Can \(n^d + 1\) unit right \(d\)-simplices cover a right \(d\)-simplex with shortest side \(n + \epsilon\)?

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Abstract

In a famous short paper, Conway and Soifer show that \(n^2 + 2\) equilateral triangles with edge length 1 can cover one with side \(n + \epsilon\). We provide a generalization to \(d\) dimensions.

1 Introduction

We denote by \(e_1, \ldots, e_d\) the unit coordinate vectors in \(\mathbb{R}^d\), and by \(e := \sum_j e_j\) the vector of ones. A unit right \(d\)-simplex is defined to be the convex hull of 0, \(e_1\), \(e_1 + e_2\), \(e_1 + e_2 + e_3\), \(e_1 + e_2 + \cdots + e_d\), or any of its images under coordinate permutations and translations. A right \(d\)-simplex is a dilation of a unit right \(d\)-simplex; if the dilation is by a factor \(\alpha > 0\), its shortest side has length \(\alpha\).

We are not able to answer the question in the title, but we first show that, if \(\epsilon \leq \delta := (n + 2)^{-1}\), then \((n + 1)^d + (n - 1)^d - n^d\) suffice. (This fails for the trivial case \(d = 1\); we assume implicitly throughout that \(d > 1\).) Notice that, under the transformation \(x \mapsto Mx\), where

\[
M := \begin{bmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix},
\]

right 2-simplices are transformed into equilateral triangles, so that our result implies that of Conway and Soifer [1]. However, while this theorem matches the known result for \(d = 2\), we observe that for larger dimensions the excess of \((n + 1)^d + (n - 1)^d - n^d\) over \(n^d\) is increasing with \(n\). Our second result is that, when \(n > d\), \(n^d + (d + 1)^d - 2d^d + (d - 1)^d\) suffice, though with a much smaller \(\epsilon\). Both of these arguments generalize those of Conway and Soifer, although inevitably they are longer. (Note that, in its original form [2], the text of the paper contains just two words: “\(n^2 + 2\) can,” although with the two figures and the usual rate of exchange, there are a total of 2002 words, exceeding the length of [3].)

We need a convenient notation for right \(d\)-simplices. For \(v \in \mathbb{R}^d\) and \(\pi\) a permutation of \(\{1, \ldots, d\}\), we use \(k(v, \pi)\) to denote the convex hull of \(v, v + e^{\pi(1)}, v + e^{\pi(1)} + e^{\pi(2)}, \ldots, v + e\). It is easy to see that

\[
k(v, \pi) = \{x \in \mathbb{R}^d : 1 \geq (x - v)_{\pi(1)} \geq (x - v)_{\pi(2)} \geq \cdots \geq (x - v)_{\pi(d)} \geq 0\}.
\]

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It is well known that the set of all \( k(0, \pi)'s, \) as \( \pi \) ranges over all permutations, triangulates the unit cube, while the set of all \( k(v, \pi)'s, \) with \( v \) an integer vector and \( \pi \) a permutation, triangulates \( \mathbb{R}^d. \) See, for example, [5]. These simplices are exactly the \( d \)-dimensional pieces when \( \mathbb{R}^d \) is partitioned by all hyperplanes of the form \( x_j = z \) or \( x_i - x_j = z, \) with \( z \) an integer. More relevant to our purposes, the set of all \( k(v, \pi)'s, \) with \( v \) an integer vector and \( \pi \) a permutation, that lie in the right \( d \)-simplex

\[
S^n := \{ x \in \mathbb{R}^d : n \geq x_1 \geq x_2 \geq \cdots \geq x_d \geq 0 \},
\]
covers (indeed, triangulates) that set. In fact, \( k(v, \pi) \) lies in this set iff \( v \in S^{n-1} \) and, if \( v_j = v_{j+1}, \) \( j \) precedes \( j+1 \) in the permutation \( \pi. \) By volume considerations, there are \( n^d \) such unit right \( d \)-simplices.

We can also easily see that the “base” of \( S^n, \) where \( x_d \) lies between 0 and 1, can also be triangulated, by \( n^d - (n-1)^d \) of these simplices, those with \( v_d = 0. \) In general, we define the base

\[
B_\alpha^\beta := \{ x \in S^\beta : x_d \leq \alpha \}
\]
of the right \( d \)-simplex

\[
S^\beta := \{ x \in \mathbb{R}^d : \beta \geq x_1 \geq x_2 \geq \cdots \geq x_d \geq 0 \},
\]
and similarly the “top”

\[
T_\alpha^\beta := \{ x \in S^\beta : x_d \geq \alpha \}.
\]
We say \( B_\alpha^\beta \) has height \( \alpha. \) The base discussed above is \( B_1^n, \) while in the sequel we consider bases \( B_\alpha^{n+\delta} \) with height \( \alpha \) either slightly more than or slightly less than 1.

## 2 The Conway Construction

The first (graphical) proof in [1, 2] — due to Conway according to Chapter 9 of [4] — shows how to cover a 2-dimensional base slightly taller than 1 with \( 2n+1 \) triangles. We generalize this construction to establish

**Theorem 1** For \( \delta := (n+2)^{-1}, \) the right \( d \)-simplex

\[
S^{n+\delta} := \{ x \in \mathbb{R}^d : n + \delta \geq x_1 \geq x_2 \geq \cdots \geq x_d \geq 0 \},
\]
with shortest side \( n + \delta, \) can be covered by

\[
(n+1)^d + (n-1)^d - n^d
\]
unit right \( d \)-simplices.

**Proof:** We divide \( S^{n+\delta} \) into its base

\[
B_{1+\delta}^{n+\delta} := \{ x \in S^{n+\delta} : x_d \leq 1 + \delta \}
\]
and its top

\[
T_{1+\delta}^{n+\delta} := \{ x \in S^{n+\delta} : x_d \geq 1 + \delta \}.
\]

Note that the top can be written as

\[
T_{1+\delta}^{n+\delta} = \{ x \in \mathbb{R}^d : n + \delta \geq x_1 \geq \cdots \geq x_d \geq 1 + \delta \},
\]
which is just the translation by \((1+\delta)e\) of \( S^{n-1}, \) and can therefore be triangulated by \((n-1)^d \) unit right \( d \)-simplices as above. The proof is completed by applying the lemma below, which we separate to contrast it to the second construction. \( \square \)
Lemma 1  For $\delta := (n + 2)^{-1}$, the base $T_{1+\delta}^{n+\delta}$ can be covered by $(n + 1)^d - n^d$ unit right $d$-simplices.

Proof: Note that the base is somewhat similar to the base

$$B_1^{n+1} := \{x \in S^{n+1} : x_d \leq 1\},$$

which as we noted above, can be triangulated by exactly this many unit right $d$-simplices. Indeed, the base we are interested in has its first $d - 1$ components squeezed in (from $n + 1$ to $n + \delta$) and its last component stretched out (from 1 to $1 + \delta$). We therefore apply an operation to the individual simplices in this triangulation, roughly as the individual cloves are transformed by squeezing the head of a roasted garlic.

As we observed above, the simplices of the triangulation of $B_1^{n+1}$ are those $k(v, \pi)$ where

$$v \in S^n \cap \mathbb{Z}^d; \quad v_d = 0; \quad \text{if } v_j = v_{j+1}, \quad j \text{ precedes } j + 1 \text{ in } \pi. \quad (1)$$

We squeeze these simplices as follows:

(a) If $\pi^{-1}(d) = d$, $\tilde{k}(v, \pi) := k((1 - \delta)v, \pi)$;

(b) if $\pi^{-1}(d) < d$, $\tilde{k}(v, \pi) := k((1 - \delta)v + \delta e, \pi)$.

We need to show that every $x \in B_1^{n+\delta}$ is covered by at least one such $\tilde{k}(v, \pi)$, where $(v, \pi)$ satisfies (1).

For any such $x$, we can choose $v \in \mathbb{Z}^d_+$, $v_d = 0$, so that all components of

$$w := x - (1 - \delta)v,$$

except possibly the last, lie between 0 and 1. We then order these components using the permutation $\pi$. Suppose first we can choose $\pi$ so that $d$ comes last:

$$1 \geq w_{\pi(1)} \geq \cdots \geq w_{\pi(d)} \geq 0, \quad \pi^{-1}(d) = d. \quad (2)$$

Note that there is some choice involved for $j < d$; if $v_j > 0$ and $0 \leq w_j \leq \delta$, we can decrease $v_j$ by 1 so that $1 - \delta \leq w_j \leq 1$ and then adjust $\pi$ accordingly. Then we have

if $v_j > 0$ for $1 \leq j < d, w_j > \delta$. \quad (3)

Moreover, if there is a set of components of $w$ that are equal, we may modify $\pi$ so that their indices appear in ascending order:

if $w_{\pi(j)} = w_{\pi(j+1)}$ for $1 \leq j < d, \pi(j) < \pi(j + 1)$. \quad (4)

We show that, if $v$ and $\pi$ can be chosen so that (2)–(4) hold, then $x$ lies in the simplex $\tilde{k}(v, \pi)$ of type (a). By the first of these conditions, it is only necessary to check (1).

First, we have $w_1 \geq 0$, so that

$$v_1 \leq (1 - \delta)^{-1}x_1 \leq (1 - \delta)^{-1}(n + \delta) = n + 1.$$

Moreover, if $v_1 = n + 1$, we have equality throughout, so that $v_1 > 0$ and $w_1 = 0$, contradicting (3). Hence $v_1 \leq n$. 3
Next, consider the condition \( v_j \geq v_{j+1} \). If \( v_{j+1} = 0 \), then this holds by default. If not, then \( j + 1 < d \) and by (3), \( w_{j+1} > \delta \), so that \( w_j < w_{j+1} + 1 - \delta \) and thus

\[
v_j > v_{j+1} - 1 + (1 - \delta)^{-1}(x_j - x_{j+1}) \geq v_{j+1} - 1,
\]

and we obtain \( v_j \geq v_{j+1} \).

Finally, if \( v_j = v_{j+1} \), then since \( x_j \geq x_{j+1} \) we have \( w_j \geq w_{j+1} \). Thus \( j \) precedes \( j + 1 \) in \( \pi \), either by (2) if these components are unequal, or by (4) if they are equal. This completes the verification of (1), and so \( x \) is covered.

Note that, if \( x_d \leq \delta \), then we can find \( v \) and \( \pi \) so that (2) holds. Indeed, we order the components of \( w \) as above, and ensure that if \( v_j > 0 \) and \( j < d \), then \( w_j > \delta \) and \( j \) precedes \( d \) in \( \pi \). But if \( v_j = 0 \) for \( j < d \), then \( x_j \geq x_d \) ensures that \( w_j \geq w_d \), and thus we can arrange that \( d \) comes last in \( \pi \). Thus the bottom sliver of the base is covered by simplices of type (a).

Now we assume that \( x \) cannot be covered by such a simplex. Then \( x_d > \delta \), and hence \( x_j > \delta \) for all \( j \). We can then find \( v \in \mathbb{Z}_d^+ \) with \( v_d = 0 \) and a permutation \( \pi \) so that, with \( w \) again defined as \( x - (1 - \delta)v \), we have

\[
1 + \delta \geq w_{\pi(1)} \geq \cdots \geq w_{\pi(d)} \geq \delta.
\]

(5)

Moreover, as above, if \( 1 \leq w_j \leq 1 + \delta \) for \( j < d \), we can increase \( v_j \) by 1 so that \( \delta \leq w_j \leq 2\delta \) and then adjust \( \pi \) accordingly, so that

\[
\text{for } j < d, w_j < 1.
\]

(6)

We can also ensure that equal components of \( w \) are suitably ordered, so that (4) holds.

If \( w_d > 1 \), then because of (6), \( \pi^{-1}(d) = 1 \). If instead \( w_d \leq 1 \), then (11) and (6) show that (2) holds, so that if \( \pi^{-1}(d) = d \), \( x \) could be covered by a simplex of type (a). Thus in either case, \( \pi^{-1}(d) < d \), so that, if \( w' := x - (1 - \delta)v - \delta \epsilon \),

\[
1 \geq w'_{\pi(1)} \geq \cdots \geq w'_{\pi(d)} \geq 0, \quad \pi^{-1}(d) < d,
\]

and \( x \) will be covered by a simplex of type (b) if we can verify (1).

Suppose (4)–(6) hold. Then \( w_1 \geq \delta \), so

\[
v_1 \leq (1 - \delta)^{-1}x_1 - (1 - \delta)^{-1}\delta < (1 - \delta)^{-1}(n + \delta) = n + 1,
\]

and we have \( v_1 \leq n \).

Next, consider the condition \( v_j \geq v_{j+1} \). If \( v_{j+1} = 0 \), then this holds by default. If not, then \( j + 1 < d \) and by (11) and (6), \( w_{j+1} \geq \delta \) and \( w_j < 1 \), so that \( w_j < w_{j+1} + 1 - \delta \) and thus

\[
v_j > v_{j+1} - 1 + (1 - \delta)^{-1}(x_j - x_{j+1}) \geq v_{j+1} - 1,
\]

and we obtain \( v_j \geq v_{j+1} \). The proof that if \( v_j = v_{j+1} \) then \( j \) precedes \( j + 1 \) in the permutation \( \pi \) is identical to that above.

Thus \( x \) is covered either by a simplex of type (a) or one of type (b), and the theorem is proved.  \( \square \)
3 The Soifer Construction

The second (graphical) proof in [1, 2], due to Soifer according to Chapter 9 of [4], demonstrates how to cover a 2-dimensional base slightly shorter than 1 with $2n - 1$ triangles. We generalize this construction to prove

**Theorem 2** For $n \geq d$ and $\delta \leq (d + 2)^{-1}d^{-(n-d)}$, $S^{n+\delta}$ can be covered by

$$n^d + (d+1)^d - 2d^d + (d-1)^d$$

unit right $d$-simplices.

**Proof:** We proceed by induction on $n$. For $n = d$, the result follows from Theorem 1. Now suppose $n > d$, and that the theorem holds for $n - 1$. Let

$$\gamma := \frac{1}{n - d} \delta \leq \delta,$$

and divide $S^{n+\delta}$ into its base $B_{1-\gamma}^{n+\delta}$ and its top $T_{1-\gamma}^{n+\delta}$. Note that

$$T_{1-\gamma}^{n+\delta} = \{x \in \mathbb{R}^d : n + \delta \geq x_1 \geq \cdots \geq x_d \geq 1 - (d - 1)\gamma\}$$

is a translation of $S^{n-1+\delta+(d-1)\gamma}$, and since

$$\delta + (d - 1)\gamma \leq d\delta \leq (d + 2)^{-1}d^{-(n-d-1)},$$

it can be covered by $(n-1)^d + (d+1)^d - 2d^d + (d-1)^d$ unit right simplices by the inductive hypothesis. Thus the result will follow from the lemma below. \(\square\)

**Lemma 2** $B_{1-\gamma}^{n+\delta}$ can be covered by $n^d - (n-1)^d$ unit right $d$-simplices.

**Proof:** In the proof of Lemma 1, we took the “bottom” simplices of the triangulation of $B_1^{n+1}$ and squeezed them together, pushing up the remaining simplices to cover a base slightly higher than 1. Now we take the bottom simplices of the triangulation of $B_1^n$ and spread them out, letting the remaining simplices rattle down filling the gaps to cover a base slightly shorter than 1.

With $\gamma$ as above, note that

$$(n-1)(1+\gamma) + 1 = n + \delta + (d - 1)\gamma.$$

(7)

The simplices of the triangulation of $B_1^n$ are those $k(v, \pi)$ where

$$v \in S^{n-1} \cap \mathbb{Z}^d; \ v_d = 0; \ \text{if } v_j = v_{j+1}, \ j \text{ precedes } j+1 \text{ in } \pi.$$  

(8)

We spread out and rattle down these simplices as follows:

(c) If $\pi^{-1}(d) = d, \hat{k}(v, \pi) := k((1+\gamma)v, \pi);$

(d) if $\pi^{-1}(d) = j < d, \hat{k}(v, \pi) := k((1+\gamma)v - (d-j)\gamma e, \pi).$

We need to show that every $x \in B_{1-\gamma}^{n+\delta}$ is covered by at least one such $\hat{k}(v, \pi)$, where $(v, \pi)$ satisfies (8). Given such an $x$, we can choose $v \in \mathbb{Z}_+^d$ with $v_d = 0$ and $v \leq (n-1)e$ so that all the components of

$$w := x - (1+\gamma)v$$
lie between $-\gamma$ and 1. We then order these components using the permutation $\pi$ so that

$$1 \geq w_{\pi(1)} \geq \cdots \geq w_{\pi(d)} > -\gamma. \quad (9)$$

Since $x \leq (n + \delta)e$, equation (7) implies that

$$\text{if } v_i = n - 1 \text{ for } 1 \leq i < d, \text{ then } w_i \leq 1 - (d - 1)\gamma. \quad (10)$$

Also, since $x \in B^{n+\delta}_{1-(d-1)\gamma}$, $w_d \leq 1 - (d - 1)\gamma$. Finally, if there is a set of components of $w$ that are equal, we may modify $\pi$ so that their indices appear in ascending order:

$$\text{if } w_{\pi(i)} = w_{\pi(i+1)} \text{ for } 1 \leq i < d, \pi(i) < \pi(i+1). \quad (11)$$

Let us first assume that $\pi^{-1}(d) = d$, Then $w_d \geq 0$, so that

$$1 \geq w_{\pi(1)} \geq \cdots \geq w_{\pi(d)} \geq 0,$$

and $x$ lies in $k((1 + \gamma)v, \pi)$, and it remains to show (8). We already know that $0 \leq v \leq (n - 1)e$ and $v_d = 0$. Since $x_i \geq x_{i+1}$,

$$v_i \geq v_{i+1} + \frac{w_{i+1} - w_i}{1 + \gamma} \geq v_{i+1} - \frac{1}{1 + \gamma} > v_{j+1} - 1$$

and so $v_i \geq v_{i+1}$, and if these are equal, then $w_i \geq w_{i+1}$ and then (9) and (11) imply that $i$ precedes $i + 1$ in $\pi$. Thus $x$ lies in a simplex of type (c) above.

Next suppose $k := \pi^{-1}(d) < d$. Let $i$ be the lowest index such that

$$w_{\pi(i)} \leq 1 - (d - k + i - 1)\gamma;$$

note that the index $k$ satisfies this inequality so that $i \leq k$. Also,

$$\text{if } i > 1, \text{ for } h < i, \quad w_{\pi(h)} \geq 1 - (d - k + i - 2)\gamma, \text{ and hence } v_{\pi(h)} < n - 1. \quad (12)$$

using (10). We now increase $v_{\pi(h)}$ by 1 for each $h < i$, to get $v'$. From the above, we still have $v' \leq (n - 1)e$. Let $w' := x - (1 + \gamma)v'$. For $j \geq i$, $w'_{\pi(j)} = w_{\pi(j)}$, while for $h < i$,

$$w'_{\pi(h)} = w_{\pi(h)} - 1 - \gamma, \text{ and so using (9) and (12), we find}$$

$$\text{for } h < i, -\gamma \geq w'_{\pi(h)} \geq -(d - k + i - 1)\gamma.$$

Thus if we order the components of $w'$, with strings of equal components in ascending order, we find the permutation $\rho$ with $\rho = (\pi(i), \ldots, \pi(d), \pi(1), \ldots, \pi(i - 1))$ with $\rho^{-1}(d) = k - i + 1 = j'$. We also have

$$1 - (d - j')\gamma \geq w'_{\rho(1)} \geq \cdots \geq w'_{\rho(d)} \geq -(d - j')\gamma,$$

and so $x$ lies in $\hat{k}(v', \rho)$, and it remains to show that $v'$ and $\rho$ satisfy (8). But this follows exactly the argument used above for the case $\pi^{-1}(d) = d$, and so $x$ lies in a simplex of type (d) and the proof is complete. □
4 Discussion

Our results are not tight. Indeed, for \( d > 2 \) and \( n = 1 \), we have shown that \( S^{1+\delta} \) can be covered by \( 2^d - 1 \) right \( d \)-simplices, while \( d + 1 \) suffice by using a construction also similar to Soifer’s construction in [1, 2]:

**Proposition 1** For \( \delta := d^{-1} \), \( S^{1+\delta} \) can be covered by \( d + 1 \) right \( d \)-simplices.

**Proof:** Let \( v^0 := 0 \), \( v^j = v^{j-1} + \delta e^j \) for \( j = 1, \ldots, d \). Let \( \iota \) denote the identity permutation \((1, 2, \ldots, d)\). We show that the \( d + 1 \) right \( d \)-simplices \( k(v^j, \iota) \) for \( j = 0, \ldots, d \) cover \( S^{1+\delta} \).

Consider any \( x \in S^{1+\delta} \). Then

\[
1 + \delta =: x_0 \geq x_1 \geq \cdots \geq x_d \geq x_{d+1} := 0.
\]

There are then \( d + 1 \) nonnegative gaps \( x_i - x_{i+1} \), \( i = 0, \ldots, d \), summing to \( 1 + \delta \), and so since \((d + 1)\delta = 1 + \delta\), one of these, say that indexed by \( i = j \), must be at least \( \delta \). But then

\[
1 \geq x_1 - \delta \geq \cdots \geq x_j - \delta \geq x_{j+1} \geq \cdots x_d \geq 0
\]

(with obvious modifications if \( j = 0 \) or \( j = d \)), so that \( x \in k(v^j, \iota) \). \( \Box \)

If Lemma 2 could be extended to all \( n \), then Proposition 1 would provide the base case to prove that \( S^{n+\delta} \) could be covered by \( n^d + d \) unit right \( d \)-simplices. However, the rather delicate arguments in Lemma 2 seem to require that the “bottom” simplices be spread out to not only cover components up to \( n + \delta \), but further up to \( n + \delta + (d - 1)\gamma \) (see (7)), and this necessitates \( n > d \).

It would be nice to complement our results with lower bounds on the number of unit right \( d \)-simplices required to cover \( S^{n+\delta} \) \((\delta > 0)\), but such results are rare even for \( d = 2 \). Indeed, volume considerations ensure that at least \( n^d + 1 \) are necessary, while for \( d = 2 \) and \( n = 1 \) or \( n = 2 \), considering all points in \( S^{n+\delta} \) all of whose components are integer multiples of \( 1 + \delta/n \), no two of which can lie in a single unit 2-simplex, shows that \( n^2 + 2 \) are necessary.

Both of these techniques are special cases of bounds from measures on \( \mathbb{R}^d \). In general, we can consider the moment problem

\[
M := \sup \{ \mu(S^{n+\delta}) : \mu \text{ is a measure on } \mathbb{R}^d \text{ with } \mu(\Sigma) \leq 1 \text{ for any right } d \text{-simplex } \Sigma \}.
\]

Then \( M \), rounded up to the next integer, provides a lower bound on the number of unit \( d \)-simplices to cover \( S^{1+\delta} \). Perhaps numerical computations on discretizations of this problem can provide insights allowing the construction of measures yielding non-trivial lower bounds on the number of simplices required.

Finally, we note that for \( d = 2 \), right \( d \)-simplices are isosceles right triangles, and that Xu, Yuan, and Ding [6] consider a different problem of covering isosceles right triangles with isosceles right triangles of possibly different sizes and allowing for rotations as well as translations and coordinate permutations.

**References**


