How should you buy plastic resin for your plastics factory?

- 80,000 pounds at $0.68 per pound
- 1,000 pounds at $0.88 per pound
How should you buy garments to stock in your store?
Why hold inventory?

- Uncertainties in external demand
- Production smoothing
- Speculation
- Inventory in transit, or in shipping/receiving
- Volume discounts from suppliers
Goal: keeping costs low

We usually consider 3 kinds of costs when managing inventory:

- Holding costs
- Order costs
- Penalty costs
Holding Costs

- All costs incurred to keep items in inventory
- Warehouse rent, taxes, insurance, climate control, lost interest on tied-up capital, etc.
- Usually written using the notation “$h$” which is the holding cost to store one item in inventory over one time unit
- Units are often $$/units inventory-unit time
Order Costs

Usually has a fixed component and a variable component:

\[ C(x) = K + cx \]

- \( K \) = fixed costs per order
- \( c \) = variable order cost
- \( x \) = number of units in order

Fixed costs can include bookkeeping, order generation, receiving & handling
Penalty Cost

Cost of not having sufficient stock on hand

- Backorder costs
  - Bookkeeping, discount price, extra shipping costs, loss of customer goodwill
- Lost sales costs
  - Lost profit, as potential customer went elsewhere

\[ p = \text{cost per unit backordered} \]

We won’t consider the length of the backorder
Three theoretical approaches to managing inventory in this course

- Economic Order Quantity (EOQ)
- Newsvendor
- Continuous Review

Each makes different assumptions, and is a good choice when those assumptions are approximately met.
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We’ll also talk about using data + simulation to apply these approaches.
Economic Order Quantity (EOQ)

Assumptions

- Known and constant demand rate
- Known and constant lead time
- Instantaneous receipt of material
- No quantity discounts
- No stock outs permitted
- No penalty costs (only order & holding costs)
Inventory level decreases linearly because the demand rate is constant.

\[ Q = \text{order size (and inventory level at time zero)} \]

\[ T = \text{time elapsed between two consecutive orders} \]

The demand rate is \[ \lambda = \frac{Q}{T} \]
When should we order more?

- We need to plan ahead. There is a lead time between when we place the order and when we receive it.
- Re-order point: when the inventory hits that level, we should place an order (we call this a “replenishment order”).
- A good time to place the order is when we are one “lead time” away from zero inventory.
Inventory Level

\[ \lambda = \frac{Q}{T} \]

Place order at this time

Get replenishment order of size \( Q \) exactly when inventory hits zero!

(Reorder point)

Lead time \( (\tau) \)
What is the reorder point?

1. $R = \lambda \tau$
2. $R = Q - \lambda \tau$
3. $R = \lambda / \tau$
4. $R = Q - \lambda / \tau$
5. none of the above
The average inventory level is $Q/2$
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Knowing the average inventory level tells us the average holding cost

- Average inventory level per unit time $= \frac{Q}{2}$
- Therefore average holding cost per unit time
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We can also calculate after how many units of time we need to order

Order size is $Q$

Demand rate is $\lambda$ items per unit time

Therefore we need to order every $T =$
We can also calculate after how many units of time we need to order.

Order size is $Q$

Demand rate is $\lambda$ items per unit time.

Therefore we need to order every $T = Q/\lambda$ time units.
Cost per unit time

Holding cost: $h$ dollars per unit of inventory per unit time

Average inventory level per unit time $= \frac{Q}{2}$

Average holding cost per unit time:

So (average) total cost per unit time:

$G = \frac{hQ}{2} + \frac{\lambda K}{Q} + \lambda c$.

This is what we want to minimize.
Cost per unit time

Holding cost: \( h \) dollars per unit of inventory per unit time

Average inventory level per unit time = \( Q/2 \)

Average holding cost per unit time: \( hQ/2 \)
Cost per unit time

Holding cost: $h$ dollars per unit of inventory per unit time

Average inventory level per unit time $= \frac{Q}{2}$

Average holding cost per unit time: $\frac{hQ}{2}$

Ordering cost: fixed cost $K$, variable cost $c$ (per item)

Ordering cost per unit time:
Cost per unit time

Holding cost: $h$ dollars per unit of inventory per unit time

Average inventory level per unit time $= \frac{Q}{2}$

Average holding cost per unit time: $h\frac{Q}{2}$

Ordering cost: fixed cost $K$, variable cost $c$ (per item)

Ordering cost per unit time: $\frac{(K + cQ)}{T}$
Cost per unit time

Holding cost: \( h \) dollars per unit of inventory per unit time

Average inventory level per unit time = \( Q/2 \)

Average holding cost per unit time: \( hQ/2 \)

Ordering cost: fixed cost \( K \), variable cost \( c \) (per item)

Ordering cost per unit time: \( (K + cQ)/T \)

\[
= (K + cQ)/(Q/\lambda) = \lambda K/Q + c\lambda
\]
Cost per unit time

Holding cost: $h$ dollars per unit of inventory per unit time

Average inventory level per unit time $= Q/2$

Average holding cost per unit time: $hQ/2$

Ordering cost: fixed cost $K$, variable cost $c$ (per item)

Ordering cost per unit time: $(K + cQ)/T$

$= (K + cQ)/(Q/\lambda) = \lambda K/Q + c\lambda$

So (average) total cost per unit time:

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Cost per unit time

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Ordering cost per unit time: \( (K + cQ)/T \)

\[ = (K + cQ)/(Q/\lambda) = \lambda K/Q + c\lambda \]

So (average) total cost per unit time:

\[ G = hQ/2 + \lambda K/Q + c\lambda. \text{ This is what we want to minimize} \]
Recall from Calculus

\[ G(Q) = \frac{hQ}{2} + \frac{\lambda K}{Q} + c\lambda. \]

We have a continuous function in one variable (\( h, \lambda, K \) and \( c \) are parameters that are given to us).

To find optimum:
Recall from Calculus

\[ G(Q) = \frac{hQ}{2} + \frac{\lambda K}{Q} + c\lambda. \]

We have a continuous function in one variable \((h, \lambda, K\) and \(c\) are parameters that are given to us).

To find optimum:

Find critical points. Check critical points (and boundary) to see which is minimizer.
Recall from Calculus

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We have a continuous function in one variable \((h, \lambda, K\) and \(c\) are parameters that are given to us).

To find optimum:

Find critical points. Check critical points (and boundary) to see which is minimizer.

Critical points are where derivative is zero.
Economic Order Quantity $Q^*$

$$G(Q) = \frac{hQ}{2} + \frac{\lambda K}{Q} + c\lambda.$$ 

Critical points: point where $Q$ is so that $\frac{d}{dQ} G(Q) = 0$. 
Economic Order Quantity $Q^*$

\[ G(Q) = \frac{hQ}{2} + \frac{\lambda K}{Q} + c\lambda. \]

Critical points: point where $Q$ is so that $\frac{d}{dQ} G(Q) = 0$.

What is $\frac{d}{dQ} G(Q)$?
Economic Order Quantity $Q^*$

\[
\frac{d}{dQ} G(Q) = h/2 - \lambda K / Q^2
\]

So critical point (solve for $Q$):

\[
h/2 - \lambda K / Q^2 = 0
\]

\[
Q^2 = 2\lambda K / h
\]

\[
Q^* = \sqrt{2\lambda K / h}
\]

(because we can only order nonnegative amounts.)
Economic Order Quantity \( Q^* \)

\[ G(Q) = \frac{hQ}{2} + \frac{\lambda K}{Q} + c\lambda. \]

\[ Q^* = \sqrt{\frac{2\lambda K}{h}} \]

Is this really minimizer?
Economic Order Quantity $Q^*$

$$G(Q) = hQ/2 + \lambda K/Q + c\lambda.$$  
$$Q^* = \sqrt{2\lambda K/h}$$

Is this really minimizer?

Yes! $\lim_{Q \to 0} G(Q) = \infty$, and $\lim_{Q \to \infty} G(Q) = \infty$. (Could also check 2nd derivative.)
There’s a tradeoff between order frequency & inventory level

- If we order large amounts infrequently, we’ll have a high average inventory level.
- Our order costs per year will be low, but our holding costs will be high.
- Once we turn cash into inventory, we’ll have to sell it to turn it into cash again.
Large inventories are risky

Risks:

- Product is a fad, and stops being popular
- Product becomes obsolete, and stops selling
- A recession hits, and consumers stop spending
- The product doesn’t sell as well as predicted
Long lead times

Be careful of long lead times, then you might have to place a replenishment order well ahead, as shown:

![Diagram of long lead times with variables R, T, and K]
Cost is not too sensitive to $Q$

- The curve for $G$ is shallow, so even if your lot size or order quantity is off from the optimal, you won’t do too badly.
- There are many reasons you might want to place a non-optimal order (maybe the product comes in boxes of 24 units...)
Cost is not too sensitive to $Q$

Let’s see if we can make this precise, by finding a nice upper bound for $G(Q)/G(Q^*)$. 
Reinterpreting $Q^*$

$$G(Q) = \frac{hQ}{2} + \frac{\lambda K}{Q} + c\lambda.$$ 

$Q^*$ is so that

$$h/2 - \frac{\lambda K}{Q^2} = 0.$$
Reinterpreting $Q^*$

$$G(Q) = \frac{hQ}{2} + \frac{\lambda K}{Q} + c\lambda.$$ 

$Q^*$ is so that

$$\frac{h}{2} - \frac{\lambda K}{Q^2} = 0.$$ 

Let's reinterpret that. Multiply $Q$:

$$\frac{hQ}{2} - \frac{\lambda K}{Q} = 0.$$ 

What does this say?
Reinterpreting $Q^*$

$Q^*$ is so that

$$hQ/2 = \lambda K/Q.$$ 

$Q^*$ is so that the holding cost per unit time and the fixed ordering costs are equal!
Total Cost of Non-Optimal \( Q \)

\[
G(Q) = \frac{hQ}{2} + \frac{\lambda K}{Q} + c\lambda.
\]

\[
G(Q^*) = \frac{hQ^*}{2} + \frac{\lambda K}{Q^*} + c\lambda = hQ^* + c\lambda = 2\frac{\lambda K}{Q^*} + c\lambda
\]

When \( c = 0 \), the ratio \( \frac{G(Q)}{G^*(Q)} \):

\[
\frac{G(Q)}{G(Q^*)} = \frac{\frac{hQ}{2} + \frac{\lambda K}{Q}}{G(Q^*)} = \frac{\frac{hQ}{2}}{G(Q^*)} + \frac{\frac{\lambda K}{Q}}{G(Q^*)} = \frac{hQ}{2\frac{Q^*}} + \frac{\lambda K}{2\frac{\lambda K}{Q^*}} \leq \frac{1}{2} \frac{Q}{Q^*} + \frac{1}{2} \frac{Q^*}{Q}
\]
Now note that \((a + t)/(b + t)\) is nonincreasing in \(t\) if \(a \geq b\) (you can check this by taking a derivative; intuitively it makes sense because \((a + t)/(b + t) \to 1\) as \(t \to \infty\) and \(a/b \geq 1\)).

We therefore have

\[
\frac{G(Q)}{G(Q^*)} \leq \frac{\tilde{G}(Q)}{\tilde{G}(Q^*)} = \frac{1}{2} \frac{Q}{Q^*} + \frac{1}{2} \frac{Q^*}{Q}.
\]
Ordering intervals

Also \( T^* = Q^*/\lambda \) and \( T = Q/\lambda \).

Therefore

\[
\frac{T}{T^*} = \frac{Q}{Q^*}.
\]
Ordering intervals

Also \( T^* = Q^*/\lambda \) and \( T = Q/\lambda \).

Therefore

\[
\frac{T}{T^*} = \frac{Q}{Q^*}.
\]

So

\[
\frac{G(Q)}{G(Q^*)} \leq \frac{1}{2} \frac{Q}{Q^*} + \frac{1}{2} \frac{Q^*}{Q} = \frac{1}{2} \frac{T}{T^*} + \frac{1}{2} \frac{T^*}{T}.
\]
Power of Two Ordering

If we have to place orders every week, or every other week, or every fourth week, then our choices are

- Weekly $= 2^0$
- Bi-weekly $= 2^1$
- Every four weeks $= 2^2$. 
Power of Two Ordering

Average total cost per time unit

Order Quantity

$1,180
$1,170
$1,160
$1,150
$1,140
$1,130
$1,120
$1,110
$1,100
$1,090
$1,080
$1,070

0
200
400
600
800
1000
1200
The worst isn’t that bad

Suppose we evaluate the cost for \( T = 2^m \) for all \( m \), and select the best one.

How bad can our total cost per time unit be, compare to the cost of \( Q^* \)?
The worst isn’t that bad

Suppose we evaluate the cost for $T = 2^m$ for all $m$, and select the best one.

How bad can our total cost per time unit be, compare to the cost of $Q^*$?

Can we give a simple (constant, independent of $T^*$) upper bound on

$$\frac{G(Q)}{G(Q^*)} \leq \min_i \left\{ \frac{1}{2} \frac{2^i}{T^*} + \frac{1}{2} \frac{T^*}{2^i} \right\}$$

for any $T^*$?
The worst isn’t that bad
The worst isn’t that bad

Let $t$ be so that $2^t \leq T^* \leq 2^{t+1}$.

Then

$$\frac{G(Q)}{G(Q^*)} \leq \min\left\{ \frac{1}{2} \frac{2^t}{T^*} + \frac{1}{2} \frac{T^*}{2^t}, \frac{1}{2} \frac{2^{t+1}}{T^*} + \frac{1}{2} \frac{T^*}{2^{t+1}} \right\}.$$  

Now want to find worst possible $T^*$ to find upper bound.
The worst isn’t that bad

\[ \frac{d}{dT^*} \left[ \frac{2^t}{T^*} + \frac{T^*}{2^t} \right] = -\frac{2^t}{(T^*)^2} + \frac{1}{2^t} > 0 \text{ for } T^* > 2^t. \]

\[ \frac{d}{dT^*} \left[ \frac{2^{t+1}}{T^*} + \frac{T^*}{2^{t+1}} \right] = -\frac{2^{t+1}}{(T^*)^2} + \frac{1}{2^{t+1}} < 0 \text{ for } T^* < 2^{t+1}. \]

So worst possible \( T^* \) is where these are equal:

\[ -\frac{2^t}{(T^*)^2} + \frac{1}{2^t} = -\frac{2^{t+1}}{(T^*)^2} + \frac{1}{2^{t+1}}. \]
The worst isn’t that bad

\[
\frac{d}{dT^*} \left[ \frac{2^t}{T^*} + \frac{T^*}{2^t} \right] = -\frac{2^t}{(T^*)^2} + \frac{1}{2^t} > 0 \text{ for } T^* > 2^t.
\]

\[
\frac{d}{dT^*} \left[ \frac{2^{t+1}}{T^*} + \frac{T^*}{2^{t+1}} \right] = -\frac{2^{t+1}}{(T^*)^2} + \frac{1}{2^{t+1}} < 0 \text{ for } T^* < 2^{t+1}.
\]

So worst possible \( T^* \) is where these are equal:

\[-\frac{2^t}{(T^*)^2} + \frac{1}{2^t} = -\frac{2^{t+1}}{(T^*)^2} + \frac{1}{2^{t+1}}.\]

This gives

\[T^* = 2^{t+\frac{1}{2}}.\]
The worst isn’t that bad

\[
\frac{G(Q)}{G(Q^*)} \leq \frac{1}{2} \frac{2^{t}}{2^{t+\frac{1}{2}}} + \frac{1}{2} \frac{2^{t+\frac{1}{2}}}{2^{t}}
\]

\[
= \frac{1}{2} \frac{1}{2^{\frac{1}{2}}} + \frac{1}{2} 2^{\frac{1}{2}}
\]

\[
= \frac{1}{2} \frac{1}{\sqrt{2}} + \frac{1}{2} \sqrt{2}
\]

\[
< 1.061.
\]
Conclusion

Power of two ordering is never more than 6.1% more expensive than if we could order any moment in time.