Quiz: question 1

What is the storage required to represent analysis solution to a matrix completion problem

\[
\text{minimize } \sum_{(i,j) \in E} (x_{ij} - b_{ij})^2 \quad \text{s.t. } \|X\| \leq \alpha
\]

with variable \( X \in \mathbb{R}^{m \times n} \) if all solutions have rank \( \leq r \)?

- \( O(mnr \log n) \)
- \( O(mnr) \)
- \( O(r(m + n) \log(m + n)) \)
- \( O(r(m + n)) \)
- \( O(r \log(m + n)) \)
Quiz: question 2

What is the main advantage of the Conditional Gradient method for designing an optimal storage algorithm?

- the rank of any iterate $\leq$ the rank of the solution
- the rank of the iterate grows by at most 1 at each iteration
- the iteration can be expressed in terms of a low dimensional “dual” variable
- the method converges linearly for underconstrained problems like matrix completion and phase retrieval
Sketchy Decisions: Convex Low-Rank Matrix Optimization with Optimal Storage

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Goal

Can we develop algorithms that provably solve a problem using \textbf{storage} bounded by the size of the \textbf{problem data} and the size of the \textbf{solution}?

\begin{align*}
\text{Problem data: } & \mathcal{O}(n) \\
\downarrow \\
\text{Working memory: } & \mathcal{O}(???) \\
\downarrow \\
\text{Solution: } & \mathcal{O}(n)
\end{align*}
Model problem: low rank matrix optimization

consider a convex problem with decision variable $X \in \mathbb{R}^{m \times n}$

compact matrix optimization problem:

$$\begin{align*}
\text{minimize} & \quad f(AX) \\
\text{subject to} & \quad \|X\|_{S_1} \leq \alpha
\end{align*}$$

(CMOP)

- $A : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^d$
- $f : \mathbb{R}^d \rightarrow \mathbb{R}$ convex and smooth
- $\|X\|_{S_1}$ is Schatten-1 norm: sum of singular values

assume

- compact specification: problem data use $O(n)$ storage
- compact solution: rank $X_\star = r$ constant

Note: Same ideas work for $X \succeq 0$
Are desiderata achievable?

\[
\begin{align*}
\text{minimize} & \quad f(AX) \\
\text{subject to} & \quad \|X\|_{S_1} \leq \alpha
\end{align*}
\]

CMOP, using any first order method:

- **Problem data:** $O(n)$
- **Working memory:** $O(n^2)$
- **Solution:** $O(n)$
Are desiderata achievable?

CMOP, using **SketchyCGM**:

- **Problem data**: $\mathcal{O}(n)$

  ![Diagram](image)

- **Solution**: $\mathcal{O}(n)$
Application: matrix completion

find $X$ matching $M$ on observed entries

$$\text{minimize} \quad \sum_{(i,j) \in \Omega} (X_{ij} - M_{ij})^2$$
subject to \quad $\|X\|_{S_1} \leq \alpha$

- $m =$ rows, $n =$ columns of matrix to complete
- $d = |\Omega|$ number of observations
- $\mathcal{A}$ selects observed entries $X_{ij}$, $(i,j) \in \Omega$
- $f(z) = \|z - \mathcal{A}M\|^2$
Matrix completion is a CMOP

find $X$ matching $M$ on observed entries

$$\text{minimize} \quad \sum_{(i,j) \in \Omega} (X_{ij} - M_{ij})^2$$
subject to \quad \|X\|_{S_1} \leq \alpha$

- compact specification if $d = \mathcal{O}(m + n)$ observations
  e.g., constant \# observations / person
- compact solution if rank($X$) constant
  i.e., constant \# parameters / person
- in practice, usually find rank $\ll 200$ even with $m$ and $n$ in the millions...
Application: Phase retrieval

- image with $n$ pixels $x_\|^n \in \mathbb{C}^n$
- acquire noisy nonlinear measurements $b_i = |\langle a_i, x_\|^\rangle|^2 + \omega_i$
- relax: if $X = x_\|^n x_\|^n$, then
  \[
  |\langle a_i, x_\|^\rangle|^2 = x_\|^a_i a_i^* x_\|^n = \text{tr}(a_i^* a_i x_\|^n x_\|^n) = \text{tr}(a_i^* a_i X)
  \]
- recover image by solving
  \[
  \begin{align*}
  &\text{minimize} & f(AX; b) \\
  &\text{subject to} & \text{tr } X = \alpha \\
  & & X \succeq 0.
  \end{align*}
  \]

1image courtesy of Manuel Guizar-Sicairos
Phase retrieval is a CMOP

find $X$ matching observations

$$\begin{align*}
\text{minimize} & \quad f(AX; b) \\
\text{subject to} & \quad \text{tr } X = \alpha \\
& \quad X \succeq 0.
\end{align*}$$

- compact specification if $d = \mathcal{O}(n)$ observations
e.g., constant # observations / pixels
- compact solution if rank($X$) constant
e.g., if correctly recover the rank-1 solution!
Why compact?

why a compact specification?

▶ data is expensive
▶ collect constant data per column (≡user or sample)
▶ if solution is compact, compact specification should suffice

why a compact solution?

▶ the world is simple and structured
▶ given \(d\) observations, there is a solution with rank \(O(\sqrt{d})\)
  (Barvinok 1995, Pataki 1998)
▶ nice latent variable models are of log rank
  (Udell & Townsend 2019)
What kind of storage bounds can we hope for?

- Assume black-box implementation of
  \[ A(uv^*) \quad u^*(A^*z) \quad (A^*z)v \]
  where \( u \in \mathbb{R}^m, \quad v \in \mathbb{R}^n, \quad \text{and} \quad z \in \mathbb{R}^d \)
- Need \( \Omega(m + n + d) \) storage to apply linear map
- Need \( \Theta(r(m + n)) \) storage for a rank-\( r \) approximate solution

**Definition.** An algorithm for the model problem has **optimal storage** if its working storage is

\[ \Theta(d + r(m + n)) \].\]
Optimal Storage

What kind of storage bounds can we hope for?

- Assume black-box implementation of

\[ \mathcal{A}(uv^*) \quad u^*(A^*z) \quad (A^*z)v \]

where \( u \in \mathbb{R}^m \), \( v \in \mathbb{R}^n \), and \( z \in \mathbb{R}^d \)

- Need \( \Omega(m + n + d) \) storage to apply linear map
- Need \( \Theta(r(m + n)) \) storage for a rank-\( r \) approximate solution

**Definition.** An algorithm for the model problem has **optimal storage** if its working storage is

\[ \Theta(d + r(m + n)). \]

If we write down \( X \), we’ve already failed.
A brief biased history of matrix optimization (I)

- **1990s: Interior-point methods**
  - Storage cost $\Theta((m + n)^4)$ for Hessian

- **2000s: Convex first-order methods (FOM)**
  - (Accelerated) proximal gradient and others
  - Store matrix variable $\Theta(mn)$

(Interior-point: Nemirovski & Nesterov 1994; ...; First-order: Rockafellar 1976; Auslender & Teboulle 2006; ...
A brief biased history of matrix optimization (I)

- **2008–Present: Storage-efficient convex FOM**
  - Conditional gradient method (CGM) and extensions
  - Store matrix in low-rank form $O(t(m + n))$ after $t$ iterations
  - Requires storage $\Theta(mn)$ for $t \geq \min(m, n)$
  - Variants: prune factorization, or seek rank-reducing steps

- **2003–Present: Nonconvex methods**
  - Burer–Monteiro factorization idea + various opt algorithms
  - Store low-rank matrix factors $\Theta(r(m + n))$
  - For guaranteed solution, need statistical assumptions or $O(n^{3/2})$ storage

The dilemma

- convex methods: slow memory hogs with guarantees
- nonconvex methods: fast, lightweight, but brittle
The dilemma

- convex methods: slow memory hogs with guarantees
- nonconvex methods: fast, lightweight, but brittle

**goal:** low memory and guaranteed convergence
Conditional gradient method (Frank-Wolfe)

\[
\begin{align*}
\text{minimize} & \quad g(w) \\
\text{subject to} & \quad w \in \mathcal{P}
\end{align*}
\]
Conditional gradient method (Frank-Wolfe)

\[
\begin{align*}
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Conditional gradient method (Frank-Wolfe)

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\minimize \quad g(w) \\
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Conditional gradient method (Frank-Wolfe)

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\begin{aligned}
& \text{minimize} & & g(w) \\
& \text{subject to} & & w \in \mathcal{P}
\end{aligned}
\]
Conditional gradient method (Frank-Wolfe)

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\begin{align*}
\text{minimize} & \quad g(w) \\
\text{subject to} & \quad w \in \mathcal{P}
\end{align*}
\]
Conditional gradient method (Frank-Wolfe)

minimize $g(w)$
subject to $w \in \mathcal{P}$
Conditional Gradient Method

\[
\begin{align*}
\text{minimize} & \quad f(AX) \\
\text{subject to} & \quad \|X\|_{S_1} \leq \alpha
\end{align*}
\]

\textbf{CGM.} set \(X^0 = 0\). for \(t = 0, 1, \ldots\)

\begin{itemize}
  \item compute \(G^t = A^* \nabla f(AX^t)\)
  \item set search direction
  \[
  H^t = \arg\max_{\|X\|_{S_1} \leq \alpha} \langle X, -G^t \rangle
  \]
  \item set stepsize \(\eta^t = 2/(t + 2)\)
  \item update \(X^{t+1} = (1 - \eta^t)X^t + \eta^t H^t\)
\end{itemize}
Conditional gradient method (CGM)

features:

- relies on efficient **linear optimization oracle** to compute

\[
H^t = \text{argmax} \langle X, -G^t \rangle \\
\|X\|_{s_1} \leq \alpha
\]

- bound on suboptimality follows from subgradient inequality

\[
f(AX^t) - f(AX^*) \leq \langle X^t - X^*, G^t \rangle \\
\leq \langle X^t - X^*, A^* \nabla f(AX^t) \rangle \\
\leq \langle AX^t - AX^*, \nabla f(AX^t) \rangle \\
\leq \langle AX^t - AH^t, \nabla f(AX^t) \rangle
\]

to provide stopping condition

- faster variants: linesearch, away steps, . . .
Linear optimization oracle for MOP

compute search direction

\[
\text{argmax} \langle X, -G \rangle \\
\|X\|_{S_1} \leq \alpha
\]

- solution given by maximum singular vector of \(-G\):

\[
-G = \sum_{i=1}^{n} \sigma_i u_i v_i^* \quad \implies \quad X = \alpha u_1 v_1^*
\]

- use Lanczos method: only need to apply \(G\) and \(G^*\)
Conditional gradient descent

Algorithm 1 CGM for the model problem (CMOP)

Input: Problem data for (CMOP); suboptimality $\varepsilon$

Output: Solution $X_*$

```plaintext
1 function CGM
2     $X \leftarrow 0$
3     for $t \leftarrow 0, 1, \ldots$ do
4         $(u, v) \leftarrow \text{MaxSingVec}(-A^*(\nabla f(A X)))$
5         $H \leftarrow -\alpha uv^*$
6         if $\langle AX - AH, \nabla f(A X) \rangle \leq \varepsilon$ then break for
7         $\eta \leftarrow 2/(t + 2)$
8         $X \leftarrow (1 - \eta)X + \eta H$
9     return $X$
```

Two crucial ideas

To solve the problem using optimal storage:

▶ Use the low-dimensional “dual” variable

\[ z_t = AX_t \in \mathbb{R}^d \]

to drive the iteration.

▶ Recover solution from small (randomized) sketch.

Never write down \( X \) until it has converged to low rank.
Conditional gradient descent

Algorithm 2 CGM for the model problem (CMOP)

**Input:** Problem data for (CMOP); suboptimality $\varepsilon$

**Output:** Solution $X_*$

```plaintext
function CGM
    $X \leftarrow 0$
    for $t \leftarrow 0, 1, \ldots$ do
        $(u, v) \leftarrow \text{MaxSingVec}(-A^*(\nabla f(A X)))$
        $H \leftarrow -\alpha u v^*$
        if $\langle A X - A H, \nabla f(A X) \rangle \leq \varepsilon$ then break for
        $\eta \leftarrow 2/(t + 2)$
        $X \leftarrow (1 - \eta)X + \eta H$
    return $X$
```

Conditional gradient descent

Introduce “dual variable” $z = AX \in \mathbb{R}^d$; eliminate $X$.

Algorithm 3 Dual CGM for the model problem (CMOP)

Input: Problem data for (CMOP); suboptimality $\varepsilon$
Output: Solution $X_*$

1. function DUALCGM
2. $z \leftarrow 0$
3. for $t \leftarrow 0, 1, \ldots$ do
4.   $(u, v) \leftarrow \text{MaxSingVec}(-A^*(\nabla f(z)))$
5.   $h \leftarrow A(-\alpha uv^*)$
6.   if $\langle z - h, \nabla f(z) \rangle \leq \varepsilon$ then break for
7.   $\eta \leftarrow 2/(t + 2)$
8.   $z \leftarrow (1 - \eta)z + \eta h$
Conditional gradient descent

Introduce “dual variable” $z = \mathcal{A}X \in \mathbb{R}^d$; eliminate $X$.

Algorithm 4 Dual CGM for the model problem (CMOP)

Input: Problem data for (CMOP); suboptimality $\varepsilon$
Output: Solution $X_*$

1. function DUALCGM
2. $z \leftarrow 0$
3. for $t \leftarrow 0, 1, \ldots$ do
4. $(u, v) \leftarrow \text{MaxSingVec}(-\mathcal{A}^*(\nabla f(z)))$
5. $h \leftarrow \mathcal{A}(-\alpha uv^*)$
6. if $\langle z - h, \nabla f(z) \rangle \leq \varepsilon$ then break for
7. $\eta \leftarrow 2/(t + 2)$
8. $z \leftarrow (1 - \eta)z + \eta h$

we’ve solved the problem... but where’s the solution?
Two crucial ideas

1. Use the low-dimensional “dual” variable

\[ z_t = A X_t \in \mathbb{R}^d \]

to drive the iteration.

2. Recover solution from small (randomized) sketch.
How to catch a low rank matrix

if $\hat{X}$ has the same rank as $X^*$,
and $\hat{X}$ acts like $X^*$ (on its range and co-range),
then $\hat{X}$ is $X^*$

use single-pass randomized sketch (Tropp Yurtsever U Cevher 2017)

► see a series of additive updates
► remember how the matrix acts on random subspace
► reconstruct a low rank matrix that acts like $X^*$
► storage cost for sketch and arithmetic cost of update are $O(r(m + n))$; reconstruction is $O(r^2(m + n))$
Single-pass randomized sketch

- Draw and fix two independent standard normal matrices
  \[ \Omega \in \mathbb{R}^{n \times k} \quad \text{and} \quad \Psi \in \mathbb{R}^{\ell \times m} \]
  with \( k = 2r + 1 \), \( \ell = 4r + 2 \).
Single-pass randomized sketch

- Draw and fix two independent standard normal matrices
  \[ \Omega \in \mathbb{R}^{n \times k} \quad \text{and} \quad \Psi \in \mathbb{R}^{\ell \times m} \]
  with \( k = 2r + 1, \ \ell = 4r + 2. \)
- The sketch consists of two matrices that capture the range and co-range of \( X: \)
  \[ Y = X\Omega \in \mathbb{R}^{n \times k} \quad \text{and} \quad W = \Psi X \in \mathbb{R}^{\ell \times m} \]
Single-pass randomized sketch

- Draw and fix two independent standard normal matrices
  \[ \Omega \in \mathbb{R}^{n \times k} \quad \text{and} \quad \Psi \in \mathbb{R}^{\ell \times m} \]
  with \( k = 2r + 1 \), \( \ell = 4r + 2 \).

- The sketch consists of two matrices that capture the range and co-range of \( X \):
  \[ Y = X\Omega \in \mathbb{R}^{n \times k} \quad \text{and} \quad W = \Psi X \in \mathbb{R}^{\ell \times m} \]

- Rank-1 updates to \( X \) can be performed on sketch:
  \[ X' = \beta_1 X + \beta_2 uv^* \]
  \[ \Downarrow \]
  \[ Y' = \beta_1 Y + \beta_2 uv^*\Omega \quad \text{and} \quad W' = \beta_1 W + \beta_2 \Psi uv^* \]
Single-pass randomized sketch

- Draw and fix two independent standard normal matrices
  \[ \Omega \in \mathbb{R}^{n \times k} \quad \text{and} \quad \Psi \in \mathbb{R}^{\ell \times m} \]
  with \( k = 2r + 1 \), \( \ell = 4r + 2 \).

- The sketch consists of two matrices that capture the range and co-range of \( X \):
  \[ Y = X\Omega \in \mathbb{R}^{n \times k} \quad \text{and} \quad W = \Psi X \in \mathbb{R}^{\ell \times m} \]

- Rank-1 updates to \( X \) can be performed on sketch:
  \[ X' = \beta_1 X + \beta_2 uv^* \]
  \[ \Rightarrow \]
  \[ Y' = \beta_1 Y + \beta_2 uv^*\Omega \quad \text{and} \quad W' = \beta_1 W + \beta_2 \Psi uv^* \]

- Both the storage cost for the sketch and the arithmetic cost of an update are \( \mathcal{O}(r(m + n)) \).
Recovery from sketch

To recover rank-$r$ approximation $\hat{X}$ from the sketch, compute

1. $Y = QR$  (tall-skinny QR)
2. $B = (\Psi Q)^\dagger W$  (small QR + backsub)
3. $\hat{X} = Q[B]_r$  (tall-skinny SVD)

Theorem (Reconstruction (Tropp Yurtsever U Cevher, 2016))

Fix a target rank $r$. Let $X$ be a matrix, and let $(Y, W)$ be a sketch of $X$. The reconstruction procedure above yields a rank-$r$ matrix $\hat{X}$ with $E\|X - \hat{X}\|_F \leq 2\|X - [X]_r\|_F$.

Similar bounds hold with high probability.

Previous work (Clarkson Woodruff 2009) algebraically but not numerically equivalent.
Recovery from sketch

To recover rank-$r$ approximation $\hat{X}$ from the sketch, compute

1. $Y = QR$ (tall-skinny QR)
2. $B = (\Psi Q)\dagger W$ (small QR + backsub)
3. $\hat{X} = Q[B]_r$ (tall-skinny SVD)

**Theorem (Reconstruction (Tropp Yurtsever U Cevher, 2016))**

*Fix a target rank $r$. Let $X$ be a matrix, and let $(Y, W)$ be a sketch of $X$. The reconstruction procedure above yields a rank-$r$ matrix $\hat{X}$ with

$$
\mathbb{E} \| X - \hat{X} \|_F \leq 2 \| X - [X]_r \|_F .
$$

Similar bounds hold with high probability.*

Previous work (Clarkson Woodruff 2009) algebraically but not numerically equivalent.
Recovery from sketch: intuition

let

\[ Y = X\Omega \in \mathbb{R}^{n \times k} \quad \text{and} \quad W = \Psi X \in \mathbb{R}^{\ell \times m} \]

- if \( Q \) is an orthonormal basis for \( \mathcal{R}(X) \), then
  \[ X = QQ^* X \]

- if \( QR = X\Omega \), then \( Q \) is (approximately) a basis for \( \mathcal{R}(X) \)

- and if \( W = \Psi X \), we can estimate
  \[
  W = \Psi X \\
  \approx \Psi QQ^* X \\
  (\Psi Q)^\dagger W \approx Q^* X
  \]

- hence we may reconstruct \( X \) as
  \[ X \approx QQ^* X \approx Q(\Psi Q)^\dagger W \]
Algorithm 5 SketchyCGM for the model problem (CMOP)

**Input:** Problem data; suboptimality $\varepsilon$; target rank $r$

**Output:** Rank-$r$ approximate solution $\hat{X} = U\Sigma V^*$

```plaintext
function SketchyCGM
    Sketch.Init($m$, $n$, $r$)
    $z \leftarrow 0$
    for $t \leftarrow 0, 1, \ldots$ do
        $(u, v) \leftarrow \text{MaxSingVec}(-A^*(\nabla f(z)))$
        $h \leftarrow A(-\alpha uv^*)$
        if $\langle z - h, \nabla f(z) \rangle \leq \varepsilon$ then break for
        $\eta \leftarrow 2/(t + 2)$
        $z \leftarrow (1 - \eta)z + \eta h$
        Sketch.CGMCUpdate($-\alpha u$, $v$, $\eta$)
    $(U, \Sigma, V) \leftarrow \text{Sketch.Reconstruct}( )$
    return $(U, \Sigma, V)$
```
Guarantees

Suppose

- $X_{cgm}^{(t)}$ is $t$th CGM iterate
- $\lfloor X_{cgm}^{(t)} \rfloor_r$ is best rank $r$ approximation to CGM solution
- $\hat{X}^{(t)}$ is SketchyCGM reconstruction after $t$ iterations

**Theorem (Convergence to CGM solution)**

After $t$ iterations, the SketchyCGM reconstruction satisfies

$$\mathbb{E} \| \hat{X}^{(t)} - X_{cgm}^{(t)} \|_F \leq 2 \| \lfloor X_{cgm}^{(t)} \rfloor_r - X_{cgm}^{(t)} \|_F.$$  

If in addition $X^* = \lim_{t \to \infty} X_{cgm}^{(t)}$ has rank $r$, then RHS $\to 0$!

(Tropp Yurtsever U Cevher, 2016)
Convergence when $\text{rank}(X_{\text{cgm}}) \leq r$
Convergence when $\text{rank}(X_{\text{cgm}}) > r$
Guarantees (II)

Theorem (Convergence rate)

Fix $\kappa > 0$ and $\nu \geq 1$. Suppose the (unique) solution $X_*$ of (CMOP) has $\text{rank}(X_*) \leq r$ and

$$f(AX) - f(AX_*) \geq \kappa \|X - X_*\|_F^\nu \quad \text{for all} \quad \|X\|_{S_1} \leq \alpha.$$  \hspace{1cm} (1)

Then we have the error bound

$$\mathbb{E} \|\hat{X}_t - X_*\|_F \leq 6 \left( \frac{2\kappa^{-1}C}{t + 2} \right)^{1/\nu} \quad \text{for } t = 0, 1, 2, \ldots$$

where $C$ is the curvature constant (Eqn. (3), Jaggi 2013) of the problem (CMOP).
SketchyCGM is scalable

(A) Memory usage for five algorithms

- **PGM** = proximal gradient (via TFOCS (Becker Candès Grant, 2011))
- **AT** = accelerated PGM (Auslander Teboulle, 2006) (via TFOCS),
- **CGM** = conditional gradient method (Jaggi, 2013)
- **ThinCGM** = CGM with thin SVD updates (Yurtsever Hsieh Cevher, 2015)
- **SketchyCGM** = ours, using $r = 1$

(B) Convergence for $n = 8 \cdot 10^6$. 
Application: Phase retrieval

- image with \( n \) pixels \( x_\mathbb{H} \in \mathbb{C}^n \)
- acquire noisy nonlinear measurements \( b_i = |\langle a_i, x_\mathbb{H} \rangle|^2 + \omega_i \)
- relax: if \( X = x_\mathbb{H}x_\mathbb{H}^* \), then
  \[
  |\langle a_i, x_\mathbb{H} \rangle|^2 = x_\mathbb{H}a_i^*a_i x_\mathbb{H}^* = \text{tr}(a_i^*a_i x_\mathbb{H}^* x_\mathbb{H}) = \text{tr}(a_i^*a_i X)
  \]
- recover image by solving
  \[
  \begin{align*}
  \text{minimize} & \quad f(AX; b) \\
  \text{subject to} & \quad \text{tr} X \leq \alpha \\
  & \quad X \succeq 0.
  \end{align*}
  \]
  compact if \( d = \mathcal{O}(n) \) observations and rank\((X^*)\) constant
SketchyCGM is reliable

Fourier ptychography:

- imaging blood cells with $A = \text{subsampled FFT}$
- $n = 25, 600$, $d = 185, 600$
- $\text{rank}(X_+) \approx 5$ (empirically)

![SketchyCGM](image1)
![Burer–Monteiro](image2)
![Wirtinger Flow](image3)

- brightness indicates phase of pixel (thickness of sample)
- red boxes mark malaria parasites in blood cells
Conclusion

SketchyCGM offers a proof-of-concept **convex method** with **optimal storage** for low rank matrix optimization using two new ideas:

- Drive the algorithm using a smaller (dual) variable.
- Sketch and recover the decision variable.

References: