Limited Memory Kelley’s Method Converges
for Composite Convex and Submodular
Objectives

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Based on joint work with
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My old 2013 Macbook Pro: 16GB Ram

macOS High Sierra
Version 10.13.6

MacBook Pro (Retina, 13-inch, Late 2013)
Processor 2.4 GHz Intel Core i5
Memory 16 GB 1600 MHz DDR3
Graphics Intel Iris 1536 MB
Serial Number C02LL39EFH04

System Report... Software Update...
Gonna buy a new model with more RAM...
Gonna buy a new model with more RAM...

Which processor is right for you?

- 2.3GHz quad-core 8th-generation Intel Core i5 processor, Turbo Boost up to 3.8GHz - $300.00

- 2.7GHz quad-core 8th-generation Intel Core i7 processor, Turbo Boost up to 4.5GHz

Memory

How much memory is right for you?

- 8GB 2133MHz LPDDR3 memory - $200.00

- 16GB 2133MHz LPDDR3 memory

nope! RAM in 13in Macbook Pro ≤ 16 GB.
Ok, so RAM isn’t smaller. Is it cheaper?

<table>
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<th>13-inch</th>
<th>15-inch</th>
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**Touch Bar and Touch ID**

- **2.2GHz 6-Core Processor**
- **256GB Storage**

  - 2.2GHz 6-core 8th-generation Intel Core i7 processor
  - Turbo Boost up to 4.1GHz
  - Radeon Pro 555X with 4GB of GDDR5 memory
  - 16GB 2400MHz DDR4 memory
  - 256GB SSD storage¹
  - Retina display with True Tone
  - Touch Bar and Touch ID
  - Four Thunderbolt 3 ports

  $2,399.00

**Touch Bar and Touch ID**

- **2.6GHz 6-Core Processor**
- **512GB Storage**

  - 2.6GHz 6-core 8th-generation Intel Core i7 processor
  - Turbo Boost up to 4.3GHz
  - Radeon Pro 560X with 4GB of GDDR5 memory
  - 16GB 2400MHz DDR4 memory
  - 512GB SSD storage¹
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  $2,799.00
Ok, so RAM isn’t smaller. Is it cheaper?

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Nope!
RIP Moore’s Law for RAM (circa 2013)
Why low memory convex optimization?

- low memory
  - Moore’s law is running out
  - low memory algorithms are often fast

- convex optimization
  - robust convergence
  - elegant analysis
Memory plateau and conditional gradient method

- (Frank & Wolfe 1956) An algorithm for quadratic programming
- (Levitin & Poljak 1966) “Conditional gradient method”
- (Clarkson 2010) Coresets, sparse greedy approximation, and the Frank-Wolfe algorithm
- (Jaggi 2013) Revisiting Frank-Wolfe: projection-free sparse convex optimization
Example: smooth minimization over $\ell_1$ ball

For $g : \mathbb{R}^n \to \mathbb{R}$ smooth, $\alpha \in \mathbb{R}$, find iterative method to solve

$$\text{minimize } g(w) \quad \text{subject to } \|w\|_1 \leq \alpha$$

What kinds of subproblems are easy?

- Projection is complicated (Duchi et al. 2008)
- Linear optimization is easy:

  $$\alpha e_i = \arg\min_x x^T w \quad \text{subject to } \|w\|_1 \leq \alpha$$

  Where $i = \text{indmax}(w)$
Conditional gradient method (Frank-Wolfe)

minimize \( g(w) \)
subject to \( w \in \mathcal{P} \)

\[
\nabla g(w(0)) - \nabla g(w(1)) = v_1
\]

\[
-v_1 = \nabla g(w(0)) - \nabla g(w(1)) = v_1
\]

\[
normal text
\]
Conditional gradient method (Frank-Wolfe)

minimize $g(w)$
subject to $w \in \mathcal{P}$

$-\nabla g(w^{(0)})$
Conditional gradient method (Frank-Wolfe)

minimize \( g(w) \)
subject to \( w \in P \)
Conditional gradient method (Frank-Wolfe)

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What’s wrong with CGM?

- slow
- complexity of iterate grows with number of iterations
Outline

Limited Memory Kelley’s Method
Submodularity primer

- **ground set** $V = \{1, \ldots, n\}$
- identify subsets of $V$ with Boolean vectors $\in \{0,1\}^n$
- $F : \{0,1\}^n \to \mathbb{R}$ is submodular if
  \[
  F(A \cup v) - F(A) \geq F(B \cup v) - F(B), \quad \forall A \subseteq B, \; v \in V
  \]
- linear functions are (sub)modular: for $w \in \mathbb{R}^n$, define
  \[
  w(A) = \sum_{i \in A} w_i
  \]
Submodular function: example

Example: cover

\[ F(S') = \left| \bigcup_{v \in S} \text{area}(v) \right| \]

\[ F(A \cup v) - F(A) \geq F(B \cup v) - F(B) \]
Submodular polyhedra

for submodular function $F$, define

- **submodular polyhedron**
  \[
P(F) = \{ w \in \mathbb{R}^n : w(A) \leq F(A), \quad \forall A \subseteq V \}\]

- **base polytope**
  \[
  B(F) = \{ w \in P(F) : w(V) = F(V) \}\]

both (generically) have exponentially many facets!
Lovász extension

define the (piecewise-linear) **Lovász extension** as

\[ f(x) = \max_{w \in B(F)} w^\top x = \sup \{ w(x) : w(A) \leq F(A) \forall A \subseteq V \} \]

the Lovász extension is the convex envelope of \( F \)

eamples:

| \( F(A) \)          | \( f(x) \)          | \( f(|x|) \)          |
|----------------------|----------------------|----------------------|
| \(|A|\) \min(|A|, 1) | \(1^\top x\) \max(x) | \(\|x\|_1\) \(\|x\|_\infty\) |
| \(\sum_{i=1}^j \min(|A \cap S_j|, 1)\) | \(\sum_{i=1}^j \max(x_{S_j})\) | \(\sum_{i=1}^j \|x_{S_j}\|_\infty\) |
Linear optimization on $B(F)$ is easy

- define the (piecewise-linear) **Lovász extension** as
  
  $$f(x) = \max_{w \in B(F)} x^\top w$$

- $f$ and $1_{B(F)}$ are Fenchel duals:
  
  $$f(x) = 1^*_{B(F)}(x)$$

- linear optimization over $B(F)$ is $O(n \log n)$ (Edmonds 1970)

  - define permutation $\pi$ so $x_{\pi_1} \geq \ldots \geq x_{\pi_n}$. then
    
    $$\max_{w \in B(F)} x^\top w = \sum_{k=1}^n x_{\pi_k} [F(\{\pi_1, \pi_2, \ldots, \pi_k\}) - F(\{\pi_1, \pi_2, \ldots, \pi_{k-1}\})]$$

- computing subgradients of $f$ require $O(n \log n)$ too!

  $$\partial f(x) = \arg\max_{w \in B(F)} x^\top w$$
Primal problem

\[
\text{minimize} \quad g(x) + f(x) \quad (\mathcal{P})
\]

- \( g : \mathbb{R}^n \rightarrow \mathbb{R} \) strongly convex
- \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) Lovász extension of submodular \( F \)
  - piecewise linear
  - homogeneous
  - (generically) exponentially many pieces
  - subgradients are easy \( O(n \log n) \)
Primal problem: example

Application to background subtraction
(Mairal, Jenatton, Obozinski, and Bach, 2010)

Input  $\ell_1$-norm  Structured norm

(Source: http://mistis.inrialpes.fr/learninria/slides/Bach.pdf)
Original Simplicial Method (OSM) (Bach 2013)

Algorithm 1 OSM (to minimize $g(x) + f(x)$)

initialize $\mathcal{V} \leftarrow \emptyset$. repeat

1. define $\hat{f}(x) = \max_{w \in \mathcal{V}} w^\top x$

2. solve subproblem

$$x \leftarrow \text{argmin } g(x) + \hat{f}(x)$$

3. compute $v \in \partial f(x) = \arg\max_{w \in B(F)} x^\top w$

4. $\mathcal{V} \leftarrow \mathcal{V} \cup v$

problem:

- $\mathcal{V}$ keeps growing!
- No known rate of convergence (Bach 2013)
Limited Memory Kelley’s Method (LM-KM)

Algorithm 2 LM-KM (to minimize $g(x) + f(x)$)

1. initialize $\mathcal{V} \leftarrow \emptyset$. repeat
2. define $\hat{f}(x) = \max_{w \in \mathcal{V}} w^\top x$
3. solve subproblem
   $$x \leftarrow \arg\min g(x) + \hat{f}(x)$$
4. compute $v \in \partial f(x) = \arg\max_{w \in B(F)} x^\top w$
5. $\mathcal{V} \leftarrow \{w \in \mathcal{V} : w^\top x = f(x)\} \cup v$

- does it converge or cycle?
- how large could $|\mathcal{V}|$ grow?
LM-KM: intuition
**L-KM converges linearly with bounded memory**

Theorem (Zhou Gupta Udell 2018)

- L-KM has bounded memory: $|\mathcal{V}| \leq n + 1$
- L-KM converges when $g$ is strong convex
- L-KM converges linearly when $g$ is smooth and strongly convex

(Corollary: OSM converges linearly, too.)
Dual problem

minimize $-g^*(-w)$
subject to $w \in B(F)$

$g^*: \mathbb{R}^n \to \mathbb{R}$ smooth (conjugate of strongly convex $g$)

$B(F)$ base polytope of submodular $F$
- (generically) exponentially many facets
- linear optimization over $B(F)$ is easy $O(n \log n)$
Conditional gradient methods for the dual

- linear optimization over constraint is easy

so use a conditional gradient method!

- away-step FW, pairwise FW, fully corrective FW (FCFW) all converge linearly (Lacoste-Julien & Jaggi 2015)

- FCFW has limited memory

- (Garber & Hazan 2015) gives linear convergence with one gradient + one linear optimization per iteration
Dual to primal

suppose $g$ is $\alpha$-strongly convex and $\beta$-smooth

- solve a dual subproblem inexactly to obtain $\hat{y} \in B(F)$ with
  \[ |g^*(-y^*) - g^*(-\hat{y})| \leq \epsilon \]

- $g^*$ is $1/\beta$-strongly convex, so
  \[ \|\hat{y} - y^*\|^2 \leq 2\beta\epsilon \]

- define $\hat{x} = \nabla_y (-g^*(-\hat{y})) = \arg\min_x g(x) + \hat{y}^\top x$
- since $g^*$ is $1/\alpha$ smooth, we have
  \[ \|\hat{x} - x^*\|^2 \leq 1/\alpha^2 \|\hat{y} - y^*\|^2 \leq 2\beta\epsilon/\alpha^2 \]

if the dual iterates converge linearly, so do the primal iterates
Fully corrective Frank-Wolfe

minimize $g(w)$
subject to $w \in \mathcal{P}$
Fully corrective Frank-Wolfe

minimize \( g(w) \)
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Fully corrective Frank-Wolfe

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**Fully corrective Frank-Wolfe**
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Fully corrective Frank-Wolfe

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& \text{minimize} & & g(w) \\
& \text{subject to} & & w \in \mathcal{P}
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\]
**Limited-memory Fully Corrective Frank Wolfe**

**L-FCFW**

**Algorithm 3** FCFW (to minimize $-g^*(-y)$ over $y \in B(F)$)

initialize $\mathcal{V} \leftarrow \emptyset$. repeat

1. solve subproblem

\begin{align*}
\text{minimize} & \quad -g^*(-y) \\
\text{subject to} & \quad y \in \text{Conv}(\mathcal{V})
\end{align*}

define solution $y = \sum_{w \in \mathcal{V}} \lambda_w w$

with $\lambda_w > 0$ and $\sum_{w \in \mathcal{V}} \lambda_w = 1$

2. compute gradient $x = \nabla(-g^*(-y))$

3. solve linear optimization $v = \arg\max_{w \in B(F)} x^\top w$

4. $\mathcal{V} \leftarrow \{w \in \mathcal{V} : \lambda_w > 0\} \cup v$
Fully corrective Frank Wolfe FCFW: properties

- **bounded memory**: Carathéodory $\implies$ can choose $\lambda_w$ so
  \[ |\{ w \in V : \lambda_w > 0 \}| \leq n + 1 \]

- **finite convergence**
  - active set changes at each iteration
  - a vertex that exits the active set is never added again

- converges linearly for smooth strongly convex objectives

- useful if linear optimization over $B(F)$ is hard
  (so convex subproblem is comparatively cheap)

- ok to solve subproblem inexactly (Lacoste-Julien & Jaggi 2015)

compare to vanilla CGM:

- memory cost similar: $w \in \mathbb{R}^n$ vs $n$ (very simple) vertices
- solving subproblems not much harder than evaluating $g^*$
Dual subproblems

► FCFW subproblem

\[
\begin{align*}
\text{maximize} & \quad -g^*(-y) \\
\text{subject to} & \quad y = \sum_{w \in \mathcal{V}} \lambda_w w \\
& \quad 1^T \lambda = 1, \quad \lambda \geq 0
\end{align*}
\]

has dual

\[
\begin{align*}
\text{minimize} & \quad g(x) + \max_{w \in \mathcal{V}} x^T w
\end{align*}
\]

which is our primal subproblem!

► first order optimality conditions show active sets match

\[
\lambda_w > 0 \iff w^T x = \max_{w \in \mathcal{V}} x^T w
\]

hence FCFW has a corresponding primal algorithm: LM-KM!
**LM-KM: numerical experiment**

- $g(x) = x^\top Ax + b^\top x + n\|x\|^2$ for $x \in \mathbb{R}^n$
- $f$ is the Lovász extension of
  
  $$F(A) = \frac{|A|(2n - |A| + 1)}{2}$$

- entries of $A \in M_n$ sampled uniformly from $[-1, 1]$
- entries of $b \in \mathbb{R}^n$ sampled uniformly from $[0, n]$
**LM-KM: numerical experiment**

Dimension $n = 10$ in upper left, $n = 100$ in others
Conclusion

LM-KM gives a new algorithm for composite convex and submodular optimization with **bounded** \(\mathcal{O}(n)\) storage with two old ideas:

- Duality
- Carathéodory
References