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SOME REMARKS ON THE RELAXATION METHOD FOR LINEAR INEQUALITIES

by

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Abstract

The relaxation method for linear inequalities iterates by projecting the current point onto (or reflecting it in) a most violated constraint. We give a condition number for inequality systems that yields a bound on the convergence ratio for relaxation methods. We also show that projection onto (or reflection in) several hyperplanes simultaneously is possible without jeopardizing convergence results. The resulting method converges finitely for a transshipment problem with unrestricted flows. Finally we show that the finite convergence of the reflection version is not bounded by a polynomial in the size of the input.

1. Introduction

Until the recent development of fast direct methods, large sparse linear systems were generally solved by iterative methods of either the Gauss-Seidel or Jacobi type. In 1954, Agmon [1] and Motzkin and Schoenberg [6] independently discovered an analogous method for solving systems of linear inequalities, called the relaxation method.

Basic versions of the method obtain each successive iterate by either projecting the current iterate onto, or reflecting it in, the hyperplane corresponding to the most violated inequality. This description makes clear why such a method might be appropriate for large, sparse systems of inequalities; only the original data and the current iterate need be stored, and any accumulated roundoff error is automatically compensated for. The usual way to attack such problems is by some variant of the simplex method. Recent techniques to exploit sparsity in the simplex method may have supplanted relaxation methods to some extent for these problems.

One very successful application of relaxation methods has been in developing bounds in combinatorial optimization problems. The subproblem of obtaining a bound is similar to finding a feasible point to an astronomically large number of linear inequalities. These inequalities are only known implicitly--for any trial solution, a most violated inequality may be obtained. See Held, Wolfe and Crowder [4] for a discussion. Our results may be of interest if a (small) group of violated inequalities may be generated for any trial solution.

The relaxation method is closely related to subgradient algorithms for nonsmooth optimization, apparently first introduced by Shor [11]; Polyak [8 , 9] obtained several results concerning the convergence of these methods.

Oettli [7] has shown how relaxation methods may be applied to linear programming problems. His description also makes clear the relationship between relaxation methods and subgradient algorithms.

This paper discusses certain aspects of the relaxation method. In part, our approach is based on the fact that there is no known polynomial algorithm for solving linear inequalities. In Section 4, therefore, we allow each step to be far more complicated than projecting on (or reflecting in) a simple hyperplane. We consider simultaneous projection onto several hyperplanes under certain conditions. Such an iteration costs the solution of a linear system of equations and therefore sacrifices the simplicity of the original methods. On the other hand, one can solve systems of equations in polynomial time.

Section 2 describes the basic relaxation method. This method generally has linear convergence rate, and the convergence ratio is a function of a certain condition number of the inequality system. Section 3 obtains a formula for such a condition number that is much simpler than that of Agmon [1]. Section 4 discusses simultaneous projection in relaxation methods. In certain cases, this extension allows finite convergence when it was not present for the original method. For example, Section 5 shows that finding unrestricted flows in a directed graph to satisfy demands from given supplies can be performed in at most a number of iterations equal to the number of nodes. When the feasible region is full-dimensional, the reflection algorithm of Motzkin and Schoenberg [6] terminates finitely; Section 6 demonstrates that the number of steps may not be polynomial in the length of the input.

It is our hope that a certain version of the simultaneous projection algorithm will exhibit finite, and possibly even polynomial, convergence.

A more modest expectation is that the material of Sections 3 and 4 could be somehow combined to give improved rates of convergence when simultaneous projections are employed. At present we see no way to achieve either goal.

2. The Relaxation Method

Here we merely describe the methods introduced by Agmon [1] and Motzkin and Schoenberg [6] and give some of their properties. We seek a point in $S = \{x \in R^n | Ax \leq b\}$ where A is an $m \times n$ real matrix and b an m-vector. We assume throughout that S is nonempty. Suppose also that the rows $a^1, a^2, \ldots, a^m$ of A have been normalized to have unit Euclidean length. The method then iterates as follows, given a relaxation parameter λ, $0 < \lambda \leq 2$:

Choose $z^1 \in R^n$ arbitrarily;

Given $z^k \in R^n$, calculate $s^k = Az^k - b$. If $s^k \leq 0$, terminate. Otherwise, choose $i = i(k)$ so that $s_i^k = \max_j s_j^k$ and set

$$z^{k+1} = z^k - \lambda s_i^k a^i.$$

Note that this method chooses $i(k)$ as the index of the furthest hyperplane defining S with respect to which z^k is on the wrong side. Then, if $\lambda = 1$, z^{k+1} is the orthogonal projection of z^k onto this hyperplane, while if $\lambda = 2$, z^{k+1} is the reflection of z^k in this hyperplane. We then have the following results.

<u>Lemma 1</u> (Agmon). Suppose the algorithm does not terminate at step k. Then, for every $w \in S$, we have

$$||z^{k+1}-w||^2 \leq ||z^k-w||^2 - [1 - (1-\lambda)^2](s_i^k)^2.$$

Theorem 1 (Agmon). For all $0 < \lambda < 2$, either the algorithm terminates or the iterates z^k converge to some $z^* \in S$. Moreover, there is some $\alpha > 0$, $0 < \beta < 1$ with $||z^k - z^*|| \leq \alpha\beta^k$. Suppose $\lambda = 1$. Then β can be taken as $(1-\phi^{-2}(A))^{1/2}$, where $\phi(A)$ is such that for all z with $a^i \cdot z \leq b_i + 1$ all i, there is some $w \in S$ with $||z-w|| \leq \phi(A)$.

Theorem 2 (Motzkin and Schoenberg). If S is n-dimensional and $\lambda = 2$, then the method terminates finitely.

Note that it is unnecessary to compute s^k at each stage; it may instead by updated. For this, calculate $\eta_{ij} = a^i \cdot a^j$ for all i,j. Then $s_j^{k+1} = s_j^k - \lambda s_{i(k)}^k \eta_{i(k)j}$. Similarly, z^k need not be updated at each iteration. We initialize $y = 0 \in R^m$; at iteration k, $y_{i(k)} \leftarrow y_{i(k)} - \lambda s_{i(k)}^k$. Then at any iteration, z^k can be retrieved by $z^k = z^1 + A^T y$.

Example 1. Let $A = \begin{bmatrix} \delta & 1 \\ \delta & -1 \end{bmatrix}$ with δ small and positive and let $b = 0$. Define $\theta = \tan^{-1}\delta$. Suppose $z^1 = (\cos\theta,,\sin\theta)^T$. Then, for $\lambda = 1$, we have the iterates $z^{k+1} = (\cos\theta(\cos 2\theta)^k, \sin\theta(-\cos 2\theta)^k)^T$. Here the rate of convergence $\beta = \cos 2\theta$. For $\lambda = 2$, we have $z^{k+1} = (\cos(2k+1)\theta, (-1)^k \sin(2k+1)\theta)^T$ for $k \leq \pi/2\theta - 1$.

3. A Lipschitz Constant for Inequality Systems

Many authors have investigated the stability of the solution set to a set of linear inequalities as the data is perturbed, from Agmon [1] and Hoffman [5] to the recent general theory of Robinson [10]. Here our aims are more modest; we simply seek a simple expression for $\phi(A)$ as in Theorem 2. Agmon [1] has given an expression, which, however, is cumbersome and hard to compute. We give below a simpler formula based on quadratic programming duality. Since some relaxation algorithms guarantee satisfaction of certain constraints (e.g., bounds on the variables) at each iteration, we consider the change in the solution set under an arbitrary relaxation of the right hand sides.

Let $S = \{x \mid Ax \leq b\}$, and for any $u \in R^m_+$, let $S(u)$ denote $\{x \mid Ax \leq b+u\}$. Our goal is a bound on $\max_{z\in S(u)} \min_{w\in S} ||z-w||$. Suppose A is $m\times n$ and has rank r; then an $r\times n$ submatrix of A with full row rank will be called a basis of A.

<u>Theorem 3</u>. Let $\phi(A,u)$ be the maximum of $(\bar{u}^T(BB^T)^{-1}\bar{u})^{1/2}$ for B a basis of A and $\bar{u}$ the corresponding subvector of u. Then

$$\max_{z\in S(u)} \min_{w\in S} ||z-w|| \leq \phi(A,u).$$

<u>Proof</u>: For any $z \in R^n$ let $f(z)$ denote the minimum of $\frac{1}{2}(z-w)^T(z-w)$ for $w \in S$. Since S is nonempty and the minimand has bounded level sets, this minimum is obtained. Also, the Kuhn-Tucker conditions are both necessary and sufficient for a minimum to this problem; hence there is a basis B with corresponding subvectors $\bar{b}$ and $\bar{u}$ such that $f(z)$ is the minimum of $\frac{1}{2}(z-w)^T(z-w)$ over $Bw \leq \bar{b}$. Now using quadratic programming duality we can write

$$f(z) = \max_v\{-\frac{1}{2}v^T(BB^T)v + (Bz-\bar{b})^Tv \mid v \geq 0\}.$$

Let $\delta = \max_{z\in S(u)} \min_{w\in S} \frac{1}{2}||z-w||^2$; then we have

$$\delta = \max_{v,z}\{-\frac{1}{2}v^T(BB^T)v + (Bz-\bar{b})^Tv \mid Az \leq b+u, \ v \geq 0\}.$$

For any $v \geq 0$ and $z \in S(u)$ the maximand is at most $-\frac{1}{2} v^T(BB)^T v + \bar{u}^T v$; hence

$$\begin{aligned}\delta &\leq \max_v \{-\frac{1}{2} v^T(BB^T)v + \bar{u}^T v \mid v \geq 0\} \\ &\leq \max_v \{-\frac{1}{2} v^T(BB^T)v + \bar{u}^T v\} \\ &= \frac{1}{2} \bar{u}^T(BB^T)^{-1}\bar{u}.\end{aligned}$$

The theorem follows.

Corollary 3.1. In Theorem 1, we may take $\phi(A) = \phi(A,e)$, where $e \in R^m$ is a vector of ones.

Note that if $r(A) = r = n$, then $\phi(A,u) = \max||B^{-1}\bar{u}||$ for B a basis of A. In this case, our bound for u the vector of ones coincides with that of Agmon [1]. For $r < n$, the bounds generally differ; examples can easily be constructed to show that neither dominates the other.

Example 2. First take $A = \begin{bmatrix}\delta & 1\\ \delta & -1\end{bmatrix}$ as in Example 1. Then A itself is its only basis. Then $\phi(A,e) = ||A^{-1}e|| = (1 + 4\delta^{-2})^{1/2}$, which is large when δ is small. The bound however is tight--take $b = 0$ and consider $z = (1,2\delta^{-1})^T \in S(e)$.

To show that the bound captures the instability of the inequality system rather than just the ill-conditioning of A, replace the second row of A by $(0,1)$. Then $||A^{-1}||$ is again large, but $\phi(A,e)$ becomes 1--again this bound is tight.

Finally let us show a situation where the bound is poor. Replace the second row of A by $(0,2)$. Then $\phi(A,e) = \frac{1}{2}(1+\delta^{-1})^{1/2}$ is large, but the inequality system is stable. Indeed $\phi(A,(1,2)^T) = 1$. A palliative is to ensure that the rows of A have roughly unit length, but it is still possible to have $\phi(A,e) >> \phi(A,\bar{u})$ for $\bar{u} \geq e$. A complete cure to this problem would probably render the bound almost impossible to compute. For the same reason, we did not insist that the basis in Theorem 3 be feasible (B is feasible if $Bx = \bar{b}$ has a solution in S), though the theorem in this stronger form remains valid.

4. Simultaneous Projection

Simple two-dimensional examples of the relaxation method show that it behaves poorly (for $\lambda = 2$ as well as $\lambda = 1$) if the iterates oscillate between two hyperplanes with $a^i \cdot a^j < 0$. In this case each move to satisfy one of these constraints increases the violation of the other; it is easily seen that this behavior can occur also in higher dimensions. Our main result in this section shows when it is possible to project orthogonally onto several constraints simultaneously without sacrificing monotonic movement towards S.

Our first result is purely matrix-theoretic. After a geometric corollary we derive our main tool, Corollary 4.2.

Theorem 4. If C is a $k{\times}n$ matrix such that CC^T has nonpositive off-diagonal entries and for some $\bar{x} \in R^n$, $C\bar{x} > 0$, then CC^T is nonsingular with nonnegative inverse.

Proof: Since R^n is the direct sum of the row space of C and its orthogonal complement, we can write $\bar{x} = C^T\bar{y} + D\bar{z}$ with $CD = 0$. Hence $\bar{y} \in R^k$ satisfies $CC^T\bar{y} > 0$.

Write $\bar{w} = CC^T\bar{y} > 0$ and let y be any solution to $CC^Ty = \bar{w}$. Splitting y into positive and negative parts, we have $y = y^+ - y^-$, $y^+ \geq 0$, $y^- \geq 0$, $y^+ \cdot y^- = 0$. Then

$$0 \leq \bar{w}^T y^- = (CC^Ty^+ - CC^Ty^-)^T y^- = (C^Ty^+)\cdot(C^Ty^-) - ||C^Ty^-||^2.$$

The first term is a nonnegative combination of inner products of different rows of C, since $y^+ \cdot y^- = 0$. Hence this term is nonpositive from our hypothesis on CC^T. Thus $C^Ty^- = 0$ and we have $\bar{w} = CC^Ty^+$. Now the hypothesis on CC^T together with the positivity of $\bar{w}$ gives $y^+ > 0$. All solutions to $CC^Ty = \bar{w}$ are therefore positive, from which CC^T must be nonsingular.

If some entry of $(CC^T)^{-1}$ were negative, we could find $w > 0$ with $(CC^T)^{-1}w \not\geq 0$. But the argument above with w replacing $\bar{w}$ shows that this is impossible. Hence $(CC^T)^{-1}$ is nonnegative.

(Note: as soon as the existence of a solution $y > 0$ to $CC^Ty > 0$ is established, we can use a theorem of Fiedler and Ptak [3] to finish the proof. The approach above is simpler and self-contained.)

Corollary 4.1. Suppose K is a finite cone in R^n such that every two extreme rays of K form an obtuse angle. If K and its polar $K^* = \{x \in R^n | x\cdot z \leq 0 \text{ for all } z \in K\}$ have nonempty interiors, then $K^* \subseteq -K$.

Proof: Write $K = \{C^Ty | y \geq 0\}$ where the rows of C are the extreme rays of K. Then $K^* = \{x | Cx \leq 0\}$. The condition on the extreme rays shows that CC^T has nonpositive off-diagonal entries. The nonemptiness of the interior of K^* implies that $C\bar{x} > 0$ has a solution, and the similar condition on K shows that any $x \in R^n$ can be written C^Ty for some y. Now if $x \in K^*$, $x = C^Ty$ and $CC^Ty \leq 0$. From the theorem, $y \leq 0$ and so $x \in -K$.

Corollary 4.2. Under the conditions of the theorem, let $T = \{x \in R^n | Cx \leq d\}$ for some $d \in R^k$. Then T is nonempty and for any $z \in R^n$ with $Cz \geq d$, the closest point in T to z is $p = z - C^T(CC^T)^{-1}(Cz-d)$.

Proof: From the theorem, CC^T is nonsingular and hence p is well-defined. Since $Cp = d$, T is nonempty. Now for any $w \in T$ we have

$$(w-p)^T(z-p) = (w-p)^TC^T(CC^T)^{-1}(Cz-d)$$

$$= (Cw-d)^T(CC^T)^{-1}(Cz-d) \leq 0,$$

since $(CC^T)^{-1}$ and $Cz-d$ are nonnegative and $Cw-d$ nonpositive. Hence $||z-w||^2 = ||z-p||^2 + ||w-p||^2 - 2(w-p)^T(z-p) \geq ||z-p||^2$.

Corollary 4.2 demonstrates the possibility of using simultaneous projection (or over- or underprojection) in relaxation methods. Specifically, given z^k and $s^k = Az^k - b$, with $i = i(k)$ such that $s_i^k = \max_j s_j^k$, choose $I = I(k) \subseteq \{1,2,\ldots,m\}$ with

(i) $i(k) \in I$,

(ii) $\eta_{ij} \equiv a^i \cdot a^j \leq 0$ for all $i,j \in I$, $i \neq j$, and

(iii) $s_I^k \geq 0$, and for some $\bar{x}$, $A_I\bar{x} > 0$.

Here of course s_I^k is the subvector of s^k and A_I the submatrix of A corresponding to the row index set I. Then set

$$z^{k+1} = z^k - \lambda A_I^T (A_I A_I^T)^{-1} s_I^k.$$

From Corollary 4.2, (ii) and (iii), z^{k+1} is well-defined.

Theorem 5. With z^{k+1} defined as above rather than as in Section 2, Lemma 1 and Theorems 1 and 2 remain valid.

Proof: We merely note that Corollary 4.2 allows the arguments of [1] and [6] to be used with obvious translations. Condition (i) on $I(k)$ ensures that the step taken has size at least s_i^k and hence the linear rate of convergence of theorem 1 remains valid. (In fact, it is easy to see that $||z^{k+1}-z^k|| \geq ||s_I^k||$ if the rows of A are normalized.)

It is natural to choose I to be any maximal set satisfying (i)-(iii), although it is sometimes preferable not to insist on maximality. The assumption that S is nonempty ensures that $s_I^k > 0$ is sufficient for (iii), for then $A_I(z^k-w) > 0$ for any $w \in S$. We will see later that it is often worthwhile including other j's with $s_j^k = 0$. Condition (iii) still holds if $\eta_{ij} < 0$ for some $i \in I$ (take $x = z^k - w - \varepsilon a^i$) or if $s_j^{k-1} > 0$ (take $\overline{x} = z^k - w + \varepsilon(z^{k-1}-w)$). The following example shows that finite termination cannot be guaranteed with $\lambda = 1$ by choosing any maximal I.

Example 3. Take $b = 0 \in R^4$ and, with δ a small positive number,

$$A = \begin{bmatrix} \delta & 1 & 2 \\ \delta & 1 & -2 \\ \delta & -1 & 2 \\ \delta & -1 & -2 \end{bmatrix}.$$

Let $\theta = \tan^{-1}\delta$ and start with $z' = (\cos\theta, \sin\theta, 0)^T$; take $\lambda = 1$. Then $s^1 = (2\sin\theta, 2\sin\theta, 0, 0)^T$. A natural choice for I is then $\{1,2\}$, leading to $z^2 = (\cos\theta\cos 2\theta, -\sin\theta\cos 2\theta, 0)^T$ and $s^2 = (0, 0, 2\sin\theta\cos 2\theta, 2\sin\theta\cos 2\theta)^T$. Again it is natural to choose $I = \{3,4\}$; continuing in this way leads to $z^{k+1} = (\cos\theta(\cos 2\theta)^k, \sin\theta(-\cos 2\theta)^k, 0)^T$ for all k. Similarly, the choice of I's with $\lambda = 2$ gives $z^{k+1} = (\cos(2k+1)\theta, (-1)^k\sin(2k+1)\theta, 0)^T$ for $k \le \pi/2\theta - 1$.

If we take $I = \{1,4\}$ for iteration 1 we find $z^2 = \sin\theta(0, \frac{4}{5}, -\frac{2}{5})^T$ and $s^2 = \sin\theta(0, \frac{8}{5}, -\frac{8}{5}, 0)^T$. Choosing $I = \{1,2\}$ then gives $z^3 = \sin 2\theta \sin\theta(-\frac{2}{5}, \frac{2\delta}{5}, 0)^T$ which lies in S. Hence unnatural but valid choices of I sometimes lead to finite convergence with $\lambda = 1$.

5. An Example of Finite Convergence

It might seem that condition (ii) above that the normals to several constraints be mutually pairwise obtuse is too restrictive to hold in practice. Here we give an application of practical importance where this condition holds.

Let A be the node-edge incidence matrix of a directed graph. Then, with b_i the net supply available at node i, $S = \{x \mid Ax \le b\}$ is the set of flows in the graph that satisfy demand without exceeding supplies. Of course, one usually adds lower and upper bounds to the flows to obtain a reasonable model, but our analysis below does not handle this case.

If a^i and a^j are two rows of A, then $a^i \cdot a^j$ is the negative of the number of edges joining nodes i and j. Hence $\eta_{ij} = a^i \cdot a^j$ is nonpositive for all i,j. Consider the following method to choose $I(k)$. Set $I(1) = \{i \mid s_i^1 > 0\}$ and for $k > 1$ set $I(k) = I(k-1) \cup \{i \mid s_i^k > 0\}$.

Theorem 6. If A is the node-edge incidence matrix of a directed graph and $I(k)$ is chosen as above, then the simultaneous projection method of Section 4 with $\lambda = 1$ yields a point in S within m iterations, where m is the number of nodes of the graph, whenever S is nonempty.

Proof: We first show that $I(k)$ satisfies conditions (i)-(iii) of the previous section if $z^k \notin S$. Clearly condition (i) is satisfied since $I(k)$ contains the indices of all violated constraints; condition (ii) is satisfied trivially. Now it follows from the derivation of z^k from z^{k-1} and s^{k-1} that $s_i^k = 0$ for $i \in I(k-1)$. Hence $s_i^k \geq 0$ for $i \in I(k)$. The fact that, for some $\bar{x}$, $A_{I(k)}\bar{x} > 0$, follows as below Theorem 5. We take $\bar{x} = z^k - w + \varepsilon(z^{k-1}-w) + \ldots + \varepsilon^{k-1}(z^1-w)$ with $w \in S$ and $\varepsilon > 0$ small.

Now note that $I(k)$ is increasing. Hence if no z^k, $1 \leq k \leq m$, lies in S, then $I(m) = \{1,\ldots,m\}$. Thus in this case $Az^{m+1} = b$ and $z^{m+1} \in S$.

6. Exponential Time of Methods with $\lambda = 2$

We have seen above (Theorems 2 and 5) that finite convergence can be assumed when S has dimension n by taking $\lambda = 2$. Here we show that the number of steps to convergence may grow exponentially with the length of input for the data.

Consider Examples 1 and 3. To make the length of the input clearer, rescale A to get

$$\begin{bmatrix} 1 & M \\ 1 & -M \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & M & 2M \\ 1 & M & -2M \\ 1 & -M & 2M \\ 1 & -M & -2M \end{bmatrix}$$

where M is a large integer. With $b = 0$, the length of the input is in both cases $O(\log M)$. However, Examples 1 and 3 show that, with $\lambda = 2$, the number of steps required is $\pi/2\theta - 1$ where $\theta = \cot^{-1} M$. For M large, the number of steps is of the order of $\pi M/2$, which grows exponentially in the length of the input.

References

[1] S. Agmon, "The Relaxation Method for Linear Inequalities," Canadian Journal of Mathematics 6 (1954), 382-392.

[2] I.I. Eremin and V.D. Mazuzov, "Iterative Method for Solving Problems of Convex Programming," Soviet Physics Doklady 11 (1967), 757-759.

[3] M. Fiedler and V. Ptak, "On Matrices with Non-positive Off-diagonal Elements and Positive Principal Minors," Czechoslovak Mathematical Journal 12 (1962), 382-400.

[4] M. Held, P. Wolfe, and H.P. Crowder, "Validation of Subgradient Optimization," Math. Programming 6 (1974), 62-88.

[5] A.J. Hoffman, "On Approximate Solutions of Systems of Linear Inequalities," J. Res. Natl. Bur. Standards 49 (1952), 263-265.

[6] T. Motzkin and I.J. Schoenberg, "The Relaxation Method for Linear Inequalities," Canadian Journal of Mathematics 6 (1954), 393-404.

[7] W. Oettli, "An Iterative Method, Having Linear Rate of Convergence, for Solving a Pair of Dual Linear Programs," Math. Programming 3 (1972), 302-311.

[8] B.T. Polyak, "A General Method of Solving Extremum Problems," Soviet Mathematics Doklady 8 (1967), 593-597.

[9] B.T. Polyak, "Minimization of Unsmooth Functionals," U.S.S.R. Computational Mathematics and Mathematical Physics 9 (1969) no. 3, 14-29.

[10] S.M. Robinson, "Regularity and Stability for Convex Multivalued Functions," Math. of Operations Research 1 (1976), 130-143.

[11] N.Z. Shor, "On the Structure of Algorithms for the Numerical Solution of Optimal Planning and Design Problems," Dissertation, Cybernetics Institute, Academy of Sciences of the U.S.S.R. (1964).