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PROJECTED SCALED STEEPEST DESCENT IN KOJIMA-MIZUNO-YOSHISE'S POTENTIAL REDUCTION ALGORITHM FOR THE LINEAR COMPLEMENTARITY PROBLEM

by

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Abstract

Kojima-Mizuno-Yoshise's potential reduction algorithm for the linear complementarity problem is shown to retain its convergence properties when the search direction is changed to projected scaled steepest descent. We also show that their algorithm for linear programming is just a specialization of their linear complementarity method.

1. *Introduction*

This note is concerned with the potential-reduction algorithm of Kojima, Mizuno and Yoshise [3] for solving the linear complementarity problem (LCP):

Find $x, y \in \mathbb{R}^n_+$ with $y = Mx + q$ and $x^T y = 0$,

where M is an $n \times n$ positive semi-definite (but not necessarily symmetric) matix and $q \in \mathbb{R}^n$. We assume that

$$S_{++} := \{(x, y) \in \mathbb{R}^{2n}: \ y = Mx + q, \quad x > 0, \quad y > 0\}$$

is nonempty. Given a suitable pair $(x^0, y^0) \in S_{++}$, their algorithm will find a pair $(x^k, y^k) \in S_{++}$ with

$$(x^k)^T y^k \leq 2^{-t}$$

in $O(\sqrt{n}t)$ iterations, and hence provides a polynomial time algorithm for such LCP's if M and q are integer.

The algorithm is motivated by seeking sufficient reduction in the potential function

$$\begin{aligned} f(x, y) &:= \rho \ell n x^T y - \textstyle\sum_j \ell n x_j y_j - n \ell n\, n \\ &= (\rho - n) \ell n\, x^T y - \textstyle\sum_j \ell n \left(\frac{n x_j y_j}{x^T y}\right) \end{aligned}$$

at each iteration, where $\rho = n + \sqrt{n}$. Guaranteeing a fixed decrease in this function at each iteration ensures the convergence result stated above; it is only necessary that the initial pair satisfy $f(x^0, y^0) = O(\sqrt{n}t)$.

At each iteration, a symmetric primal-dual scaling is (implicitly) made, and then a search direction is chosen. However, this search direction is not necessarily the projected steepest descent direction for the potential function in the scaled space in the usual sense (although Kojima, Mizuno and Yoshise [3, Section 3(B)] show that it is such a direction with respect to a suitable norm on the subspace $\{(x,y) \in \mathbb{R}^{2n}: y = Mx\}$). In section 2, we will show that the directions do coincide when M is skew symmetric (as in the case where the LCP arises from a linear programming problem), and that if the search direction is changed to the projected steepest descent direction, then the complexity analysis of [3] remains valid, with only some constants changed.

Other potential reduction algorithms for the LCP do use the scaled steepest descent direction, but with separate primal and dual scaling (Kojima, Megiddo and Ye [1], Ye [5], and Ye and Pardalos [6], for example). We show that this direction can also be viewed as a sort of Newton direction for minimizing the potential, where the Hessian comes from the barrier part $-\sum_j \ell n\, x_j y_j$ of the potential function. However, for this algorithm at best a bound of $O(n^2 t)$ iterations has been proved [1, 5, 6], even for the positive semi-definite case, with a larger value for ρ. We comment briefly on this difference.

In section 3(A), Kojima, Mizuno and Yoshise [3] state that their analysis remains valid for the linear programming problem with suitable changes to the definitions. In section 3, we will show that the situation is not merely analogous; if the appropriate LCP is set up, the two algorithms are identical. This was probably known to several researchers, including Kojima, Mizuno and Yoshise, although to our knowledge it has not been stated explicitly. We demonstrate the equivalence in the context of convex quadratic programming.

As far as possible, we use the notation of [3].

2. Scaled steepest descent

Let the current iterate be $(x, y) \in S_{++}$. Kojima, Mizuno and Yoshise let the next iterate be

$$(\bar{x}, \bar{y}) := (x, y) - \theta(\Delta x, \Delta y) \tag{2.1}$$

for a suitable direction $(\Delta x, \Delta y)$ (with $\Delta y = M\Delta x$) and step size θ. To ensure that $(\bar{x}, \bar{y}) \in S_{++}$, they require

$$\theta\|X^{-1}\Delta x\|_\infty \le \tau, \;\; \theta\|Y^{-1}\Delta y\|_\infty \le \tau \tag{2.2}$$

where $\tau \in (0,1)$ and an upper-case letter (like X) denotes the diagonal matrix with diagonal entries the components of the corresponding lower-case letter (like x). They show that

$$f(\bar{x}, \bar{y}) \leq f(x, y) + \theta g_1(\Delta x, \Delta y) + \theta^2 g_2(\Delta x, \Delta y), \tag{2.3}$$

where

$$g_1(\Delta x, \Delta y) := -u^T(V^{-1}Y\Delta x + V^{-1}X\Delta y); \tag{2.4}$$

$$V = (XY)^{1/2}, \quad v = Ve; \tag{2.5}$$

$$u = \frac{\rho}{v^T v} v - V^{-1}e; \tag{2.6}$$

and

$$g_2(\Delta x, \Delta y) := \frac{\rho}{x^T y} \Delta x^T \Delta y + \frac{\|X^{-1}\Delta x\|^2 + \|Y^{-1}\Delta y\|^2}{2(1-\tau)} \tag{2.7}$$

Consider the so-called symmetric scaling:

$$\begin{aligned} \hat{x} \to \tilde{x} &:= V^{-1}Y\hat{x} = (XY^{-1})^{-1/2}\hat{x}, \\ \hat{y} \to \tilde{y} &:= V^{-1}X\hat{y} = (XY^{-1})^{1/2}\hat{y}. \end{aligned} \tag{2.8}$$

Note that $f(\tilde{x}, \tilde{y}) = f(\hat{x}, \hat{y})$, and that $(\hat{x}, \hat{y}) \in S_{++}$ iff

$$(\tilde{x}, \tilde{y}) \in \tilde{S}_{++} := \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^{2n}: \tilde{y} = \tilde{M}\tilde{x} + \tilde{q},\ \tilde{x} > 0,\ \tilde{y} > 0\},$$

where

$$\tilde{M} = (XY^{-1})^{1/2}M(XY^{-1})^{1/2}, \quad \tilde{q} = (XY^{-1})^{1/2}q. \tag{2.10}$$

We remark that $\tilde{M}$, as a symmetric scaling of M, is skew-symmetric or positive semi-definite exactly when M is.

Under the symmetric scaling, the current iterate (x, y) is transformed into (v, v) and the direction $(\Delta x, \Delta y)$ is transformed into $(\tilde{\Delta x}, \tilde{\Delta y}) := (V^{-1}Y\Delta x, V^{-1}X\Delta y)$ (cf. (2.4)). The gradient of f at the point (v, v) is

$$\nabla f(v, v) = (u, u). \tag{2.11}$$

In order to decrease the bound (2.3) on f as much as possible, Kojima, Mizuno and Yoshise choose $\tilde{\Delta}x$ and $\tilde{\Delta}y$ as the unique solution of

$$\begin{aligned} \tilde{\Delta}x + \tilde{\Delta}y &= \Delta u := \frac{u}{\|u\|}, \\ \tilde{\Delta}y &= \tilde{M}\,\tilde{\Delta}x; \end{aligned} \tag{2.12}$$

they assume M (and hence $\tilde{M}$) is positive semi-definite, so that $I + \tilde{M}$ is positive definite and $\tilde{\Delta}x = (I + \tilde{M})^{-1}\Delta u$, $\tilde{\Delta}y = \tilde{M}(I + \tilde{M})^{-1}\Delta u$ solves (2.12).

Instead, we will use the projected steepest descent direction for f in this scaling. Using (2.11), we want the projection of (u, u) into the null space of $(-\tilde{M}, I)$. Using standard techniques this gives

$$\begin{aligned} \tilde{\Delta}x &= (I + \tilde{M}^T\tilde{M})^{-1}(I + \tilde{M}^T)u \\ \tilde{\Delta}y &= \tilde{M}(I + \tilde{M}^T\tilde{M})^{-1}(I + \tilde{M}^T)u; \end{aligned} \tag{2.13}$$

then

$$\tilde{\Delta}x + \tilde{\Delta}y = u' := (I + \tilde{M})(I + \tilde{M}^T\tilde{M})^{-1}(I + \tilde{M}^T)u; \tag{2.14}$$

This is the form of the step given (with a different scaling) in [1, section 4]. An alternative representation

$$\begin{aligned} \tilde{\Delta}x &= (I + \tilde{M}^T(I + \tilde{M}\tilde{M}^T)^{-1}(I - \tilde{M}))u \\ \tilde{\Delta}y &= (I - (I + \tilde{M}\tilde{M}^T)^{-1}(I - \tilde{M}))u \end{aligned}$$

is used in [1, section 7], and also in [5,6], but this is less convenient for our purposes. The equivalence is easily shown. Note that, for any $\tilde{M}$, $I + \tilde{M}\tilde{M}^T$ and $I + \tilde{M}^T\tilde{M}$ are positive definite and hence invertible, so that $(\tilde{\Delta}x, \tilde{\Delta}y)$ in (2.13) is well-defined; but (2.12) requires conditions, since it may have no solution (e.g., if $\tilde{M} = -I$ and $\Delta u \neq 0$).

Actually, we shall scale $\tilde{\Delta}x$ and $\tilde{\Delta}y$ in (2.13) and define them (for $\tilde{M}$ positive semi-definite) as the unique solution to

$$\tilde{\Delta}x + \tilde{\Delta}y = \Delta u' := \frac{u'}{\|u'\|}, \qquad (2.15)$$
$$\tilde{\Delta}y = \tilde{M}\tilde{\Delta}x.$$

The next result parallels theorem 2.2 of Kojima, Mizuno and Yoshise [3], with $\sqrt{3}/4$ replacing $\sqrt{3}/2$.

Theorem 1. Suppose $\tilde{M}$ is positive semi-definite, and let $v_{min} := \min\{v_1, v_2, \ldots, v_n\}$ and

$$\theta = v_{min}\tau \text{ for some } \tau \in (0,1).$$

Suppose the direction $(\tilde{\Delta}x, \tilde{\Delta}y)$ satisfies (2.15) and $\Delta x = VY^{-1}\tilde{\Delta}x$, $\Delta y = VX^{-1}\tilde{\Delta}y$. Then (2.2) holds, and

$$\theta g_1(\Delta x, \Delta y) \le -(\sqrt{3}/4)\tau, \qquad (2.16)$$

$$\theta^2 g_2(\Delta x, \Delta y) \le \max\left\{\frac{n+\sqrt{n}}{2n}, \frac{1}{2(1-\tau)}\right\}\tau^2. \qquad (2.17)$$

As in [3], this implies that, if $n \ge 2$ and $\tau = .4$, $f(\bar{x}, \bar{y}) \le f(x, y) - .03$, giving the required complexity result.

The proof of theorem 1 follows that of theorem 2.2 in [3]; we assume the reader is familiar with the argument there. Lemma 2.3-2.5 in [3] remain valid, with $\Delta u'$ replacing Δu in the proof of lemma 2.4. Then (2.2) and (2.17) follow as in [3]. However, for (2.16) we find

$$\begin{aligned}\theta g_1(\Delta x, \Delta y) &= -u^T(\tilde{\Delta}x + \tilde{\Delta}y)v_{min}\tau \\ &= -\frac{u^\tau u'}{\|u'\|} v_{min}\tau \\ &= -\frac{u^T(I+\tilde{M})(I+\tilde{M}^T\tilde{M})^{-1}(I+\tilde{M}^T)u}{\|(I+\tilde{M})(I+\tilde{M}^T\tilde{M})^{-1}(I+\tilde{M}^T)u\|} v_{min}\tau \\ &\le -\frac{\lambda_{min}\|u\|^2}{\lambda_{max}\|u\|} v_{min}\tau \le -\left(\frac{\lambda_{min}}{\lambda_{max}}\frac{\sqrt{3}}{2}\right)\tau,\end{aligned}$$

where λ_{min} and λ_{max} are the minimum and maximum eigenvalues of the positive definite matrix

$$H := (I + \tilde{M})(I + \tilde{M}^T\tilde{M})^{-1}(I + \tilde{M}^T). \tag{2.18}$$

Hence the proof will be completed if we establish

Lemma 1. For any $\tilde{M}$, define H by (2.18). Then

a) The eigenvalues of H are all at most 2;

b) If $\tilde{M}$ is positive semi-definite, the eigenvalues of H are all at least 1;

c) If $\tilde{M}$ is skew-symmetric, $H = I$.

Proof. Note that $(I + \tilde{M}^T)(I + \tilde{M}) = (I + \tilde{M}^T\tilde{M}) + (\tilde{M} + \tilde{M}^T)$. If $\tilde{M}$ is skew-symmetric, the last term vanishes, so that $(I + \tilde{M}^T\tilde{M})^{-1} = (I + \tilde{M})^{-1}(I + \tilde{M}^T)^{-1}$ and so $H = I$. If $\tilde{M}$ is positive semi-definite, $\tilde{M} + \tilde{M}^T \geq 0$ ($A \geq B$ means $A - B$ is positive semi-definite) and so $(I + \tilde{M}^T)(I + \tilde{M}) \geq I + \tilde{M}^T\tilde{M}$. It follows that $(I + \tilde{M}^T\tilde{M})^{-1} \geq (I + \tilde{M})^{-1}(I + \tilde{M}^T)^{-1}$ and hence $H \geq I$. This proves (b).

For (a), we have $(I - \tilde{M}^T)(I - \tilde{M}) = I + \tilde{M}^T\tilde{M} - (\tilde{M} + \tilde{M}^T) \geq 0$, so

$$\begin{aligned} 2(I + \tilde{M}^T\tilde{M}) &\geq I + \tilde{M}^T\tilde{M} + \tilde{M} + \tilde{M}^T \\ &= (I + \tilde{M}^T)(I + \tilde{M}). \end{aligned}$$

Thus $\hat{H} := (I + \tilde{M}^T\tilde{M})^{-1/2}(I + \tilde{M}^T)(I + \tilde{M})(I + \tilde{M}^T\tilde{M})^{-1/2} \leq 2I$, so $\hat{H}$ has all eigenvalues at most 2 and hence so does H.

Not only does lemma 1 prove the theorem; it also shows that, when $\tilde{M}$ (or M) is skew-symmetric (as in the case of an LCP arising from a linear programming problem), $u' = u$ so that the projected steepest descent direction given by (2.15) coincides with the Kojima-Mizuno-Yoshise direction given by (2.12).

If a steepest descent step is good, a Newton step should be even better. Such a step can be defined by minimizing a local quadratic model of f subject to remaining in the affine hull of S_{++}. Unfortunately, the Hessian of f is not necessarily positive definite. If we instead use the Hessian of $-\sum_j \ell n\, x_j y_j$, which is

$$\begin{bmatrix} X^{-2} & 0 \\ 0 & Y^{-2} \end{bmatrix},$$

we find $(\Delta x, \Delta y)$ is the solution to

$$\min -\nabla_x f^T \Delta x - \nabla_y f^T \Delta y + \tfrac{1}{2}\, \Delta x^T X^{-2} \Delta x + \tfrac{1}{2}\, \Delta y^T Y^{-2} \Delta y$$
$$\Delta y = M \Delta x$$

(the negative signs are present because we are taking a step in the direction $(-\Delta x, -\Delta y)$). To compute this direction, we recall that the Newton direction is invariant under affine transformations, and in particular under diagonal scalings. So we scale space so that our approximate Hessian becomes the identity. This is the separate primal and dual scaling:

$$\hat{x} \to \tilde{x} := X^{-1}\hat{x},$$
$$\hat{y} \to \tilde{y} := Y^{-1}\hat{y}.$$

In this scaled space, the projected Newton direction is exactly the projected steepest descent step, which is the direction of Kojima, Megiddo and Ye [1], Ye [5], and Ye and Pardalos [6]. It can be computed from (2.13), using now

$$\tilde{M} := Y^{-1} M X, \quad u = \frac{\rho}{x^T y}\, XYe - e.$$

However, note that $\tilde{M}$ loses properties like symmetry, skew-symmetry or positive semi-definiteness possessed by M. This invalidates several of the arguments used by Kojima-Mizuno-Yoshise.

We would argue that, while the "Newton" or Kojima-Megiddo-Ye direction is good when M has no structure, it is not appropriate otherwise. Indeed, the philosophy behind the Newton step is to make a transformation of the space so that the resulting problem is as nicely behaved as possible. While in many numerical analysis problems this idea translates into making a single Jacobian or Hessian matrix into the identity, we believe that in the present context the simplicity of the nonlinear function f allows one also to pay attention to preserving structure in M. Given that we wish to confine ourselves to symmetric scalings, that in (2.8) makes the Hessian of the barrier term as well-conditioned as possible. Moreover, in this scaled space, a large entry in the Hessian (which indicates that second-order effects might preclude sufficient decrease in the function) corresponds to a small entry of v; hence small steps are necessary in this case to preserve feasibility, but the gradient u has large norm, and so an adequate reduction in f is still possible.

3. *Equivalence of LP, QP and LCP algorithms*

In this section we will show that the LCP framework not only suggests analogous algorithms for linear and convex quadratic programming, but that in fact these algorithms are identical. We will show this equivalence in the context of Kojima-Mizuno-Yoshise's potential reduction algorithm, but the arguments used apply also to related path-following methods. This analysis may be well-known to many, but it seemed worthwhile to make it explicit.

We write the primal convex quadratic programming problem in the form

$$\text{(QP)} \qquad \begin{aligned} &\min\ c^T w + \tfrac{1}{2}\, w^T Q w \\ &Aw = b \\ &w \geq 0 \end{aligned}$$

with dual

$$\text{(QD)} \qquad \begin{aligned} &\max\ b^T r - \tfrac{1}{2}\omega^T Q \omega \\ &-Q\omega + A^T r + s = c \\ &s \geq 0, \end{aligned}$$

where A is $m \times n$ and Q is symmetric and positive semi-definite (possibly 0, which yields linear programming). The statement analogous to that of Kojima-Mizuno-Yoshise in section 3(A) of [3] (they only treated LP) is that the analysis of the LCP remains valid for the QP if we use

$$f(w,s) = \rho \, \ell n \, w^T s - \sum \ell n \, w_j s_j - n \, \ell n \, n \tag{3.1}$$

with $\rho = n + \sqrt{n}$, define

$$\begin{aligned} \tilde{S}_{++} = \{(w,s) \in \mathbb{R}^{2n}&: (w,s) > 0, \; Aw = b, \\ -Qw + A^T r + s = c &\text{ for some } r \in \mathbb{R}^m\} \end{aligned} \tag{3.2}$$

(assumed to be nonempty) and define the direction $(\Delta w, \Delta s)$ via

$$\begin{aligned} \tilde{V}^{-1} S \Delta w + \tilde{V}^{-1} W \Delta s &= \Delta u \\ A \Delta w = 0, \; -Q \Delta w + A^T \Delta r + \Delta s &= 0 \end{aligned} \tag{3.3}$$

for some Δr, where

$$\tilde{V} = (WS)^{1/2}, \; v = \tilde{V} e, \tag{3.4}$$

and Δu and u are as in (2.6), (2.12).

We first note that, if $\tilde{S}_{++}$ is nonempty, then any dependent rows can be removed from A without changing (QP), $\tilde{S}_{++}$ or the direction $(\Delta w, \Delta s)$ solving (3.3). Indeed, the removal of such rows does not change either the set of feasible (w,s) or the corresponding objective function value in (QD). We therefore assume that A has full rank m.

By permuting the columns of A if necessary (which also leaves the algorithm invariant), we can assume that the last m columns are linearly independent, and partition A into $[A_1, A_2]$, with A_2 square and nonsingular, and similarly c^T, w^T, ω^T and s^T into $[c_1^T, c_2^T]$, $[w_1^T, w_2^T]$, $[\omega_1^T, \omega_2^T]$ and $[s_1^T, s_2^T]$, and finally Q into

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}. \tag{3.5}$$

Then $Aw = b$ implies $A_1w_1 + A_2w_2 = b$ or

$$\bar{A}w = \bar{A}_1w_1 + w_2 = \bar{b}, \text{ with } \bar{A} = A_2^{-1}A, \ \bar{A}_1 = A_2^{-1}A_1, \ \bar{b} = A_2^{-1}b. \tag{3.6}$$

We can change A and b into $\bar{A}$ and $\bar{b}$ without changing (QP), (QD), $\tilde{S}_{++}$, or the direction $(\Delta w, \Delta s)$ solving (3.3). Indeed, r and Δr just become $\bar{r} = A_2^T r$ and $\bar{\Delta}r = A_2^T \Delta r$.

Now we may use (3.6) to solve for w_2 in terms of w_1 and substitute it in (QP). We find

$(\bar{QP})$

$$\begin{aligned} &\min \bar{c}_1^T w_1 + \tfrac{1}{2} w_1^T \bar{Q}_{11} w_1 \\ &\bar{A}_1 w_1 + w_2 = \bar{b} \\ &w_1, w_2 \geq 0 \end{aligned}$$

with dual

$(\bar{QD})$

$$\begin{aligned} &\max \bar{b}^T \bar{r} - \tfrac{1}{2} \omega_1^T \bar{Q}_{11} \omega_1 \\ &- \bar{Q}_{11} \omega_1 + \bar{A}_1^T \bar{r} + s_1 = \bar{c}_1 \\ &\bar{r} + s_2 = 0 \\ &s_1, s_2 \geq 0, \end{aligned}$$

from which we can remove the constraint $\bar{r} + s_2 = 0$ by replacing $\bar{r}$ by $-s_2$. Here

$$\begin{aligned} \bar{c}_1 &= c_1 - \bar{A}_1^T c_2 + Q_{12}\bar{b} - \bar{A}_1^T Q_{22} \bar{b}, \\ \bar{Q}_{11} &= Q_{11} - Q_{12}\bar{A}_1 - \bar{A}_1^T Q_{21} + \bar{A}_1^T Q_{22} \bar{A}_1. \end{aligned} \tag{3.7}$$

Note that $(\bar{QD})$ is not immediately equivalent to (QD) unless ω satisfies the constraints of (28), in particular $A\omega = b$. However, only such dual solutions are considered in both the LCP formulation of a quadratic programming problem or the interior-point algorithms designed for such problems (cf. (3.2)).

Now observe that $\tilde{S}_{++}$ is unchanged if we derive it from $(\bar{QP})$ and $(\bar{QD})$ instead of (QP) and (QD) (because of the presence of the constraint $Aw = b$). Hence

$$\tilde{S}_{++} = \{(w,s) \in \mathbb{R}^{2n}: (w,s) > 0,\ \bar{A}_1 w_1 + w_2 = \bar{b},\ -\bar{Q}_{11} w_1 - \bar{A}_1^T s_2 + s_1 = \bar{c}_1\}. \tag{3.8}$$

Similarly, the direction $(\Delta w, \Delta s)$ which solves (3.3) also solves the analogous system derived from $(\bar{QP})$, $(\bar{QD})$:

$$\begin{aligned} \tilde{V}^{-1} S \Delta w + \tilde{V}^{-1} W \Delta s &= \Delta u, \\ \bar{A}_1 \Delta w_1 + \Delta w_2 = 0,\quad -\bar{Q}_{11} \Delta w_1 - \bar{A}_1^T \Delta s_2 + \Delta s_1 &= 0. \end{aligned} \tag{3.9}$$

But the optimality conditions for $(\bar{QP})$ and $(\bar{QD})$ can be written as a linear complementarity problem:

$$\text{find } x := \begin{pmatrix} w_1 \\ s_2 \end{pmatrix} \text{ and } y := \begin{pmatrix} s_1 \\ w_2 \end{pmatrix} \in \mathbb{R}^n_+ \text{ with } y = Mx + q = \begin{bmatrix} \bar{Q}_{11} & \bar{A}_1^T \\ -\bar{A}_1 & 0 \end{bmatrix} x + \begin{pmatrix} \bar{c}_1 \\ \bar{b} \end{pmatrix} \tag{3.10}$$

and $x^T y = 0$.

Moreover, if we define S_{++} from this LCP as in the introduction,

$$(x,y) = \left(\begin{pmatrix} w_1 \\ s_2 \end{pmatrix}, \begin{pmatrix} s_1 \\ w_2 \end{pmatrix} \right) \in S_{++} \text{ iff } (w,s) \in \tilde{S}_{++}.$$

Next, $V = (XY)^{1/2} = (WS)^{1/2} = \tilde{V}$. And since

$$\begin{aligned} S\Delta w + W\Delta s &= \begin{pmatrix} S_1 \Delta w_1 \\ S_2 \Delta w_2 \end{pmatrix} + \begin{pmatrix} W_1 \Delta s_1 \\ W_2 \Delta s_2 \end{pmatrix} \\ &= \begin{pmatrix} S_1 \Delta w_1 \\ W_2 \Delta s_2 \end{pmatrix} + \begin{pmatrix} W_1 \Delta s_1 \\ S_2 \Delta w_1 \end{pmatrix} \\ &= Y\Delta x + X\Delta y, \end{aligned}$$

we see that system (3.9) to determine the direction $(\Delta w, \Delta s)$ corresponds exactly to system (2.12) to determine the direction $(\Delta x, \Delta y)$ for the LCP (3.10).

Hence we have

Theorem 2. The interior-point algorithm for (QP) and (QD) that generates sequences of pairs in $\tilde{S}_{++}$ of (3.2) by moving in the direction $(\Delta w, \Delta s)$ solving (3.3) yields identical iterates to the Kojima-Mizuno-Yoshise potential-reduction algorithm applied to the related LCP (3.10), given corresponding initial solutions and step sizes. If $Q = 0$ (so the programming problems are linear), the algorithm for the LCP (3.10) using the projected steepest descent direction given by the solution to (2.15) also generates identical iterates.

As we mentioned at the beginning of this section, analogous theorems hold for other interior-point methods for quadratic programming and linear complementarity problems, which often differ only in the choice of Δu in (3.3); for instance, the primal-dual path-following algorithms of Monteiro-Adler [4] for QP (see their equations (3.1)) and Kojima-Mizuno-Yoshise [2] for LCP's (see their equation (2.1)) are similarly equivalent, given corresponding μ's.

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