Can $n^d + 1$ unit right $d$-simplices cover a right $d$-simplex with shortest side $n + \epsilon$?

Michael J. Todd *

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Abstract

In a famous short paper, Conway and Soifer show that $n^2 + 2$ equilateral triangles with edge length 1 can cover one with side $n + \epsilon$. We provide a generalization to $d$ dimensions.

1 Introduction

We denote by $e^1, \ldots, e^d$ the unit coordinate vectors in $\mathbb{R}^d$, and by $e := \sum_j e^j$ the vector of ones. A unit right $d$-simplex is defined to be the convex hull of $0$, $e^1$, $e^1 + e^2$, \ldots, $e^1 + e^2 + \cdots + e^d$, or any of its images under coordinate permutations and translations. A right $d$-simplex is a dilation of a unit right $d$-simplex; if the dilation is by a factor $\alpha > 0$, its shortest side has length $\alpha$.

We are not able to answer the question in the title, but we do show that, if $\epsilon \leq \delta := (n + 2)^{-1}$, then $(n + 1)^d + (n - 1)^d - n^d$ suffice. (This fails for the trivial case $d = 1$; we assume implicitly throughout that $d > 1$.) Notice that, under the transformation $x \mapsto Mx$, where

$$
M := \begin{bmatrix}
1 & -1/2 \\
0 & \sqrt{3}/2
\end{bmatrix},
$$

right 2-simplices are transformed into equilateral triangles, so that our result implies that of Conway and Soifer [1].

We need a convenient notation for right $d$-simplices. For $v \in \mathbb{R}^d$ and $\pi$ a permutation of $\{1, \ldots, d\}$, we use $k(v, \pi)$ to denote the convex hull of $v$, $v + e^{\pi(1)}$, $v + e^{\pi(1)} + e^{\pi(2)}$, \ldots, $v + e$. It is easy to see that

$$
k(v, \pi) = \{ x \in \mathbb{R}^d : 1 \geq (x - v)_{\pi(1)} \geq (x - v)_{\pi(2)} \geq \cdots \geq (x - v)_{\pi(d)} \geq 0 \}.
$$

It is well known that the set of all $k(0, \pi)$’s, as $\pi$ ranges over all permutations, triangulates the unit cube, while the set of all $k(v, \pi)$’s, with $v$ an integer vector and $\pi$ a permutation, triangulates $\mathbb{R}^d$. See, for example, [2]. These simplices are exactly the $d$-dimensional pieces when $\mathbb{R}^d$ is partitioned by all hyperplanes of the form $x_j = z$ or $x_i - x_j = z$, with $z$
an integer. More relevant to our purposes, the set of all \( k(v, \pi) \)'s, with \( v \) an integer vector and \( \pi \) a permutation, that lie in the right \( d \)-simplex

\[
S^n := \{ x \in \mathbb{R}^d : n \geq x_1 \geq x_2 \geq \cdots \geq x_d \geq 0 \},
\]
covers (indeed, triangulates) that set. In fact, \( k(v, \pi) \) lies in this set iff \( v \in S^{n-1} \) and, if \( v_j = v_{j+1} \), \( j \) precedes \( j+1 \) in the permutation \( \pi \). By volume considerations, there are \( n^d \) such unit right \( d \)-simplices.

We can also easily see that the “base” of \( S^n \), where \( x_d \) lies between 0 and 1, can also be triangulated, by \( n^d - (n-1)^d \) of these simplices, those with \( v_d = 0 \).

2 The Result

**Theorem 1** For \( \delta := (n + 2)^{-1} \), the right \( d \)-simplex

\[
S^{n+\delta} := \{ x \in \mathbb{R}^d : n + \delta \geq x_1 \geq x_2 \geq \cdots \geq x_d \geq 0 \},
\]
with shortest side \( n + \delta \), can be covered by

\[
(n + 1)^d + (n - 1)^d - n^d
\]
unit right \( d \)-simplices.

**Proof:** We divide \( S^{n+\delta} \) into its base

\[
S_1^{n+\delta} := \{ x \in S^{n+\delta} : 0 \leq x_d \leq 1 + \delta \}
\]
and its top

\[
S_2^{n+\delta} := \{ x \in S^{n+\delta} : x_d \geq 1 + \delta \}.
\]

Note that the top can be written as

\[
S_2^{n+\delta} = \{ x \in \mathbb{R}^d : n + \delta \geq x_1 \geq x_2 \geq \cdots \geq x_d \geq 1 + \delta \},
\]
which is just the translation by \((1 + \delta)e\) of \( S^{n-1} \), and can therefore be triangulated by \((n - 1)^d \) unit right \( d \)-simplices as above.

It remains to cover the base \( S_1^{n+\delta} \) by \((n + 1)^d - n^d \) unit right \( d \)-simplices. Note that this base is somewhat similar to the base

\[
S'_1 := \{ x \in S^{n+1} : 0 \leq x_d \leq 1 \},
\]
which as we noted above, can be triangulated by exactly this many unit right \( d \)-simplices. Indeed, the base we are interested in has its first \( d - 1 \) components squeezed in (from \( n + 1 \) to \( n + \delta \)) and its last component stretched out (from 1 to 1 + \( \delta \)). We therefore apply an operation to the simplices in this triangulation, roughly as the individual cloves are transformed by squeezing the head of a roasted garlic.

As we observed above, the simplices of the triangulation of \( S'_1 \) are those \( k(v, \pi) \) where

\[
v \in S^n \cap \mathbb{Z}^d, \quad v_d = 0; \quad \text{if} \ v_j = v_{j+1}, \ j \text{ precedes } j+1 \text{ in } \pi.
\]  
(1)

We squeeze these simplices as follows:

(a) If \( \pi^{-1}(d) = d, k(v, \pi) := k((1 - \delta)v, \pi) \);
(b) if \( \pi^{-1}(d) < d, \tilde{k}(v, \pi) := k((1 - \delta)v + \delta e, \pi) \).

We need to show that every \( x \in S_1^{n+\delta} \) is covered by at least one such \( \tilde{k}(v, \pi) \), where \((v, \pi)\) satisfies (1).

For any such \( x \), we can choose \( v \in \mathbb{Z}_+^d \), \( v_d = 0 \), so that all components of

\[
     w := x - (1 - \delta)v,
\]

except possibly the last, lie between 0 and 1. We then order these components using the permutation \( \pi \). Suppose first we can choose \( \pi \) so that \( d \) comes last:

\[
     1 \geq w_{\pi(1)} \geq \cdots \geq w_{\pi(d)} \geq 0, \quad \pi^{-1}(d) = d. \tag{2}
\]

Note that there is some choice involved for \( j < d \); if \( v_j > 0 \) and \( 0 \leq w_j \leq \delta \), we can decrease \( v_j \) by 1 so that \( 1 - \delta \leq w_j \leq 1 \) and then adjust \( \pi \) accordingly. Then we have

\[
     \text{if } v_j > 0 \text{ for } 1 \leq j < d, w_j > \delta. \tag{3}
\]

Moreover, if there is a set of components of \( w \) that are equal, we may modify \( \pi \) so that their indices appear in ascending order:

\[
     \text{if } w_{\pi(j)} = w_{\pi(j+1)} \text{ for } 1 \leq j < d, \pi(j) < \pi(j+1). \tag{4}
\]

We show that, if \( v \) and \( \pi \) can be chosen so that (2)–(4) hold, then \( x \) lies in the simplex \( \tilde{k}(v, \pi) \) of type (a). By the first of these conditions, it is only necessary to check (1).

First, we have \( w_1 \geq 0 \), so that

\[
     v_1 \leq (1 - \delta)^{-1}x_1 \leq (1 - \delta)^{-1}(n + \delta) = n + 1.
\]

Moreover, if \( v_1 = n + 1 \), we have equality throughout, so that \( v_1 > 0 \) and \( w_1 = 0 \), contradicting (3). Hence \( v_1 \leq n \).

Next, consider the condition \( v_j \geq v_{j+1} \). If \( v_{j+1} = 0 \), then this holds by default. If not, then \( j + 1 < d \) and by (3), \( w_{j+1} > \delta \), so that \( w_j < w_{j+1} + 1 - \delta \) and thus

\[
     v_j > v_{j+1} - 1 + (1 - \delta)^{-1}(x_j - x_{j+1}) \geq v_{j+1} - 1,
\]

and we obtain \( v_j \geq v_{j+1} \).

Finally, if \( v_j = v_{j+1} \), then since \( x_j \geq x_{j+1} \) we have \( w_j \geq w_{j+1} \). Thus \( j \) precedes \( j + 1 \) in \( \pi \), either by (2) if these components are unequal, or by (4) if they are equal. This completes the verification of (1), and so \( x \) is covered.

Note that, if \( x_d \leq \delta \), then we can find \( v \) and \( \pi \) so that (2) holds. Indeed, we order the components of \( w \) as above, and ensure that if \( v_j > 0 \) and \( j < d \), then \( w_j > \delta \) and \( j \) precedes \( d \) in \( \pi \). But if \( v_j = 0 \) for \( j < d \), then \( x_j \geq x_d \) ensures that \( w_j \geq w_d \), and thus we can arrange that \( d \) comes last in \( \pi \). Thus the bottom sliver of the base is covered by simplices of type (a).

Now we assume that \( x \) cannot be covered by such a simplex. Then \( x_d > \delta \), and hence \( x_j > \delta \) for all \( j \). We can then find \( v \in \mathbb{Z}_+^d \) with \( v_d = 0 \) and a permutation \( \pi \) so that, with \( w \) again defined as \( x - (1 - \delta)v \), we have

\[
     1 + \delta \geq w_{\pi(1)} \geq \cdots \geq w_{\pi(d)} \geq \delta. \tag{5}
\]
Moreover, as above, if $1 \leq w_j \leq 1+\delta$ for $j < d$, we can increase $v_j$ by 1 so that $\delta \leq w_j \leq 2\delta$ and then adjust $\pi$ accordingly, so that

$$\text{for } j < d, w_j < 1.$$ \hspace{1cm} (6)

We can also ensure that equal components of $w$ are suitably ordered, so that (4) holds.

If $w_d > 1$, then because of (6), $\pi^{-1}(d) = 1$. If instead $w_d \leq 1$, then (5) and (6) show that (2) holds, so that if $\pi^{-1}(d) = d$, $x$ could be covered by a simplex of type (a). Thus in either case, $\pi^{-1}(d) < d$, so that, if $w' := x - (1-\delta)v - \delta e$,

$$1 \geq w'_{\pi(1)} \geq \cdots \geq w'_{\pi(d)} \geq 0, \quad \pi^{-1}(d) < d,$$

and $x$ will be covered by a simplex of type (b) if we can verify (1).

Suppose (4)–(6) hold. Then $w_1 \geq \delta$, so

$$v_1 \leq (1-\delta)^{-1}x_1 - (1-\delta)^{-1}\delta < (1-\delta)^{-1}(n+\delta) = n + 1,$$

and we have $v_1 \leq n$.

Next, consider the condition $v_j \geq v_{j+1}$. If $v_{j+1} = 0$, then this holds by default. If not, then $j + 1 < d$ and by (5) and (6), $w_{j+1} \geq \delta$ and $w_j < 1$, so that $w_j < w_{j+1} + 1 - \delta$ and thus

$$v_j > v_j + 1 + (1-\delta)^{-1}(x_j - x_{j+1}) \geq v_{j+1} - 1,$$

and we obtain $v_j \geq v_{j+1}$. The proof that if $v_j = v_{j+1}$ then $j$ precedes $j + 1$ in the permutation $\pi$ is identical to that above.

Thus $x$ is covered either by a simplex of type (a) or one of type (b), and the theorem is proved.

$\square$

References
