Can $n^d + 1$ unit right d-simplices cover a right d-simplex with shortest side $n + \epsilon$?

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Abstract

In a famous short paper, Conway and Soifer show that n^2+2 equilateral triangles with edge length 1 can cover one with side $n+\epsilon$. We provide a generalization to d dimensions.

1 Introduction

We denote by $e^1, \ldots, e^d$ the unit coordinate vectors in $\mathbb{R}^d$, and by $e := \sum_j e^j$ the vector of ones. A unit right d-simplex is defined to be the convex hull of 0, e^1, $e^1 + e^2$, ... , $e^1 + e^2 + \cdots + e^d$, or any of its images under coordinate permutations and translations. A right d-simplex is a dilation of a unit right d-simplex; if the dilation is by a factor $\alpha > 0$, its shortest side has length α.

We are not able to answer the question in the title, but we first show that, if $\epsilon \leq \delta := (n+2)^{-1}$, then $(n+1)^d + (n-1)^d - n^d$ suffice. (This fails for the trivial case $d = 1$; we assume implicitly throughout that $d > 1$.) Notice that, under the transformation $x \mapsto Mx$, where

$$M := \begin{bmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix},$$

right 2-simplices are transformed into equilateral triangles, so that our result implies that of Conway and Soifer [1]. However, while this theorem matches the known result for $d = 2$, we observe that for larger dimensions the excess of $(n+1)^d + (n-1)^d - n^d$ over n^d is increasing with n. Our second result is that, when $n > d$, $n^d + (d+1)^d - 2d^d + (d-1)^d$ suffice, though with a much smaller ϵ. Both of these arguments generalize those of Conway and Soifer, although inevitably they are longer. (Note that, in its original form [2], the text of the paper contains just two words: "$n^2 + 2$ can," although with the two figures and the usual rate of exchange, there are a total of 2002 words, exceeding the length of [3].)

We need a convenient notation for right d-simplices. For $v \in \mathbb{R}^d$ and π a permutation of $\{1, \ldots, d\}$, we use $k(v, \pi)$ to denote the convex hull of v, $v + e^{\pi(1)}$, $v + e^{\pi(1)} + e^{\pi(2)}$, ...,$v + e$. It is easy to see that

$$k(v, \pi) = \{x \in \mathbb{R}^d : 1 \geq (x - v)_{\pi(1)} \geq (x - v)_{\pi(2)} \geq \cdots \geq (x - v)_{\pi(d)} \geq 0\}.$$

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It is well known that the set of all $k(0,\pi)$'s, as π ranges over all permutations, triangulates the unit cube, while the set of all $k(v,\pi)$'s, with v an integer vector and π a permutation, triangulates $\mathbb{R}^d$. See, for example, [5]. These simplices are exactly the d-dimensional pieces when $\mathbb{R}^d$ is partitioned by all hyperplanes of the form $x_j = z$ or $x_i - x_j = z$, with z an integer. More relevant to our purposes, the set of all $k(v,\pi)$'s, with v an integer vector and π a permutation, that lie in the right d-simplex

$$S^n := \{x \in \mathbb{R}^d : n \geq x_1 \geq x_2 \geq \cdots \geq x_d \geq 0\},$$

covers (indeed, triangulates) that set. In fact, $k(v,\pi)$ lies in this set iff $v \in S^{n-1}$ and, if $v_j = v_{j+1}$, j precedes $j+1$ in the permutation π. By volume considerations, there are n^d such unit right d-simplices.

We can also easily see that the "base" of S^n, where x_d lies between 0 and 1, can also be triangulated, by $n^d - (n-1)^d$ of these simplices, those with $v_d = 0$. In general, we define the base

$$B^\beta_\alpha := \{x \in S^\beta : x_d \leq \alpha\}$$

of the right d-simplex

$$S^\beta := \{x \in \mathbb{R}^d : \beta \geq x_1 \geq x_2 \geq \cdots \geq x_d \geq 0\},$$

and similarly the "top"

$$T^\beta_\alpha := \{x \in S^\beta : x_d \geq \alpha\}.$$

We say B^β_α has height α. The base discussed above is B^n_1, while in the sequel we consider bases $B^{n+\delta}_\alpha$ with height α either slightly more than or slightly less than 1.

2 The Conway Construction

The first (graphical) proof in [1, 2] — due to Conway according to Chapter 9 of [4] — shows how to cover a 2-dimensional base slightly taller than 1 with $2n+1$ triangles. We generalize this construction to establish

Theorem 1 *For $\delta := (n+2)^{-1}$, the right d-simplex*

$$S^{n+\delta} := \{x \in \mathbb{R}^d : n+\delta \geq x_1 \geq x_2 \geq \cdots \geq x_d \geq 0\},$$

with shortest side $n+\delta$, can be covered by

$$(n+1)^d + (n-1)^d - n^d$$

unit right d-simplices.

Proof: We divide $S^{n+\delta}$ into its base

$$B^{n+\delta}_{1+\delta} := \{x \in S^{n+\delta} : x_d \leq 1+\delta\}$$

and its top

$$T^{n+\delta}_{1+\delta} := \{x \in S^{n+\delta} : x_d \geq 1+\delta\}.$$

Note that the top can be written as

$$T^{n+\delta}_{1+\delta} = \{x \in \mathbb{R}^d : n+\delta \geq x_1 \geq \cdots \geq x_d \geq 1+\delta\},$$

which is just the translation by $(1+\delta)e$ of S^{n-1}, and can therefore be triangulated by $(n-1)^d$ unit right d-simplices as above. The proof is completed by applying the lemma below, which we separate to contrast it to the second construction. □

Lemma 1 *For $\delta := (n+2)^{-1}$, the base $T_{1+\delta}^{n+\delta}$ can be covered by $(n+1)^d - n^d$ unit right d-simplices.*

Proof: Note that the base is somewhat similar to the base

$$B_1^{n+1} := \{x \in S^{n+1} : x_d \le 1\},$$

which as we noted above, can be triangulated by exactly this many unit right d-simplices. Indeed, the base we are interested in has its first $d-1$ components squeezed in (from $n+1$ to $n+\delta$) and its last component stretched out (from 1 to $1+\delta$). We therefore apply an operation to the individual simplices in this triangulation, roughly as the individual cloves are transformed by squeezing the head of a roasted garlic.

As we observed above, the simplices of the triangulation of B_1^{n+1} are those $k(v,\pi)$ where

$$v \in S^n \cap \mathbb{Z}^d; \quad v_d = 0; \quad \text{if } v_j = v_{j+1},\ j \text{ precedes } j+1 \text{ in } \pi. \tag{1}$$

We squeeze these simplices as follows:

$$(a) \text{ If } \pi^{-1}(d) = d, \tilde{k}(v,\pi) := k((1-\delta)v, \pi);$$

$$(b) \text{ if } \pi^{-1}(d) < d, \tilde{k}(v,\pi) := k((1-\delta)v + \delta e, \pi).$$

We need to show that every $x \in B_{1+\delta}^{n+\delta}$ is covered by at least one such $\tilde{k}(v,\pi)$, where (v,π) satisfies (1).

For any such x, we can choose $v \in \mathbb{Z}_+^d$, $v_d = 0$, so that all components of

$$w := x - (1-\delta)v,$$

except possibly the last, lie between 0 and 1. We then order these components using the permutation π. Suppose first we can choose π so that d comes last:

$$1 \ge w_{\pi(1)} \ge \cdots \ge w_{\pi(d)} \ge 0, \quad \pi^{-1}(d) = d. \tag{2}$$

Note that there is some choice involved for $j < d$; if $v_j > 0$ and $0 \le w_j \le \delta$, we can decrease v_j by 1 so that $1 - \delta \le w_j \le 1$ and then adjust π accordingly. Then we have

$$\text{if } v_j > 0 \text{ for } 1 \le j < d, w_j > \delta. \tag{3}$$

Moreover, if there is a set of components of w that are equal, we may modify π so that their indices appear in ascending order:

$$\text{if } w_{\pi(j)} = w_{\pi(j+1)} \text{ for } 1 \le j < d, \pi(j) < \pi(j+1). \tag{4}$$

We show that, if v and π can be chosen so that (2)–(4) hold, then x lies in the simplex $\tilde{k}(v,\pi)$ of type (a). By the first of these conditions, it is only necessary to check (1).

First, we have $w_1 \ge 0$, so that

$$v_1 \le (1-\delta)^{-1} x_1 \le (1-\delta)^{-1}(n+\delta) = n+1.$$

Moreover, if $v_1 = n+1$, we have equality throughout, so that $v_1 > 0$ and $w_1 = 0$, contradicting (3). Hence $v_1 \le n$.

Next, consider the condition $v_j \geq v_{j+1}$. If $v_{j+1} = 0$, then this holds by default. If not, then $j+1 < d$ and by (3), $w_{j+1} > \delta$, so that $w_j < w_{j+1} + 1 - \delta$ and thus

$$v_j > v_{j+1} - 1 + (1-\delta)^{-1}(x_j - x_{j+1}) \geq v_{j+1} - 1,$$

and we obtain $v_j \geq v_{j+1}$.

Finally, if $v_j = v_{j+1}$, then since $x_j \geq x_{j+1}$ we have $w_j \geq w_{j+1}$. Thus j precedes $j+1$ in π, either by (2) if these components are unequal, or by (4) if they are equal. This completes the verification of (1), and so x is covered.

Note that, if $x_d \leq \delta$, then we can find v and π so that (2) holds. Indeed, we order the components of w as above, and ensure that if $v_j > 0$ and $j < d$, then $w_j > \delta$ and j precedes d in π. But if $v_j = 0$ for $j < d$, then $x_j \geq x_d$ ensures that $w_j \geq w_d$, and thus we can arrange that d comes last in π. Thus the bottom sliver of the base is covered by simplices of type (a).

Now we assume that x cannot be covered by such a simplex. Then $x_d > \delta$, and hence $x_j > \delta$ for all j. We can then find $v \in \mathbb{Z}_+^d$ with $v_d = 0$ and a permutation π so that, with w again defined as $x - (1-\delta)v$, we have

$$1 + \delta \geq w_{\pi(1)} \geq \cdots \geq w_{\pi(d)} \geq \delta. \tag{5}$$

Moreover, as above, if $1 \leq w_j \leq 1+\delta$ for $j < d$, we can increase v_j by 1 so that $\delta \leq w_j \leq 2\delta$ and then adjust π accordingly, so that

$$\text{for } j < d, w_j < 1. \tag{6}$$

We can also ensure that equal components of w are suitably ordered, so that (4) holds.

If $w_d > 1$, then because of (6), $\pi^{-1}(d) = 1$. If instead $w_d \leq 1$, then (11) and (6) show that (2) holds, so that if $\pi^{-1}(d) = d$, x could be covered by a simplex of type (a). Thus in either case, $\pi^{-1}(d) < d$, so that, if $w' := x - (1-\delta)v - \delta e$,

$$1 \geq w'_{\pi(1)} \geq \cdots \geq w'_{\pi(d)} \geq 0, \quad \pi^{-1}(d) < d,$$

and x will be covered by a simplex of type (b) if we can verify (1).

Suppose (4)–(6) hold. Then $w_1 \geq \delta$, so

$$v_1 \leq (1-\delta)^{-1}x_1 - (1-\delta)^{-1}\delta < (1-\delta)^{-1}(n+\delta) = n+1,$$

and we have $v_1 \leq n$.

Next, consider the condition $v_j \geq v_{j+1}$. If $v_{j+1} = 0$, then this holds by default. If not, then $j+1 < d$ and by (11) and (6), $w_{j+1} \geq \delta$ and $w_j < 1$, so that $w_j < w_{j+1} + 1 - \delta$ and thus

$$v_j > v_{j+1} - 1 + (1-\delta)^{-1}(x_j - x_{j+1}) \geq v_{j+1} - 1,$$

and we obtain $v_j \geq v_{j+1}$. The proof that if $v_j = v_{j+1}$ then j precedes $j+1$ in the permutation π is identical to that above.

Thus x is covered either by a simplex of type (a) or one of type (b), and the theorem is proved. □

3 The Soifer Construction

The second (graphical) proof in [1, 2], due to Soifer according to Chapter 9 of [4], demonstrates how to cover a 2-dimensional base slightly shorter than 1 with $2n-1$ triangles. We generalize this construction to prove

Theorem 2 *For $n \geq d$ and $\delta \leq (d+2)^{-1}d^{-(n-d)}$, $S^{n+\delta}$ can be covered by*

$$n^d + (d+1)^d - 2d^d + (d-1)^d$$

unit right d-simplices.

Proof: We proceed by induction on n. For $n = d$, the result follows from Theorem 1. Now suppose $n > d$, and that the theorem holds for $n-1$. Let

$$\gamma := \frac{1}{n-d}\delta \leq \delta,$$

and divide $S^{n+\delta}$ into its base $B^{n+\delta}_{1-(d-1)\gamma}$ and its top $T^{n+\delta}_{1-(d-1)\gamma}$. Note that

$$T^{n+\delta}_{1-(d-1)\gamma} = \{x \in \mathbb{R}^d : n+\delta \geq x_1 \geq \cdots \geq x_d \geq 1-(d-1)\gamma\}$$

is a translation of $S^{n-1+\delta+(d-1)\gamma}$, and since

$$\delta + (d-1)\gamma \leq d\delta \leq (d+2)^{-1}d^{-(n-d-1)},$$

it can be covered by $(n-1)^d + (d+1)^d - 2d^d + (d-1)^d$ unit right simplices by the inductive hypothesis. Thus the result will follow from the lemma below. □

Lemma 2 *$B^{n+\delta}_{1-(d-1)\gamma}$ can be covered by $n^d - (n-1)^d$ unit right d-simplices.*

Proof: In the proof of Lemma 1, we took the "bottom" simplices of the triangulation of B^{n+1}_1 and squeezed them together, pushing up the remaining simplices to cover a base slightly higher than 1. Now we take the bottom simplices of the triangulation of B^n_1 and spread them out, letting the remaining simplices rattle down filling the gaps to cover a base slightly shorter than 1.

With γ as above, note that

$$(n-1)(1+\gamma) + 1 = n + \delta + (d-1)\gamma. \tag{7}$$

The simplices of the triangulation of B^n_1 are those $k(v, \pi)$ where

$$v \in S^{n-1} \cap \mathbb{Z}^d; \quad v_d = 0; \quad \text{if } v_j = v_{j+1},\ j \text{ precedes } j+1 \text{ in } \pi. \tag{8}$$

We spread out and rattle down these simplices as follows:

$$(c)\text{ If } \pi^{-1}(d) = d, \hat{k}(v,\pi) := k((1+\gamma)v, \pi);$$

$$(d)\text{ if } \pi^{-1}(d) = j < d, \hat{k}(v,\pi) := k((1+\gamma)v - (d-j)\gamma e, \pi).$$

We need to show that every $x \in B^{n+\delta}_{1-(d-1)\gamma}$ is covered by at least one such $\hat{k}(v,\pi)$, where (v,π) satisfies (8). Given such an x, we can choose $v \in \mathbb{Z}^d_+$ with $v_d = 0$ and $v \leq (n-1)e$ so that all the components of

$$w := x - (1+\gamma)v$$

lie between $-\gamma$ and 1. We then order these components using the permutation π so that

$$1 \geq w_{\pi(1)} \geq \cdots \geq w_{\pi(d)} > -\gamma. \tag{9}$$

Since $x \leq (n+\delta)e$, equation (7) implies that

$$\text{if } v_i = n-1 \text{ for } 1 \leq i < d, w_i \leq 1-(d-1)\gamma. \tag{10}$$

Also, since $x \in B^{n+\delta}_{1-(d-1)\gamma}$, $w_d \leq 1-(d-1)\gamma$. Finally, if there is a set of components of w that are equal, we may modify π so that their indices appear in ascending order:

$$\text{if } w_{\pi(i)} = w_{\pi(i+1)} \text{ for } 1 \leq i < d, \pi(i) < \pi(i+1). \tag{11}$$

Let us first assume that $\pi^{-1}(d) = d$, Then $w_d \geq 0$, so that

$$1 \geq w_{\pi(1)} \geq \cdots \geq w_{\pi(d)} \geq 0,$$

and x lies in $k((1+\gamma)v, \pi)$, and it remains to show (8). We already know that $0 \leq v \leq (n-1)e$ and $v_d = 0$. Since $x_i \geq x_{i+1}$,

$$v_i \geq v_{i+1} + \frac{w_{i+1} - w_i}{1+\gamma} \geq v_{i+1} - \frac{1}{1+\gamma} > v_{j+1} - 1$$

and so $v_i \geq v_{i+1}$, and if these are equal, then $w_i \geq w_{i+1}$ and then (9) and (11) imply that i precedes $i+1$ in π. Thus x lies in a simplex of type (c) above.

Next suppose $k := \pi^{-1}(d) < d$. Let i be the lowest index such that

$$w_{\pi(i)} \leq 1-(d-k+i-1)\gamma;$$

note that the index k satisfies this inequality so that $i \leq k$. Also,

$$\text{if } i > 1, \quad \text{for } h < i, \quad w_{\pi(h)} \geq 1-(d-k+i-2)\gamma, \text{ and hence } v_{\pi(h)} < n-1. \tag{12}$$

using (10). We now increase $v_{\pi(h)}$ by 1 for each $h < i$, to get v'. From the above, we still have $v' \leq (n-1)e$. Let $w' := x-(1+\gamma)v'$. For $j \geq i$, $w'_{\pi(j)} = w_{\pi(j)}$, while for $h < i$, $w'_{\pi(h)} = w_{\pi(h)} - 1 - \gamma$, and so using (9) and (12), we find

$$\text{for } h < i, -\gamma \geq w'_{\pi(h)} \geq -(d-k+i-1)\gamma.$$

Thus if we order the components of w', with strings of equal components in ascending order, we find the permutation ρ with $\rho = (\pi(i), \ldots, \pi(d), \pi(1), \ldots, \pi(i-1))$ with $\rho^{-1}(d) = k-i+1 =: j'$. We also have

$$1-(d-j')\gamma \geq w'_{\rho(1)} \geq \cdots \geq w'_{\rho(d)} \geq -(d-j')\gamma,$$

and so x lies in $\hat{k}(v', \rho)$, and it remains to show that v' and ρ satisfy (8). But this follows exactly the argument used above for the case $\pi^{-1}(d) = d$, and so x lies in a simplex of type (d) and the proof is complete. $\square$

4 Discussion

Our results are not tight. Indeed, for $d > 2$ and $n = 1$, we have shown that $S^{1+\delta}$ can be covered by $2^d - 1$ right d-simplices, while $d+1$ suffice by using a construction also similar to Soifer's construction in [1, 2]:

Proposition 1 *For $\delta := d^{-1}$, $S^{1+\delta}$ can be covered by $d+1$ right d-simplices.*

Proof: Let $v^0 := 0$, $v^j = v^{j-1} + \delta e^j$ for $j = 1, \ldots, d$. Let ι denote the identity permutation $(1, 2, \ldots, d)$. We show that the $d+1$ right d-simplices $k(v^j, \iota)$ for $j = 0, \ldots, d$ cover $S^{1+\delta}$.

Consider any $x \in S^{1+\delta}$. Then

$$1 + \delta =: x_0 \geq x_1 \geq \cdots \geq x_d \geq x_{d+1} := 0.$$

There are then $d+1$ nonnegative gaps $x_i - x_{i+1}$, $i = 0, \ldots, d$, summing to $1 + \delta$, and so since $(d+1)\delta = 1 + \delta$, one of these, say that indexed by $i = j$, must be at least δ. But then

$$1 \geq x_1 - \delta \geq \cdots \geq x_j - \delta \geq x_{j+1} \geq \cdots x_d \geq 0$$

(with obvious modifications if $j = 0$ or $j = d$), so that $x \in k(v^j, \iota)$. □

If Lemma 2 could be extended to all n, then Proposition 1 would provide the base case to prove that $S^{n+\delta}$ could be covered by $n^d + d$ unit right d-simplices. However, the rather delicate arguments in Lemma 2 seem to require that the "bottom" simplices be spread out to not only cover components up to $n + \delta$, but further up to $n + \delta + (d-1)\gamma$ (see (7)), and this necessitates $n > d$.

It would be nice to complement our results with lower bounds on the number of unit right d-simplices required to cover $S^{n+\delta}$ ($\delta > 0$), but such results are rare even for $d = 2$. Indeed, volume considerations ensure that at least $n^d + 1$ are necessary, while for $d = 2$ and $n = 1$ or $n = 2$, considering all points in $S^{n+\delta}$ all of whose components are integer multiples of $1 + \delta/n$, no two of which can lie in a single unit 2-simplex, shows that $n^2 + 2$ are necessary.

Both of these techniques are special cases of bounds from measures on $\mathbb{R}^d$. In general, we can consider the moment problem

$$M := \sup\{\mu(S^{n+\delta}) : \mu \text{ is a measure on } \mathbb{R}^d \text{ with } \mu(\Sigma) \leq 1 \text{ for any right } d\text{-simplex } \Sigma\}.$$

Then M, rounded up to the next integer, provides a lower bound on the number of unit d-simplices to cover $S^{1+\delta}$. Perhaps numerical computations on discretizations of this problem can provide insights allowing the construction of measures yielding non-trivial lower bounds on the number of simplices required.

Finally, we note that for $d = 2$, right d-simplices are isosceles right triangles, and that Xu, Yuan, and Ding [6] consider a different problem of covering isosceles right triangles with isosceles right triangles of possibly different sizes and allowing for rotations as well as translations and coordinate permutations.

References

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