Recitation II

Selim Erdogan

Neighborhood polyhedron.

A polyhedron $P$ is called neighborhood (or 1-neighborhood) if for every pair of vertices $v$ and $w$ of $P$, $[v,w]$ is an edge of $P$. ($\exists \ c \in \mathbb{R}^d$ with $[v,w] = \text{argmin} \left\{ c^T x : x \in \mathbb{R}^d \right\}$)

Example: $P = \{ x : e^T x \leq 1, \ x \geq 0 \}$ = conv $\{ 0, e_1, e_2, \ldots, e_d \}$ with $e_i = \left[ \begin{array}{c} 0 \\ 1 \\ \vdots \\ 1 \end{array} \right]_i$.

Note: $P$ has $d+1$ vertices.

Definition: The diameter of $P$, $\delta(P)$, is the largest over all vertices $v,w$ of $P$ of the smallest # of edges of $P$ in a path from $v$ to $w$: $\delta(P) = \max_{v,w \in P} \left\{ \min \{ k : \exists v_0 = v, v_1, \ldots, v_k = w \text{ edge of } P, v_1, \ldots, v_k \} \right\}$

Fact: $P$ is neighborhood iff $\delta(P) = 1$

Remark: A low value for $\delta(P)$ indicates that a LP problem with feasible region $P$ may have a good variant of the simplex method, while if $\delta(P)$ is high simplex method must be bad for some initial vertex.

Note, simplices in $\mathbb{R}^d$ are neighborhood, but have only $d+1$ vertices.

But:

Theorem: For $d\geq 4$, $\exists$ a polytope $P$ with $n$ vertices, for any $n \geq d+1$, that is neighborhood.