Last time we saw that every bounded polyhedra is a polytope in the set of convex combination of its vertices.
Now we will extend the theory to pointed polyhedra (i.e., those that contain no lines).

**Definition 1** Let $C$ be a nonempty convex set: then the **recession cone** of $C$, $\text{rec}(C)$, is
\[
\{ d \in R^m : \forall x \in C, \forall \alpha \geq 0, x + \alpha d \in C \}.
\]

**Proposition 1** If $C$ is a nonempty set then $\text{rec}(C)$ is a nonempty convex cone.

**Proof:**
Let $d_1, d_2 \in \text{rec}(C), \lambda_1, \lambda_2 \geq 0$. We want to show that $\lambda_1 d_1 + \lambda_2 d_2 \in \text{rec}(C)$. For any $x \in C$ and any $\alpha \geq 0$
\[
x + \alpha (\lambda_1 d_1 + \lambda_2 d_2) = [x + (\alpha \lambda_1) d_1] + (\alpha \lambda_2) d_2.
\]
The quantity in brackets lies in $C$ since $\alpha \lambda_1 \geq 0$ and $d_1 \in \text{rec}(C)$, and then the desired vector lies in $C$ because $\alpha \lambda_1 \geq 0$ and $d_2 \in \text{rec}(C)$. Also, $0 \in \text{rec}(C)$ by definition. \(\square\)

**Proposition 2** For $Q := \{ y \in R^m : A_x^T y \leq c_x, A_w^T y = c_w \}$ then (if $Q$ is nonempty)
\[
\text{rec}(Q) = \{ d \in R^m : A_x^T d \leq 0, A_w^T d = 0 \}.
\]

**Proof:**
\(\supseteq\):
if $A_x^T d \leq 0, A_w^T d = 0$ then for any $y \in Q, \alpha \geq 0$.
\[
A_x^T (y + \alpha d) = A_x^T y + \alpha A_x^T d \leq c_x + 0 = c_x,
\]
and similarly
\[
A_w^T (y + \alpha d) = c_w,
\]
hence $(y + \alpha d) \in Q$.

\(\subseteq\):
Suppose $d \in \text{rec}(Q)$, and choose any $y \in Q$. Then $\forall \alpha \geq 0$
\[
A_x^T (y + \alpha d) = A_x^T y + \alpha A_x^T d \leq c_x;
\]
and then
\[
A_x^T y \leq c_x \Rightarrow A_x^T d \leq 0
\]
(otherwise, the inequality would fail for large $\alpha$); similarly
\[
A_w^T d = 0.
\]
\(\square\)
Theorem 1 (Representation of Pointed Polyhedra). Let $Q$ (defined as in Proposition 2) be a nonempty pointed polyhedron, and let $P$ be the set of all convex combinations of its vertices and $K$ be its recession cone. Then

$$Q = P + K := \{ p + d : p \in P, d \in K \}.$$ 

Proof:

$\supseteq:$
Every vertex of $Q$ satisfies all linear constraints of $Q$ so $p$ also does for any $p \in P$.
So any $p + d \in P + K$ has

$$A^T_x (p + d) = A^T_x p + A^T_x d \leq c^x + 0 = c^x;$$

$$A^T_w (p + d) = A^T_w p + A^T_w d = c^w + 0 = c^w.$$ 

$\subseteq:$
The proof is by induction on $\{ m - ra(y) \}$.

True for $\{ m - ra(y) = 0 \} \iff y$ is itself a vertex of $Q$ and $d = 0 \in \text{rec}(C)$.

Suppose true if $\{ m - ra(y) < k \}$ for some $k > 0$ and consider $y \in Q$ with $ra(y) = m - k < m$.
Choose $0 \neq d \in \mathbb{R}^m$ with $a^T_j d = 0, \forall j \in I(y)$ and consider $y + \alpha d, \alpha \in \mathbb{R}$. Since $Q$ is pointed there are three cases to consider.

(1) $\alpha$ is bounded above and below, say by $\underline{\alpha} < 0 \ & \ \bar{\alpha} > 0$.
As in the previous theorem

$$y = \frac{\bar{\alpha}}{\bar{\alpha} - \underline{\alpha}} (y + \alpha d) + \frac{\underline{\alpha}}{\bar{\alpha} - \underline{\alpha}} (y + \bar{\alpha} d),$$

and $(y + \bar{\alpha} d)$ has $m - ra(y + \bar{\alpha} d) < k$, so

$$(y + \bar{\alpha} d) = \bar{p} + \bar{d} \ , \ \bar{p} \in P \ , \ \bar{d} \in K;$$

and similarly

$$(y + \underline{\alpha} d) = \underline{p} + \underline{d} \ , \ \underline{p} \in P \ , \ \underline{d} \in K;$$

so

$$y = \frac{\bar{\alpha}}{\bar{\alpha} - \underline{\alpha}} (p + d) + \frac{\underline{\alpha}}{\bar{\alpha} - \underline{\alpha}} (p + \bar{d})$$

$$= \left[ \frac{\bar{\alpha}}{\bar{\alpha} - \underline{\alpha}} p + \frac{\underline{\alpha}}{\bar{\alpha} - \underline{\alpha}} \bar{p} \right] + \{ \ldots \underline{d} + \ldots \bar{d} \}.$$ 

The vector in brackets is a point of $P$ and that in braces a point in $K$.

(2) $\alpha$ is bounded below but not above. Then $d \in K$ and $y = [y + \alpha d] + (-\alpha) d$, with $\alpha$ defined as before. The vector in brackets lies in $P + K$ as in the first part by the inductive hypothesis. Therefore
\begin{align*}
y &= (p + d) + (-\alpha) d \\
    &= p + (d + (-\alpha) d)
\end{align*}

lies in \( P + K \).

(3) \( \alpha \) is bounded above but not below. Then we can simply switch \( d \) to \(-d\) and \( \alpha \) to \(-\alpha \), and we get back to case(2).

This completes the proof. \( \square \)

**Theorem 2** (*Fundamental theorem of LP*). Consider the LP problem \( \max \{ b^T y : y \in Q \} \) with \( Q \) being a pointed polyhedron. Then

1. if there is a feasible solution, there is a vertex solution (basic feasible solution);
2. if there is a feasible solution and \( b^T y \) is unbounded above on \( Q \), then there is a ray or halfline: \( \{ y + \alpha d : \alpha \geq 0 \} \in Q \) on which \( b^T y \) is unbounded above; and
3. if \( b^T y \) is bounded above on \( Q \), then the max is attained and attained at a vertex \( Q \).

**Proof:**

(1): If \( Q \neq \emptyset \), so there exists a vertex.

(2) & (3):

Assume \( P \neq \emptyset \) & \( P \) is a set of convex combinations of \( v_1, v_2, v_3, ..., v_k \).

\[
\sup \{ b^T y : y \in Q \} = \sup \{ b^T y : y \in P + K \} = \sup \{ b^T p + b^T d : p \in P, d \in K \} = \sup \{ b^T p : p \in P \} + \sup \{ b^T d : d \in K \}.
\]

If there is some \( d \in K \) with \( b^T d > 0 \) then by considering \( \alpha d \), \( \alpha \to +\infty \), see that \( \sup \{ b^T d : d \in K \} = +\infty \). Then \( b^T y \) is unbounded above on \( Q \) and clearly unbounded above on \( \{ y + \alpha d : \alpha \geq 0 \} \) for any \( y \in Q \).

If there is no such \( d \in K \), then \( \sup \{ b^T d : d \in K \} = 0 \), attained by \( d = 0 \). Then

\[
\sup \{ b^T y : y \in Q \} = \sup \{ b^T p : p \in P \} = \sup \{ \sum_{i=1}^{k} \lambda_i (b^T v_i) : \sum_{i=1}^{k} \lambda_i = 1, \text{ all } \lambda_i \geq 0 \} = \max_{1 \leq i \leq k} b^T v_i.
\]

In this case \( \sup \{ b^T y : y \in Q \} \) is attained by \( y = v_i \) where \( i \) attains the maximum. \( \square \)