EOQ?

1. 0
2. 1
3. $\frac{1}{2}$
4. $\frac{Q}{\lambda}$
5. $\frac{\tau}{T}$
Achieving a Fill Rate

To achieve a fill rate of $\beta$, we solve for the desired $n(R)$:

$$n(R) = (1 - \beta)Q.$$
Achieving a Fill Rate

If we have inverse of \( n(R) \) we can use this inverse to calculate \( R \), otherwise we can again use bisection search to find \( R \) such that \( n(R) = (1 - \beta)Q \) (because \( n(R) \) is nonincreasing in \( R \)).
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If we have inverse of $n(R)$ we can use this inverse to calculate $R$, otherwise we can again use bisection search to find $R$ such that $n(R) = (1 - \beta)Q$ (because $n(R)$ is nonincreasing in $R$). Note that we (again) need $Q$ to determine $R$!

But now $R$ does not depend on $p$ (it now depends on $\beta$ instead).
We want to find $R$ such that $n(R)$ is within $\varepsilon$ of $(1 - \beta)Q$.

- Set $L = 0$.
- Find an integer $U$ large enough that $n(U) \leq (1 - \beta)Q$. To do this, guess at $U$, check $n(R)$, and keep increasing $U$ until $n(R) \leq (1 - \beta)Q$.
- While $U - L > \varepsilon$:
  - Choose $M = (L + U)/2$
  - If $n(R) \leq (1 - \beta)Q$, set $U = R$
  - If $n(R) > (1 - \beta)Q$, set $L = R$

Set our final value to $R = U$.
Same idea as before: iteratively finding $R$ and $Q$

We will once again find an iterative procedure for finding $R$ and $Q$.

We now know how to find $R$ given a value $Q$ (using $\beta$, not $p$). How about find $Q$ given a value of $R$?
Once we have $R$, here is how we get $Q$

- Previously, $Q$ was calculated from $R$ and the other parameters, including $p$:
  \[ Q = \sqrt{\frac{2}{h}} \left( K\lambda + p\lambda n(R) \right) \]  
  (slide 54 of lecture 9)
- We now don’t have $p$ — how to get around this?
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- Previously, $Q$ was calculated from $R$ and the other parameters, including $p$:
  \[
  Q = \sqrt{\frac{2}{h} \left( K\lambda + p\lambda n(R) \right)}
  \]
  (slide 54 of lecture 9)

- We now don’t have $p$ — how to get around this?

- For a given $R$ and $Q$, there is an implicitly defined $p$. Recall:
  \[
  F(R) = 1 - \frac{Qh}{p\lambda}
  \]
  (slide 49 of lecture 9).
Once we have $R$, here is how we get $Q$

Recall: $F(R) = 1 - \frac{Qh}{p\lambda}$ (slide 49 of lecture 9).

Solve for $p$:

$$p = \frac{Qh}{\lambda(1 - F(R))}$$

and use this in the equation for $Q = \sqrt{\frac{2}{h} \left( K\lambda + p\lambda n(R) \right)}$ (slide 54 of lecture 9):

$$Q = \sqrt{\frac{2}{h} \left( K\lambda + \frac{Qh}{\lambda(1 - F(R))}\lambda n(R) \right)}$$
Once we have $R$, here is how we get $Q$

$$Q = \sqrt{\frac{2}{h} \left( K\lambda + \frac{Qh}{\lambda(1 - F(R))}\lambda n(R) \right)}$$

Now we have to solve for $Q$:

$$Q^2 = \frac{2}{h} \left( K\lambda + \frac{Qh}{\lambda(1 - F(R))}\lambda n(R) \right)$$

$$Q^2 - \frac{2n(R)}{1 - F(R)} Q - \frac{2K\lambda}{h} = 0$$

So

$$Q = \frac{n(R)}{1 - F(R)} + \sqrt{\left( \frac{n(R)}{1 - F(R)} \right)^2 + \frac{2K\lambda}{h}}.$$
Iterative Procedure for Finding $Q$ and $R$ for a given fill rate $\beta$

- Find $Q_0 = EOQ$
- Find $R_0$ from $n(R_0) = (1 - \beta)Q_0$ [using bisection search]
- Plug $R_0$ into the equation for $Q$ to get $Q_1$.
  [use equation at bottom of previous slide. In that formula, use $n(R_0) = (1 - \beta)Q_0$.]
- Find $R_1$ from $n(R_1) = (1 - \beta)Q_1$ [using bisection search]
- etc.
- Stop when $R_{n+1} \approx R_n$ (stop when the $R_{n+1}$ and $R_n$ both give the same value when rounded to the nearest integer)
Iterative Procedure for Finding $Q$ and $R$ for a given fill rate $\beta$ — Summary

- Lots of ugly formulas!
- Point is:
  
  Can start with $Q_0 = \text{EOQ}$.
  
  - A value for $Q$ and $\beta$ determines a value for $R$ ($\rightarrow R_0$).
  - A value for $Q$ and for $R$ determines an (implicit) value for $p$.
  - A value for $R$ and $p$ determines a (new) value for $Q$ ($\rightarrow Q_1$).
  - Rinse and repeat.
Special Case: Normally Distributed Demand

Computing $Q$, $R$ policies for most demand distributions is a lot of work.

You need to be able to calculate:

$$F(R) = P(C(\tau) \leq R)$$

and

$$n(R) = E[(C(\tau) - R)^+]$$

and then repeatedly use bisection search to find the value of $R$. 
Special Case: Normally Distributed Demand

When $D_1, D_2, \ldots$ are i.i.d. normally distributed with mean $\lambda$ and variance $\sigma^2$:

$C(\tau)$ is distributed

1. Normal with mean $\lambda\tau$ and variance $\sigma^2\tau$
2. Normal with mean $\lambda\tau$ and variance $\sigma^2\tau^2$
3. Normal with mean $\lambda\tau^2$ and variance $\sigma^2\tau^2$
4. Poisson with mean $\lambda\tau$ and variance $\sigma^2\tau^2$
5. Poisson with mean $\lambda\tau^2$ and variance $\sigma^2\tau^2$
So we can use this for cdf of $C(\tau)$:

$$
\mathbb{P}(C(\tau) \leq x) = \mathbb{P}
\left( \frac{C(\tau) - \lambda \tau}{\sigma} \leq \frac{x - \lambda \tau}{\sigma} \right)
= \Phi \left( \frac{x - \lambda \tau}{\sigma} \right)
$$

where $\Phi(\cdot)$ is the cdf of the standard normal.
What about $n(R) = E[(C(\tau) - R)^+]$?
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Can interpret this as the expected cost for the newvendor when...
Special Case: Normally Distributed Demand

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Can interpret this as the expected cost for the newvendor when...

- demand is \( C(\tau) \)
- inventory level is \( R \) (this was \( Q \) in Newsvendor)
- \( c_u = 1, \ c_o = 0 \)
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- \( c_u = 1, \ c_o = 0 \)

Using the formula from last slide Newsvendor Lecture:

\[
n(R) = -\sigma \sqrt{\tau} (z \Phi(-z) - \phi(z))
\]

where \( z = \frac{R - \lambda \tau}{\sigma \sqrt{\tau}} \).