1. Assume $X \succ 0$, $S \succ 0$, and $P$ is invertible. Suppose that the scaled matrices $PX^TP$ and $P^{-T}SP^{-1}$ commute. Show that $MXS$, where $M = P^TP$, is symmetric and positive definite, and that the MZ search directions given by this $P$ at the iterate $(X,y,S)$ are well-defined.

2. In this question, all operators are restricted to $\mathbb{M}^n$.
   a) If $G$ and $H$ are $n \times n$ matrices, show that the adjoint of $G \odot H$ is $G^T \odot H^T$, i.e., $U \cdot (G \odot H)V = (G^T \odot H^T)U \cdot V$ for all $U, V \in \mathbb{M}^n$.
   b) If $G$ and $H$ are $n \times n$ matrices, show that $(G \odot H)(J \odot J) = (GJ \odot HJ)$ and $(J \odot J)(G \odot H) = (JG \odot JH)$.
   c) Suppose $G, H \in \mathbb{M}^n$, with eigenvalues $(\lambda_i)_1^n$ and $(\mu_j)_1^n$ respectively. Show that, if $G$ and $H$ commute, then $G \odot H$ has eigenvalues $\frac{1}{2}(\lambda_i\mu_j + \lambda_j\mu_i)$, $1 \leq i \leq j \leq n$. (You may want to start first with the case that $G$ and $H$ are diagonal.)

3. It appears that computing the scaling point $W$ for $X$ and $S$ requires two eigenvalue decompositions, one of $S$ to get $S^{\pm1/2}$ and then a second of $S^{1/2}XS^{1/2}$ to get its square root. Show that it suffices to compute a Cholesky factorization of $S$ and then a single eigenvalue decomposition.

4. Suppose you know that all feasible solutions of a standard primal form SDP problem $(P)$ satisfy $I \cdot X \leq \gamma$ for some positive $\gamma$. Show that $(P)$ can be written in an equivalent way so that its dual is the problem of minimizing the maximum eigenvalue of a matrix depending linearly on some parameters.

5. a) Suppose $q$ is a unit eigenvector corresponding to the largest eigenvalue of $C - A^*y$. Let $f(y) := \lambda_{\max}(C - A^*y)$. Show that $-A(qq^T)$ is a subgradient of $f$ at $y$.
   b) More generally, suppose that $q$ is a unit vector with $q^T(C - A^*y)q \geq \lambda_{\max}(C - A^*y) - \epsilon$ for some nonnegative $\epsilon$. Show that $-A(qq^T)$ is an $\epsilon$-subgradient of $f$ at $y$, i.e., that $f(z) \geq f(y) + (-A(qq^T))^T(z - y) - \epsilon$ for all $z$. 